



Multicriteria optimization of variable thickness plates using adaptive weighted sum method

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MS received 3 November 2020; revised 15 January 2021; accepted 2 March 2021

Abstract. In this paper, a multicriteria design framework for variable thickness isotropic plates using the adaptive weighted sum method is developed. The design objectives are the minimization of weight and static displacement and the design variables are the elemental thicknesses of plates modelled using finite elements. Here, the multicriteria optimization framework is constructed by integrating the finite element method, analytical sensitivity technique along with optimization algorithms. The first-order shear deformation theory is used in the static and dynamic analyses of plates. Both single and multiobjective optimization studies are conducted to study the optimal thickness distributions of variable thickness plates under static and dynamic constraints. To study multicriteria optimization of plates, the weighted sum method is first applied which gives sparsely distributed Pareto optimal solutions. Then, the adaptive weighted sum method is employed where a coarser representation of Pareto optimal solutions is generated using the weighted sum method and less populated regions are identified for further refinement. The suboptimization problems are solved in these regions to determine a new set of Pareto optimal solutions. The Pareto optimal curves obtained using the adaptive weighted sum method are also compared with the conventional weighted sum method under different constraints. The effect of boundary conditions on the Pareto optimal solutions and thickness distributions of plates is also investigated.

Keywords. Multicriteria optimization; Weighted sum method; Adaptive weighted sum method; Pareto front; Variable thickness plate; Finite elements.

1. Introduction

Structural design process using mathematical models is extensively used in aircraft, automotive and civil industries for the design and development of its structures. It consists of structural modeling, design parameterization, finite element analysis, design sensitivity analysis and design optimization. In general, the structural design optimization problems are formulated as a single objective under certain behaviour constraints. However, in practical applications, considering only one objective function rarely gives a representative measure of the performance of the structure. There exist several design objectives (usually conflicting) which must be considered in the optimal design of structures. In order to include many design objectives, one can solve each single objective optimization sequentially. However, this method cannot find an optimal solution because only one objective is solved in each optimization process. These objective functions should be optimized simultaneously to obtain solutions. Here, the interaction

among objectives gives rise to optimal solutions that trade different objectives against each other. In multicriteria optimization, there may not exist a solution that is best with respect to all objectives. Instead, these are equally good solutions which are known as Pareto optimal solutions or non-dominated solutions. Here, an effort is first made to find a set of trade-off optimal solutions. After a set of such trade-off solutions is found, the decision maker then uses higher level qualitative considerations to make a choice of solution.

Many optimization studies have been conducted by the researchers earlier considering weight (cost) or performance of the structures (static displacement, fundamental frequency or buckling load) as a single design objective. Armand [1] analytically addressed the minimum weight design of a simply supported, rectangular shear plate with a prescribed fundamental frequency of free vibration. He extended the classical methods of optimal control theory to a two-dimensional structural optimization problem. Kamat [2] optimized thin rectangular plates for optimum fundamental frequency using the finite element approach. The results suggested the non-uniqueness of solutions,

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Published online: 21 April 2021

and explained about the difficulty in determining a true global optimum. Haug *et al* [3, 4] studied the optimal design of structures with a natural frequency constraint using the generalized steepest descent method. He also studied optimization of a simply supported square plate using a collocation technique wherein only a finite number of structural elements were used. The sensitivities were evaluated by using finite difference techniques. Leal and Soares [5] discussed the capacities of the mixed finite element method in optimization of bending of plates. Three different methods of sensitivity analysis namely analytical, semi-analytical and finite difference methods were used for the optimal design of bending of plates. The ADS (Automated Design Synthesis) program was used to optimize the weight of structure by considering plate thickness as design variable under displacement, natural frequency and stress constraints. Grandhi *et al* [6] discussed the minimum weight design of isotropic plate structures with single and multiple frequency constraints, using the generalized compound scaling algorithm for reaching the optimum. Moradi *et al* [7] discussed the maximization of fundamental frequency of rectangular orthotropic composite plates subject to an equality constraint on plate volume and inequality constraints on the lower and upper bounds of allowable range of thicknesses using nonlinear constrained optimization theory. Hinton *et al* [8] studied thickness optimization of variable thickness isotropic plates and shells by integrating Coons patch technique for thickness definition, structural analysis using the finite element method, sensitivity evaluation using the global finite difference method and the sequential quadratic programming method. Both strain energy and weight minimization of plates and shells were investigated with constraints on volume and von Mises stresses respectively and optimal thickness variations were obtained under different loading, boundary and design variable linking conditions. Later, Hinton *et al* [9, 10] also developed a computational tool for structural shape optimization of isotropic shells and folded plates using curved, variable thickness finite strips in which the strain energy or the weight of the structure was minimised subject to certain constraints. Optimal shapes were presented for various shells and folded plates of variable thickness resting on elastic foundation. Narita and Robinson [11] used layerwise optimization approach to determine the optimum fiber orientation angles for the maximum fundamental frequency of cylindrically curved laminated panels under general edge conditions. Faria and Almeida [12] studied buckling optimization of variable thickness plates where the variability in the loading distribution was also taken into account. A min-max strategy was used to handle the loading variability such that the resulting optimal design could withstand an entire class of linear piecewise loadings. Pyrz [13] investigated optimal material distribution of plates by minimizing the structural strain

energy under constant volume constraint. He implemented genetic algorithms to study the optimal design of piecewise constant thickness plates subjected to bending.

In multi-objective optimization studies, different metrics frequently require different optimal designs. Several multiobjective optimization studies on plates and shells have been conducted by the researchers to improve the performance of the structures. Topal and Uzman [14] studied multiobjective optimization of symmetrically angle-ply square laminated plates subjected to biaxial compressive and uniform thermal loads. The design objective was the maximization of the buckling load for a weighted sum of the biaxial compressive and thermal loads for a given laminate by optimally determining the fiber orientation. The modified feasible direction (MFD) method was used for all optimization studies. The effect of weighting factors, number of layers, aspect ratios, load ratios and boundary conditions on the optimal design was also investigated. Topal [15] investigated multiobjective optimization of laminated cylindrical shells under external load. The design objective was to maximize a weighted sum of the frequency and buckling load for a given laminate by optimally determining the fiber orientation. The effect of different weighting ratios, shell aspect ratio, shell thickness-to-radius ratios and boundary conditions on the optimal designs was also investigated. Walker [16] studied multiobjective design of laminated plates for maximum stability using the golden section method along with the finite element method. The objective of the design optimization problem was to maximize a weighted sum of the critical buckling load and resonance frequency for a given laminate thickness by optimally determining the fibre orientation. Walker and Smith [17] investigated multiobjective optimization of laminated composite plates using genetic algorithms along with the finite element method. The design objective was to minimise a weighted sum of the mass and deflection of fibre reinforced structures with fibre orientations and laminae thicknesses as discrete design variables. Results were investigated for different load distributions, aspect ratio, and various combinations of clamped, simply supported and free boundary conditions. Abouhamze and Shakeri [18] presented a multi-objective optimization strategy for optimal stacking sequence of laminated cylindrical panels with respect to the fundamental frequency and critical buckling load, using the weighted summation method. To improve the speed of the optimization process, artificial neural networks were used to reproduce the behavior of the structure both in free vibration and buckling conditions. A genetic algorithm was used for all optimization studies. Serhat and Basdogan [19] presented a multi-objective design methodology for laminated composite plates with dynamic and load-carrying requirements. Lamination parameters were used to characterize laminate stiffness matrices. The weighted sum method was used to maximize the fundamental frequency, buckling load and equivalent stiffness metrics to improve the dynamic and load carrying

performance of laminated composite plates. Kim and Park [20] introduced an interactive multiobjective optimization technique called satisficing trade-off method to avoid difficulty in parameter selection for updating finite element model. Nicholas *et al* [21] investigated multiobjective optimization of laminated composite plates using genetic algorithms (NSGA II). The design objective was to maximize buckling load factor and minimize weight of composite structures with fibre orientations, stacking sequence and number of layers as design variables. Grandhi *et al* [22] discussed multiobjective optimization of large-scale structures using compound scaling algorithm. The behavioral constraints such as stress, displacement and fundamental frequency were treated as objectives to get the Pareto optimal front. A reliability based decision criterion was used for selecting the best compromise design. Borri and Speranzini [23] discussed the multicriteria optimization design of laminated composite shells based on analytical optimization methodology (trade-off method). Two objective functions, the maximum displacement and the total volume of the uniform thickness composite structures were considered to determine the Pareto-optimal curve. Madeira *et al* [24, 25] discussed the optimal design of laminated composite panels with objectives as minimization of weight and maximization of modal damping. The design variables were the number and position of constrained layer damping (CLD) treatments on the surface of the laminated plate. The problem was solved using Direct Multi Search solver and trade-off Pareto optimal fronts and the respective treatment configurations were obtained.

From the above literature, it can be observed that the conventional weighted sum method has been extensively used to solve various multicriteria optimization problems of plates and shells. Since, there are no general rules for selecting the weighting factors, optimization problems need to be solved repeatedly by varying the values of weighting factors which is very time consuming. Further, continuous variation of the weighting factors does not guarantee complete representation of Pareto optimal solutions and also it is not possible to obtain solutions in non-convex regions of the Pareto front.

In this work, a multicriteria optimization technique for variable thickness plates based on the adaptive weighted sum method is presented to avoid such difficulty. To the authors' best knowledge this technique has not been attempted by the researchers to study multicriteria optimization of variable thickness plates and shells. In this technique, an adaptive weighing factor selection procedure is adopted, which effectively determines a fairly well distribution of Pareto optimal solutions on a Pareto front. This technique can also be applied to optimization problems with both convex and non-convex Pareto front. The complete optimization framework is constructed by integrating the finite element method, analytical sensitivity technique along with optimization algorithms. The finite element formulation of plate is done using the first order shear

deformation theory and an analytical design sensitivity technique is employed to evaluate the derivatives of objective and constraint functions. To formulate multicriteria optimization problems, two objectives, the minimization of weight (cost) and static deflection (performance) are considered with variable thicknesses of plate as design variables. First, a coarser representation of Pareto solutions is generated using the weighted sum method and sparsely populated regions are identified for further refinement. The suboptimization problems are solved in these regions to determine a new set of optimal solutions by applying additional inequality constraints in the objective space. The applicability of the present approach is demonstrated by solving Pareto optimal solutions for variable thickness plates under various constraints and boundary conditions. The Pareto optimal curves obtained using the adaptive weighted sum method are also compared with the conventional weighted sum method under different constraints.

2. Multicriteria optimization model

Multicriteria optimization of variable thickness isotropic plates is formulated by considering two objectives, the minimization of weight (cost) and static displacement (performance) with constraints on fundamental frequency and maximum displacement. Here, the structural domain is discretized into finite elements and the thickness of each element is treated as a design variable. The equations for bi-objective optimization of plates can be written as:

$$\begin{aligned}
 &\text{Minimize} && F_1 = \rho \sum_{i=1}^{NE} a_i h_i \\
 &&& F_2 = \delta_{\max} \\
 &\text{subjected to} && g_1 = \delta_{\max} - \delta_{\lim} \leq 0 \\
 &&& g_2 = \omega_d - \omega_f \leq 0 \\
 &\text{and} && h^L \leq h_i \leq h^U
 \end{aligned} \tag{1}$$

where F_1 and F_2 are two objective functions to be minimized and g_1 and g_2 are two inequality constraints. NE denotes the number of finite elements whereas a_i and h_i represent the area and thickness of each element of the plate respectively. δ_{\max} and ω_f are the maximum nodal displacement and the fundamental frequency whereas δ_{\lim} and ω_d are their permissible values respectively. h^L and h^U are the lower and upper limits of each design variable. In the following sections, both weighted sum and adaptive weighted sum methods are discussed to find Pareto optimal solutions.

2.1 Weighted sum method

Multicriteria optimization of plates can be represented as a single objective optimization problem by scalarization

using the weighted sum method. Here, the single objective optimization problem is formulated as a sum of objective functions F_i multiplied by weighting coefficients α_i . Hence, bi-objective optimization of plates can be represented using the weighted sum method as [26]:

$$\begin{aligned} & \text{Minimize} && \alpha_1 F_1 + \alpha_2 F_2 \\ & \text{subjected to} && g_1 = \delta_{\max} - \delta_{\lim} \leq 0 \\ & && g_2 = \omega_d - \omega_f \leq 0 \\ & \text{and} && h^L \leq h_i \leq h^U (i = 1, \dots, NE) \\ & && (\alpha_1, \alpha_2) \in [0, 1] \end{aligned} \quad (2)$$

The normalization of the weighting coefficients can be performed as:

$$\alpha_1 + \alpha_2 = 1 \quad (3)$$

As different objective functions can have different magnitudes, the normalization of objectives is generally done to get a Pareto optimal solution consistent with the weights. In the case of bi-objective optimization, the objective functions are normalized as:

$$\bar{F}_i = \frac{F_i - F_i^U}{F_i^N - F_i^U} \quad (4)$$

where the normalized objectives $\bar{F}_i \in [0, 1]$, use the same design space with the non-normalized ones. Here, F_i^U and F_i^N are the utopian and nadir points defined as:

$$\begin{aligned} F_i^U &= [F_1(h^{1*}), F_2(h^{2*})], \\ F_i^N &= \max[F_i(h^{1*}), F_i(h^{2*})] \end{aligned}$$

where \mathbf{h}^{i*} is the optimal solution vector for single objective optimization of F_i .

By varying the weights, it is possible to generate a set of Pareto optimal solutions for Eq. (2). Here, every combination of weighting coefficients corresponds to a single objective optimization problem. The sequential quadratic programming algorithm is used to solve the single objective optimization problems in MATLAB. Each optimization problem gives an optimal solution on a Pareto front. Thus, a set of Pareto optimal solutions is generated by solving optimization problems using a predefined set of weighting coefficients.

2.2 Adaptive weighted sum method

The adaptive weighted sum method is an extension of the conventional weighted sum method. This method increases the number of solutions on a Pareto front by changing weights adaptively. Here, it approximates the shape of the Pareto curve more effectively by putting computational effort in the region where new solutions are most required. Thus, the proposed method provides relatively uniformly spaced and widely distributed Pareto front for multicriteria optimization of variable

thickness plates. This technique also helps to find Pareto optimal solutions in non-convex regions which cannot be found using the conventional weighted sum method [27].

Figure 1 shows the procedure used for bi-objective optimization of plates based on the adaptive weighted sum method. First, the weighted sum method is conducted to obtain a coarse profile of a Pareto front. Then, the Euclidian distances are calculated between neighboring solutions on the Pareto front and sparsely populated regions are identified for further refinement. These regions are now the feasible regions for suboptimization with additional constraints in the objective space. Here, two additional inequality constraints are formed in each region parallel to the objective function axes such that their distances from the solutions are d_1 and d_2 in the inward directions of F_1 and F_2 . The suboptimization problem is solved in each region using the weighted sum method, and a new set of solutions is obtained. Again, new regions are identified for further refinement by calculating the distances between two neighboring solutions. This procedure is repeated until the segment length in all the regions converge to a pre-defined maximum length. The step-by-step procedure for bi-objective optimization of plates using the adaptive weighted sum method is given below.

Step 1: Construct the normalized objective function \bar{F}_i in the objective space.

Step 2: Conduct multicriteria optimization of plates using the conventional weighted sum method. Initially, use a small number of divisions, $n_{\text{int}} (n_{\text{int}} = 4 \sim 6)$ and find the uniform step size of the weighting factor α as:

$$\Delta\alpha_{\text{int}} = \frac{1}{n_{\text{int}}} \quad (5)$$

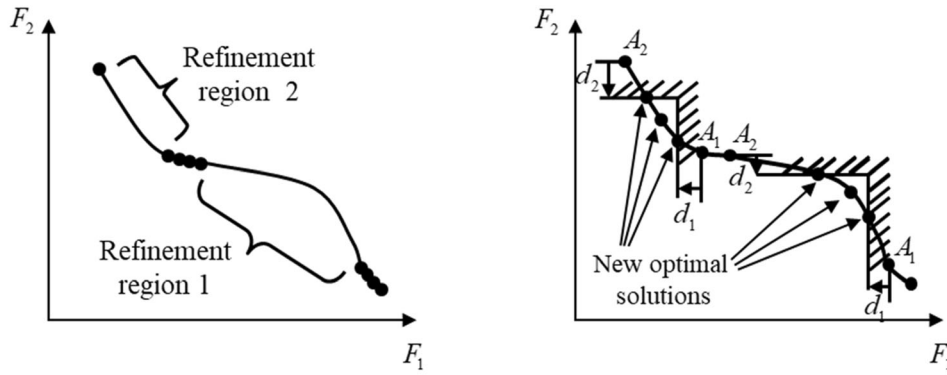
Note that a large step size, $\Delta\alpha_{\text{int}}$, leads to small number of solutions.

Step 3: Calculate the distances between two neighboring solutions and identify the regions for refinement. Remove nearly overlapping solutions which generally occurs with the weighted sum method. If the Euclidian distances are very small between these solutions (less than the tolerance), then only one solution is kept to represent the Pareto front.

Step 4: Calculate the number of refinements required in each region based on the relative length of the segment as:

$$n_i = \text{roundoff} \left(C \frac{l_i}{l_{\text{avg}}} \right) \quad (6)$$

where n_i denotes the number of refinements required for the i th segment, l_i is the length of the i th segment and l_{avg} is the average length of all the segments. The function 'roundoff' gives the nearest integer. Here, C is a constant multiplier whose value is considered between 1 and 2. If, in any segment $n_i \leq 1$, no further refinement is required in the segment. If $n_i > 1$, go to the next step. Note that more



(a) Region identification (b) Constraint imposition and region refinement

Figure 1. The bi-objective optimization procedure based on adaptive weighted sum method.

refinement is required if the length of segment is longer relative to the average length of all segments.

Step 5: Calculate the offset distances between the two end points of each segment. First, a piecewise linearized secant line is drawn between the points, A_1 and A_2 . Then, the offset distance (d_o) is defined along the piecewise linearized Pareto front which determines the final distribution of Pareto solutions. To find the offset distances parallel to the objective axes, the angle θ can be defined as:

$$\theta = \tan^{-1} \left(-\frac{A_1^y - A_2^y}{A_1^x - A_2^x} \right) \quad (7)$$

where A_i^x and A_i^y are the x and y locations of the end points, A_1 and A_2 , respectively, in the objective space. Then, d_1 and d_2 are calculated using d_o and θ as:

$$d_1 = d_o \cos \theta \text{ and } d_2 = d_o \sin \theta$$

Step 6: Perform suboptimization by applying additional inequality constraints in each feasible region using the weighted sum method. The feasible region is offset from A_1 and A_2 by the distances of d_1 and d_2 in the inward directions of F_1 and F_2 as shown in figure 1 (b). The suboptimization problem in each region can be defined as:

$$\begin{aligned} & \text{Minimize} && \alpha \bar{F}_1 + (1 - \alpha) \bar{F}_2 \\ & \text{subjected to} && \bar{F}_1 \leq A_1^x - d_1 \\ & && \bar{F}_2 \leq A_2^y - d_2 \\ & \text{and} && g_i \leq 0 \quad (i = 1, 2) \\ & && h^L \leq h_i \leq h^U \quad (i = 1, \dots, NE) \\ & && \alpha \in [0, 1] \end{aligned} \quad (8)$$

where A_i^x and A_i^y are the x and y positions of the end points and d_1 and d_2 are the offset distances calculated in Step 5. The uniform step size of the weighting factor α_i for each feasible region is calculated based on the number of refinements n_i as:

$$\Delta \alpha_i = \frac{1}{n_i} \quad (9)$$

Step 7: Calculate the segment lengths between all the neighboring solutions and remove nearly overlapping solutions. If any segment length is greater than the defined maximum length, go to Step 4. Terminate the optimization process if all segment lengths are less than the defined maximum length.

3. Design sensitivity estimation

The present gradient based optimization approach requires the estimation of the derivatives of various functions with respect to design variables. In this study, an analytical approach for the design sensitivity analysis of variable thickness plates is considered. Here, the sensitivity parameter of interest is the thickness of each element of plate modelled using the finite element method. The finite element formulation of plate is done based on the first order shear deformation theory [28].

3.1 Sensitivity estimation for static deflection

For the linear static analysis of variable thickness plates, the governing finite element equations can be expressed as:

$$K_{ij} \Delta_j = f_i \quad (i, j = 1, 2, \dots, N) \quad (10)$$

where K_{ij} is the global stiffness matrix, f_i is the external load vector and Δ_j is the displacement vector. N denotes the total number of degrees of freedom of the system.

The derivatives of displacement with respect to design variables x_r can be obtained by differentiating Eq. (10) as [29]:

$$K_{ij}\Delta_j^r = f_i^r - K_{ij}^r\Delta_j \quad (i, j = 1, 2, \dots, N; r = 1, 2, \dots, R) \quad (11)$$

where, Δ_j^r , K_{ij}^r and f_i^r represent the derivatives of displacement vector, stiffness matrix and force vector with respect to design variables. Here, R denotes the total number of design variables which is equal to the total number of finite elements (NE) considered in the analysis. The calculation of Δ_j^r requires K_{ij}^r which can be computed analytically or by using the finite difference technique. It is preferred to evaluate the derivatives analytically, since it will reduce considerable computation time during the optimization process. If the external force is independent of design variables, the term f_i^r becomes zero.

3.2 Sensitivity estimation for free vibration

For the free vibration analysis of variable thickness plates, the governing finite element equations can be expressed as:

$$(K_{ij} - \lambda_k M_{ij})\phi_j^k = 0 \quad (i, j, k = 1, 2, \dots, N) \quad (12)$$

where K_{ij} and M_{ij} are the global stiffness and mass matrices respectively. ϕ_j^k is the k th mode shape and λ_k is the k th eigenvalue with the corresponding natural frequency $\lambda_k = \omega_k^2$.

The derivatives of eigenvalue with respect to design variable x_j can be obtained by differentiating Eq. (12) as [30]:

$$(K_{ij} - \lambda_k M_{ij})\phi_j^{k,r} + (K_{ij}^r - \lambda_k M_{ij}^r - \lambda_k^r M_{ij})\phi_j^k = 0 \quad (i, j, k = 1, 2, \dots, N, r = 1, 2, \dots, R) \quad (13)$$

where, M_{ij}^r , $\phi_j^{k,r}$ and λ_k^r represent the derivatives of mass matrix, k^{th} mode shape vector and eigenvalue with respect to design variables respectively.

Pre-multiplying the above equation by ϕ_i^{kT} and applying the equilibrium condition of Eq. (12) along with mass-orthonormality condition (*i.e.* $\phi_i^{kT} M_{ij} \phi_j^k = 1$), the expression for the eigenvalue derivatives can be written as:

$$\lambda_k^r = \phi_i^{kT} (K_{ij}^r - \lambda_k M_{ij}^r) \phi_j^k \quad (i, j, k = 1, 2, \dots, N, r = 1, 2, \dots, R) \quad (14)$$

Hence, the design sensitivity of natural frequency $\lambda_k = \omega_k^2$ with respect to design variables x_r can be calculated as:

$$\omega_k^r = \frac{1}{2\omega_k} \lambda_k^r \quad (k = 1, 2, \dots, N, r = 1, 2, \dots, R) \quad (15)$$

The above sensitivity formulation is applicable to structural system having real and distinct eigenvalues. In this work, the sensitivity of the lowest natural frequency

(fundamental frequency) of variable thickness plates is considered which is real and distinct.

4. Numerical studies

The multicriteria optimization procedures discussed in the preceding sections are employed to demonstrate the optimal behavior of variable thickness square isotropic plates. The dimensions and material properties of the isotropic square plate are given below:

$$E = 207 \text{ GPa}, \nu = 0.3, \rho = 7940.324 \text{ Kg/m}^3, a = b = 0.254 \text{ m}$$

First, the design sensitivities of responses with respect to elemental thickness of square isotropic plates are presented under static and dynamic conditions. Then, single objective weight optimization of variable thickness plates is demonstrated under static and dynamic constraints. The optimal weight and thickness distributions obtained from the present approach are also compared with those available in the literature. Finally, multicriteria optimization of plates is studied to obtain Pareto optimal solutions considering weight and static response as objectives under static and dynamic constraints. The capability of both weighted sum and adaptive weighted sum methods are also discussed. The effect of different boundary conditions on the optimum thickness distributions of plates is also presented.

4.1 Sensitivity studies of variable thickness plates

In this section, the sensitivities of maximum displacement and fundamental frequency of square isotropic plates are calculated considering thickness of each element as design variable under various boundary conditions. The plate is discretized into 20x20 finite elements (NE) which corresponds to 400 design variables.

First, the sensitivities of maximum displacement and fundamental frequency of all edges simply supported (SSSS) plates are shown in figure 2. The maximum displacement derivatives are obtained considering uniformly distributed load of magnitude 68948 N/m^2 . It can be seen from the figure that the maximum value of fundamental frequency sensitivity is observed around the center and at the corners of the plate. Contrary to the case of fundamental frequency, the maximum value of displacement sensitivity appears around the center of the plate. A very low displacement sensitivity value is also observed at the corners of the plate.

The sensitivities of maximum displacement and fundamental frequency of all edges clamped (CCCC) and cantilever (CFFF) plates are also studied and shown in figures 3 and 4, respectively. The maximum displacement

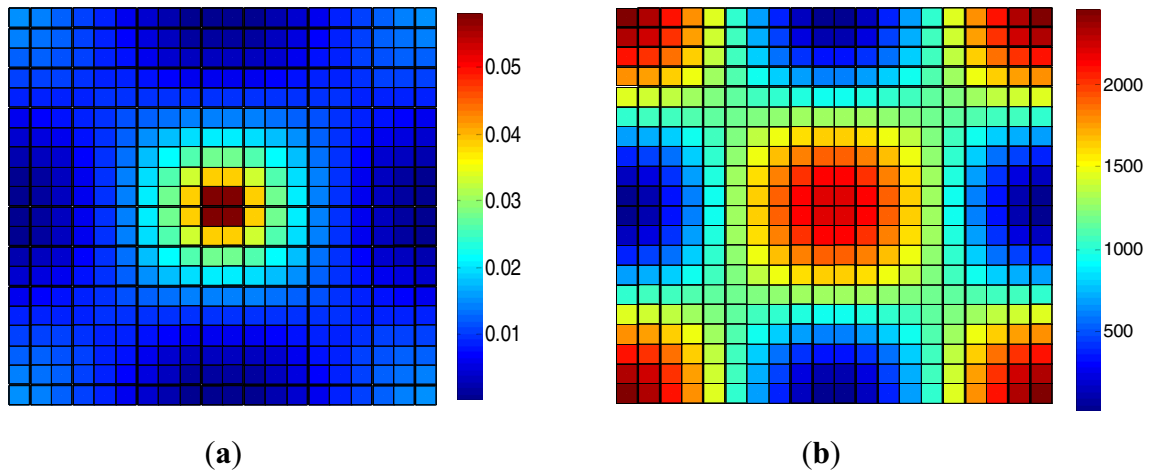


Figure 2. Sensitivities of response due to thickness variation in each element of simply supported (SSSS) plate (a) maximum displacement, (b) fundamental frequency.

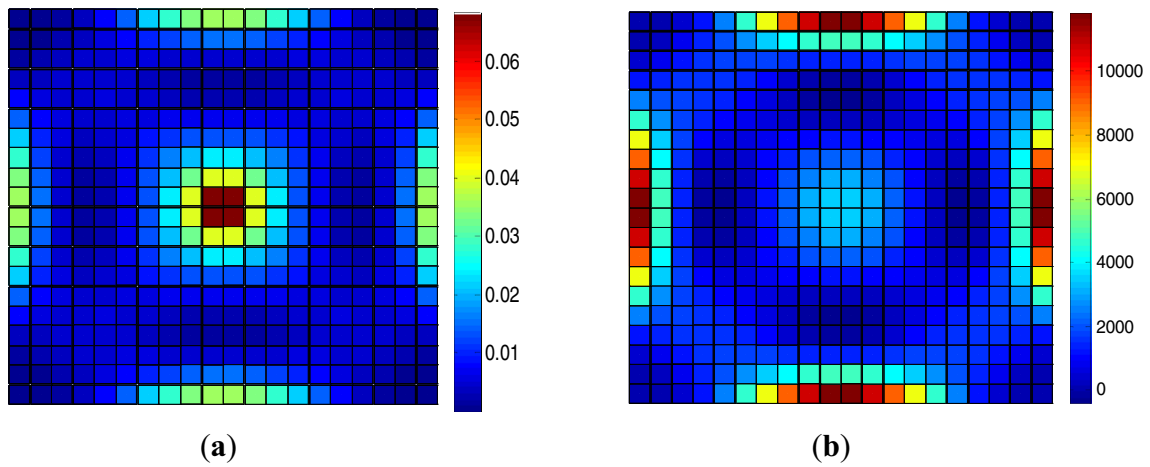


Figure 3. Sensitivities of response due to thickness variation in each element of clamped (CCCC) plate (a) maximum displacement and (b) fundamental frequency.

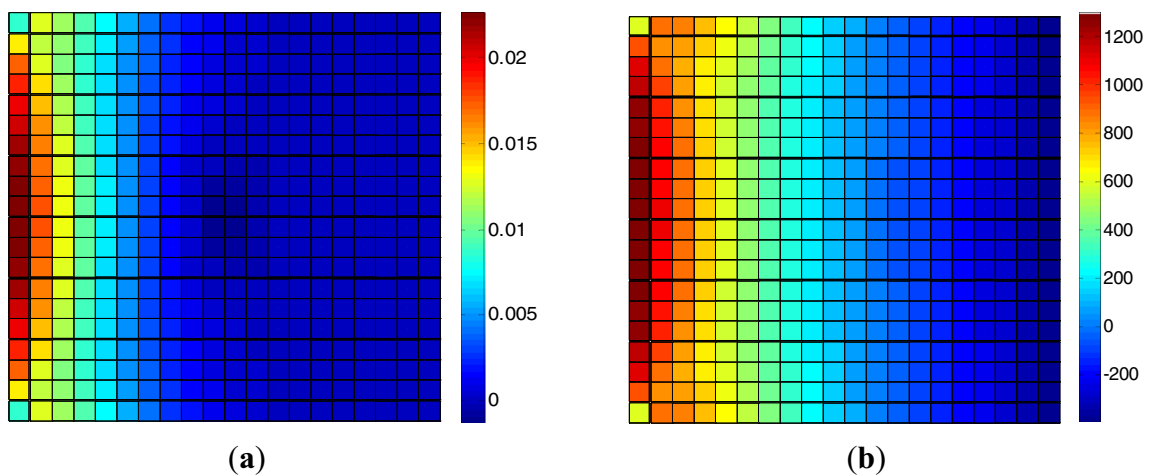


Figure 4. Sensitivities of response due to thickness variation in each element of cantilever (CFFF) plate (a) maximum displacement, (b) fundamental frequency.

derivatives are obtained considering uniformly distributed load of magnitudes 206844 N/m^2 and 2068.44 N/m^2 acting on clamped and cantilever plates respectively. It is observed that a very high value of fundamental frequency sensitivity appears near the boundary edges of the clamped plate with a low sensitivity value around the center of the plate. Contrary to the case of fundamental frequency, a high displacement sensitivity appears around the center of the clamped plate with a very small sensitivity value near the boundary edges of the plate. In the case of cantilever plates, a high value of displacement and fundamental frequency sensitivities exist near the root which decrease as we move towards the tip of the plate.

4.2 Single objective optimization of variable thickness plates

In this section, single objective optimization of variable thickness square isotropic plates is discussed. Here, the weight of isotropic plates is minimized subjected to maximum displacement and fundamental frequency constraints. The dimensions and material properties of plates are same as defined earlier. The plate is discretized using 20×20 rectangular elements with each element thickness treated as an independent design variable. A uniform thickness of 2.54 mm is taken as an initial design thickness with the lower and upper bounds of each design variable as $h^L = 2.54 \text{ mm}$ and $h^U = 7.62 \text{ mm}$ respectively. The single objective multi-variable optimization problems are solved using the sequential quadratic programming approach to obtain optimum weights and thickness distributions of variable thickness plates. The present optimization results are also validated with those available in the literature. Parametric studies are also conducted to investigate the effect of different boundary conditions on the optimal weight and thickness distribution of variable thickness plates.

4.3 Single objective optimization of plates with maximum displacement constraint

In this section, single objective optimization of square isotropic plates is studied, by minimizing weight, subjected to maximum displacement constraint. The permissible value for the maximum displacement of plates is taken as $\delta_{lim} = 1.905 \text{ mm}$. First, a simply supported plate subjected to uniformly distributed load of magnitude 68948 N/m^2 is considered. The maximum central displacement of the plate is found to be $\delta_{max} = 3.751978 \text{ mm}$ which is higher than the permissible value. Single objective multi-variable optimization is then carried out and the optimum weight of the plate is shown in Table 1. The result obtained by Haug *et al* [3] is also shown in the table and a good agreement between the two is observed. The optimal thickness

Table 1. Comparison of optimum weight for simply supported (SSSS) plate with maximum displacement constraint.

Optimum weight of SSSS plate in kg	
Present approach	Haug <i>et al</i> [3]
1.4741	1.4834

distribution obtained for the simply supported (SSSS) plate is shown in figure 5. The thickness profile of the resulting optimum plate shows a growth in thickness around the center and at the corners of the plate. It is also observed that the central elements are slightly thicker compared to the corner elements of the plate. Symmetry in the thickness distribution about the central lines of the plate is also observed. The present thickness distribution follows the same pattern as reported in [3].

Further, the weight minimization of clamped and cantilever plates is also conducted under uniformly distributed load of magnitudes 206844 N/m^2 and 2068.44 N/m^2 respectively which results in the maximum central displacements of $\delta_{max} = 3.5075 \text{ mm}$ and 3.5783 mm respectively. The permissible value for the maximum displacement is taken as $\delta_{lim} = 1.905 \text{ mm}$. Single objective multi-variable optimization gives optimum weights of 1.4092 kg and 1.4494 kg for the clamped and cantilever plates respectively. The optimal thickness distributions achieved for both clamped (CCCC) and cantilever (CFFF) plates are also shown in figure 6. The thickness profile of the clamped plate shows a growth in thickness around the center and at the middle edges of the plate. The thickness profile of the cantilever plate shows thicker elements near

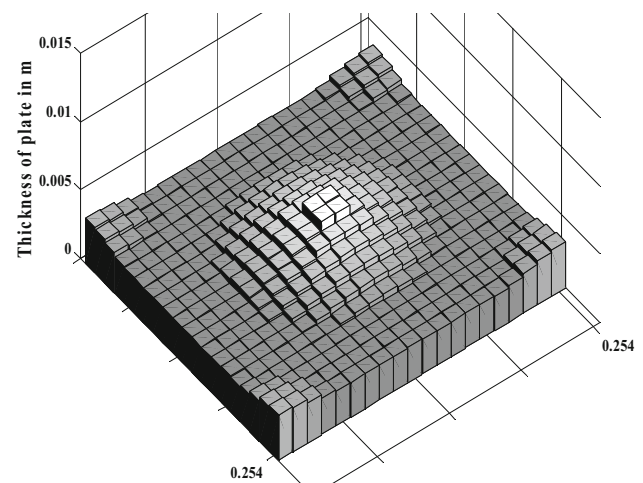


Figure 5. Optimal thickness distribution of simply supported (SSSS) plate with maximum displacement constraint.

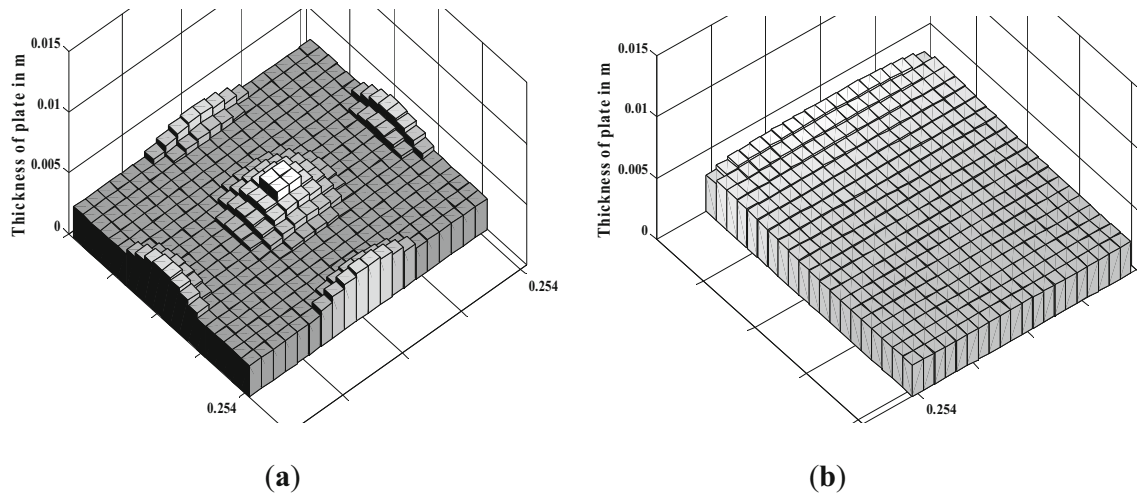


Figure 6. Optimal thickness distribution of plates with maximum displacement constraint (a) clamped and (b) cantilever.

the root which decreases as we move towards the tip of the plate.

4.4 Single objective optimization of plates with fundamental frequency constraints

In this section, single objective optimization of square plates is studied, by minimizing weight, subjected to fundamental frequency constraint. First, a simply supported (SSSS) plate is considered whose fundamental frequency is found to be $\omega_f = 1203.613$ rad/sec. The permissible value of fundamental frequency is taken as $\omega_d = 1375$ rad/sec. Single objective multi-variable optimization is then carried out and the optimum weight obtained for the plate is shown in Table 2. The result obtained by Haug *et al* [3] is also shown in the table and a good agreement between the two results is observed. The optimal thickness distribution obtained for the plate is also shown in figure 7. The thickness profile of the resulting optimum plate shows a growth in thickness around the center and at the corners of the plate. It is also observed that the corner elements are slightly thicker compared to central elements of the plate. The thickness distribution obtained from the present approach follows the same pattern as given in [3].

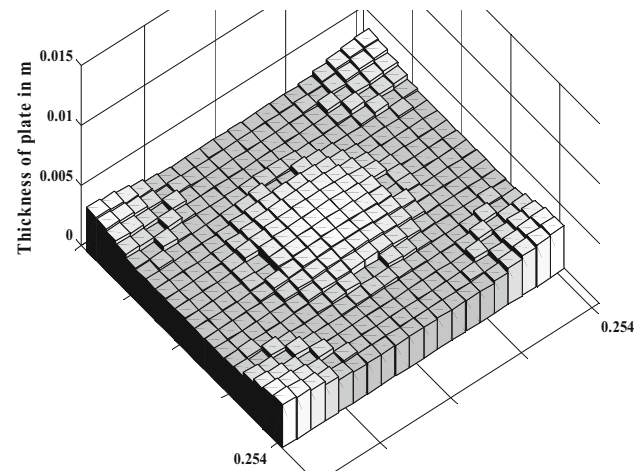


Figure 7. Optimal thickness distribution of simply supported (SSSS) plate with fundamental frequency constraint.

Table 2. Comparison of optimum weight for simply supported (SSSS) plate with fundamental frequency constraint.

Optimum weight of SSSS plate in kg	
Present approach	Haug <i>et al</i> [3]
1.4242	1.4196

Similarly, the weight minimization of clamped (CCCC) and cantilever (CFFF) plates under fundamental frequency constraint is performed. The permissible values of fundamental frequency for the clamped and cantilever plates are taken as $\omega_d = 2420$ rad/sec and 235 rad/sec respectively. The fundamental frequencies obtained for the plates are $\omega_f = 2200.63$ rad/sec and 211.183 rad/sec respectively which are lower than the permissible values. Single objective multi-variable optimization gives optimum weights of 1.3321 kg and 1.3318 kg for the clamped and cantilever plates respectively. The optimal thickness distributions achieved for both plates are also shown in figure 8. Contrary to displacement constraint, the thickness profile of the clamped plate shows a growth in thickness only at the middle edges. There is no growth in thickness around the center of the plate. The thickness profile of the cantilever

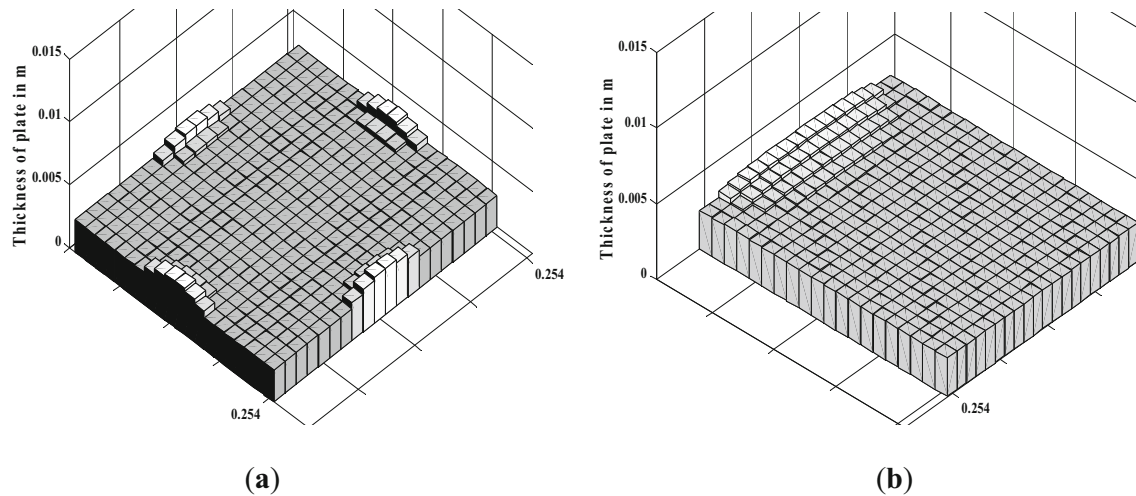


Figure 8. Optimal thickness distribution of plates with fundamental frequency constraint (a) clamped and (b) cantilever.

plate shows thicker elements near the root which decreases as we move towards the tip of the plate.

4.5 Multicriteria optimization studies of variable thickness plates

This section investigates multicriteria optimization of varying thickness isotropic plates with minimization of weight and displacement as objective functions. First, multicriteria optimization of plates is studied with displacement constraint under various boundary conditions. Further, an additional frequency constraint is added in the multicriteria optimization problem to study its effect on Pareto optimal solutions. Both weighted sum and adaptive weighted sum optimization algorithms are employed to obtain optimal solutions on a Pareto front. The adaptive weighted sum method is performed using the parameter values given in Table 3. The dimensions and material properties of plates are same as defined earlier. Here, the finite element discretization of plates is done using 10x10 rectangular elements with element thicknesses treated as independent design variables. A uniform thickness of 2.54 mm is taken as an initial design thickness with the lower and upper bounds of each design variable as $h^L = 2.54$ mm and $h^U = 7.62$ mm respectively.

Table 3. Different parameters used for adaptive weighted sum method.

Various parameters	Parameter values
Initial step size for weighting factor ($\Delta\alpha_{int}$)	0.25
Constant multiplier (C)	1.5
Offset distance (d_o)	0.1

4.6 Multicriteria optimization of plates with maximum displacement constraint

In this section, multicriteria optimization of square plates is studied, by minimizing weight and displacement, subjected to maximum displacement constraint. First, a simply supported (SSSS) plate under uniformly distributed design load of magnitude 68948 N/m^2 is considered. The permissible limit on the deflection is taken as $\delta_{lim} = 1.905 \text{ mm}$. Both weighted sum and adaptive weighted sum methods are applied to solve the multicriteria optimization problem. Figure 9 shows the Pareto optimal solutions obtained by the conventional weighted sum method. The number of optimal solutions taken on the Pareto front is 9, but most of the solutions are found in the central region of the Pareto front.

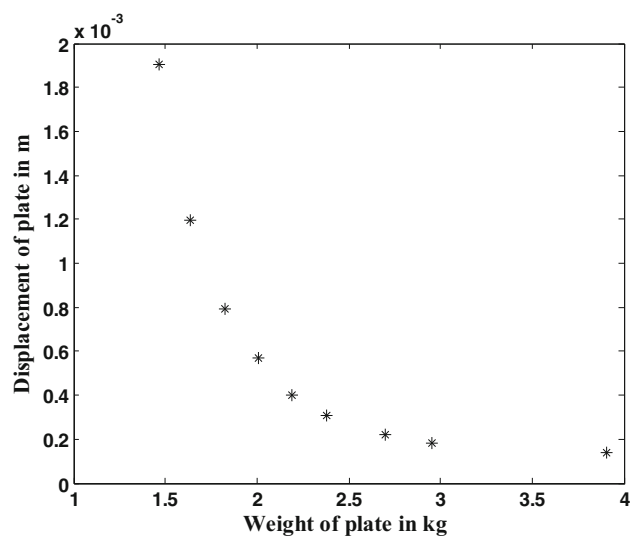


Figure 9. Pareto front for simply supported (SSSS) plate using weighted sum method.

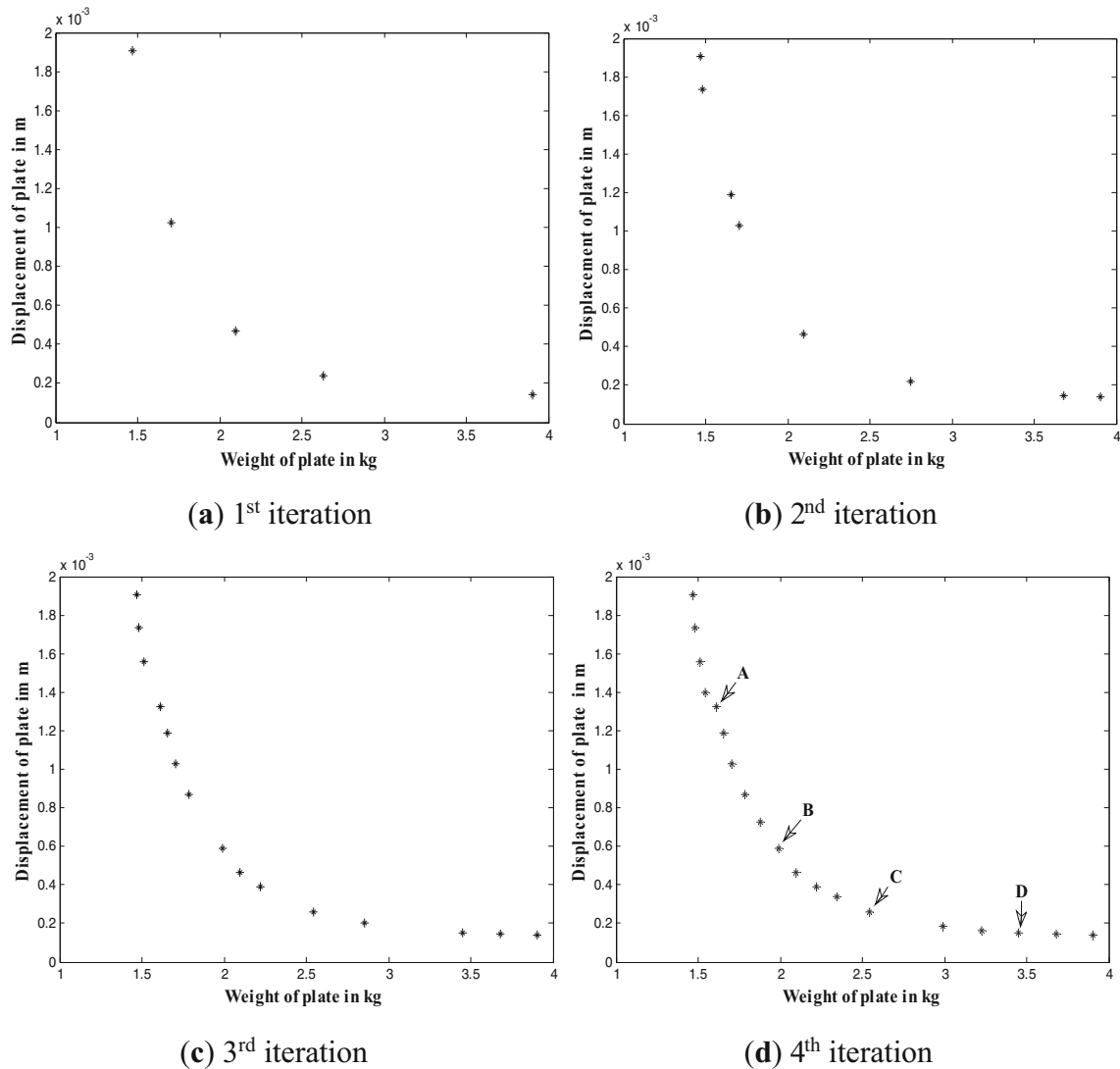


Figure 10. Pareto fronts for simply supported (SSSS) plate at different iterations of adaptive weighted sum method.

Figure 10 shows the Pareto optimal solutions obtained by the adaptive weighted sum method. Here, the Pareto front obtained at each iteration of the adaptive weighted sum method is also presented. The adaptive method converges in four iterations, obtaining fairly well distributed solutions on the Pareto front. Figure 11 shows the thickness distributions obtained at various points A, B, C and D on the Pareto front at the 4th iteration of the adaptive weighted sum method. Absolute symmetry in the thickness distribution about the central lines of the plate is observed.

Next, multicriteria optimization of clamped (CCCC) plates under uniformly distributed design load of magnitude 206844 N/m^2 is studied. Figure 12(a) shows the Pareto optimal solutions obtained by the conventional weighted sum method. The number of optimal solutions taken is 11 which are mostly concentrated in the central region of the

Pareto front. Figure 12(b) shows the Pareto optimal solutions obtained at the 5th iteration of the adaptive weighted sum method. The adaptive method converges in five iterations, representing fairly well distributed solutions on the Pareto front. Figure 13 shows the thickness distributions obtained at various points A, B, C and D on the Pareto front. Absolute symmetry in the thickness distribution about the central lines of the plate is observed.

Similarly, multicriteria optimization of cantilever (CFFF) plates is also investigated under uniformly distributed design load of magnitude 2068.44 N/m^2 . Figure 14(a) shows the Pareto optimal solutions obtained by the weighted sum method. The number of optimal solutions taken is 11 which are mostly concentrated in the central region of the Pareto front. Figure 14(b) shows the Pareto optimal solutions obtained at the 4th iteration of the

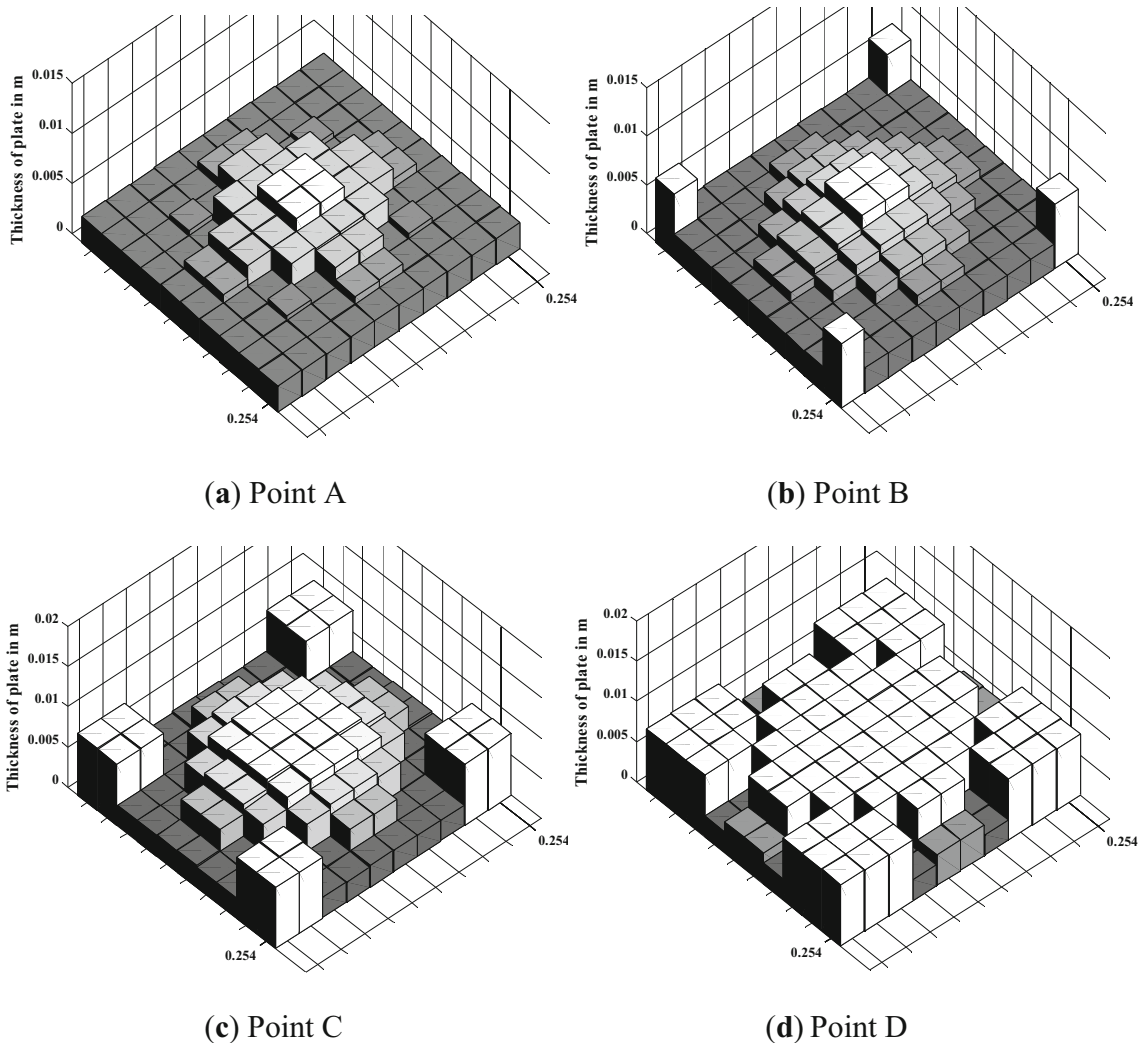


Figure 11. Thickness distributions of simply supported (SSSS) plate located at points A, B, C and D on the Pareto front obtained at the 4th iteration of adaptive weighted sum method

adaptive weighted sum method. In this case, the adaptive method converges in four iterations, showing fairly well distributed solutions on the Pareto front. Figure 15 shows the thickness distributions obtained at various points on the Pareto front. Absolute symmetry in the thickness distribution about the central line of the plate is observed.

4.7 Multicriteria optimization of plates with maximum displacement and fundamental frequency constraints

In this section, multicriteria optimization of simply supported (SSSS) plates is studied, by minimizing weight and displacement as objectives, subjected to maximum displacement and fundamental frequency constraints. The plate is subjected to uniformly distributed load of

magnitude 68948 N/m^2 . The permissible limits on the maximum deflection and fundamental frequency are $\delta_{lim} = 1.905 \text{ mm}$ and $\omega_d = 1375 \text{ rad/s}$ respectively. Figure 16 shows the Pareto optimal solutions obtained by the conventional weighted sum and adaptive weighted sum methods. Here, the adaptive method converges in four iterations, obtaining well distributed solutions on the Pareto front compared to weighted sum method. Figure 17 shows the thickness distributions obtained at various points on the Pareto front using the adaptive weighted sum method. It is observed from the results that the corner elements of the simply supported plate become thicker by applying additional frequency constraint as compared to the plate having only displacement constraint. Thus, the trade-off's for weight obtained for this case is slightly higher compared to the displacement constraint based multicriteria optimization.

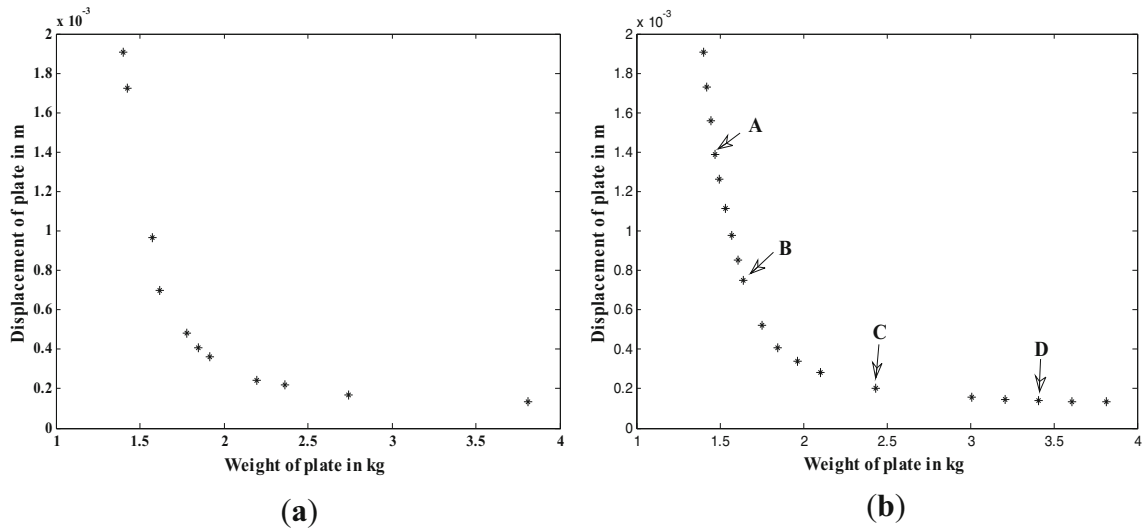


Figure 12. Pareto fronts for clamped (CCCC) plate using (a) weighted sum and (b) adaptive weighted sum methods at 5th iteration.

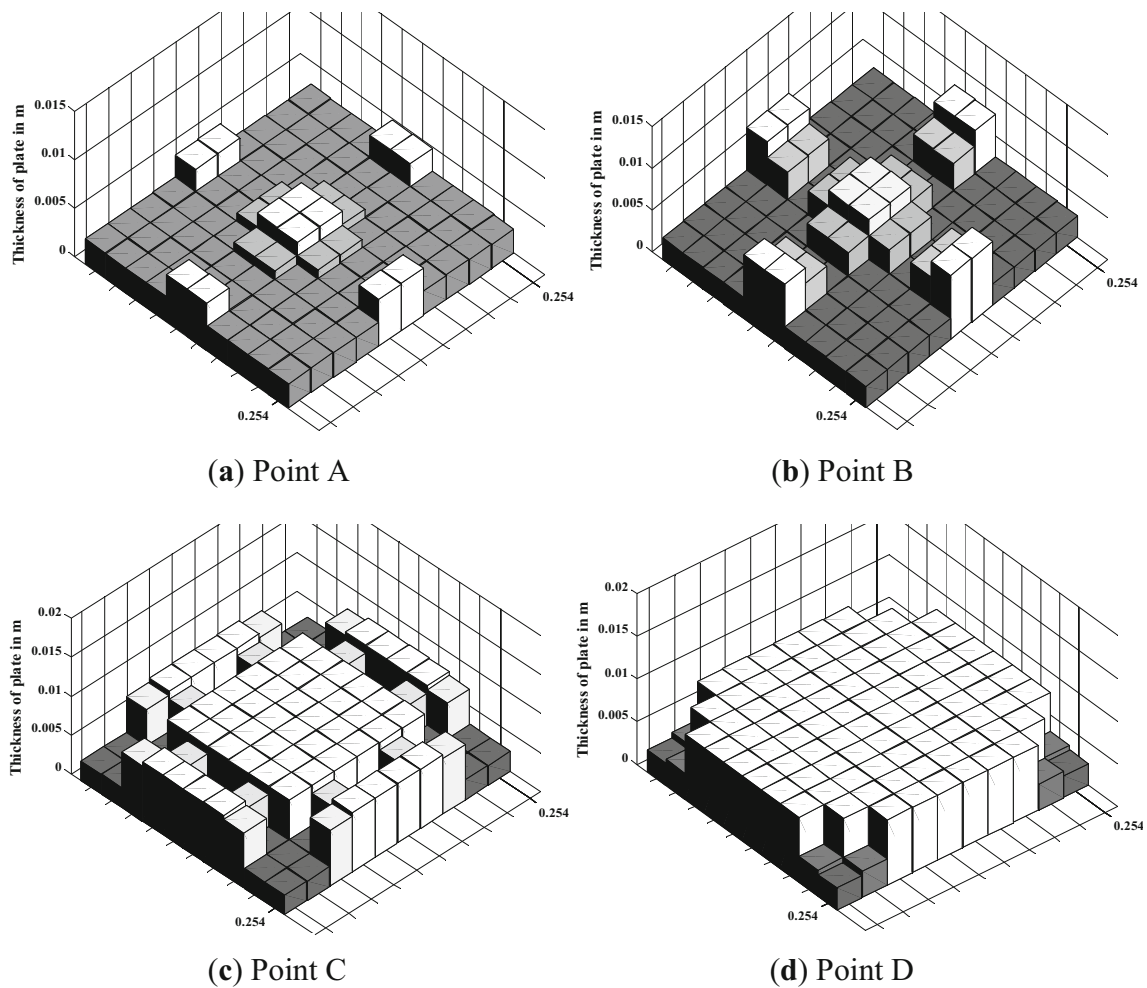


Figure 13. Thickness distributions of clamped (CCCC) plate located at points A, B, C and D on the Pareto front obtained at the 5th iteration of adaptive weighted sum method.

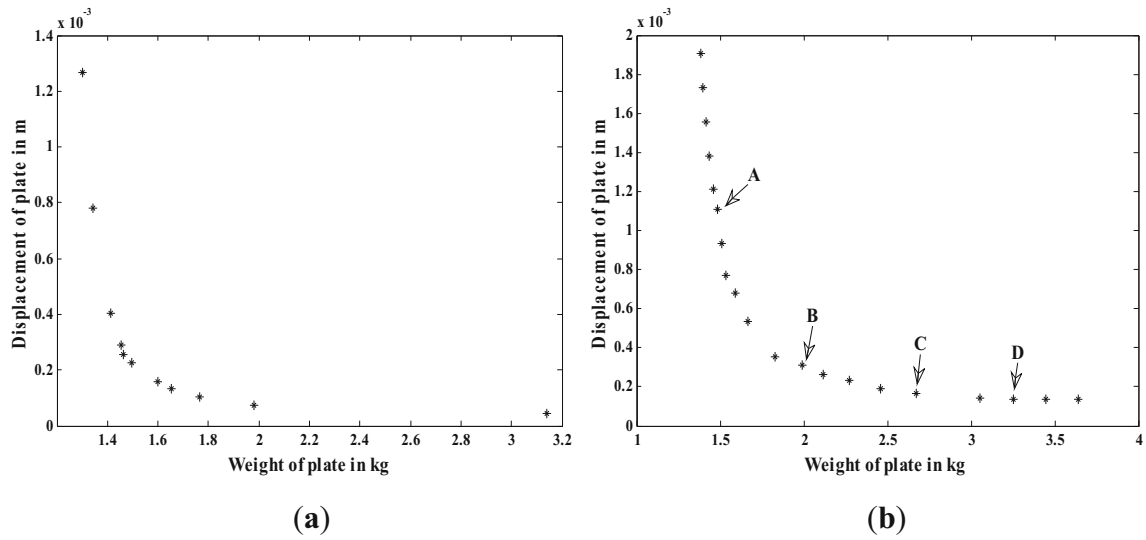


Figure 14. Pareto fronts for cantilever (CFFF) plate using (a) weighted sum and (b) adaptive weighted sum methods at 4th iteration.

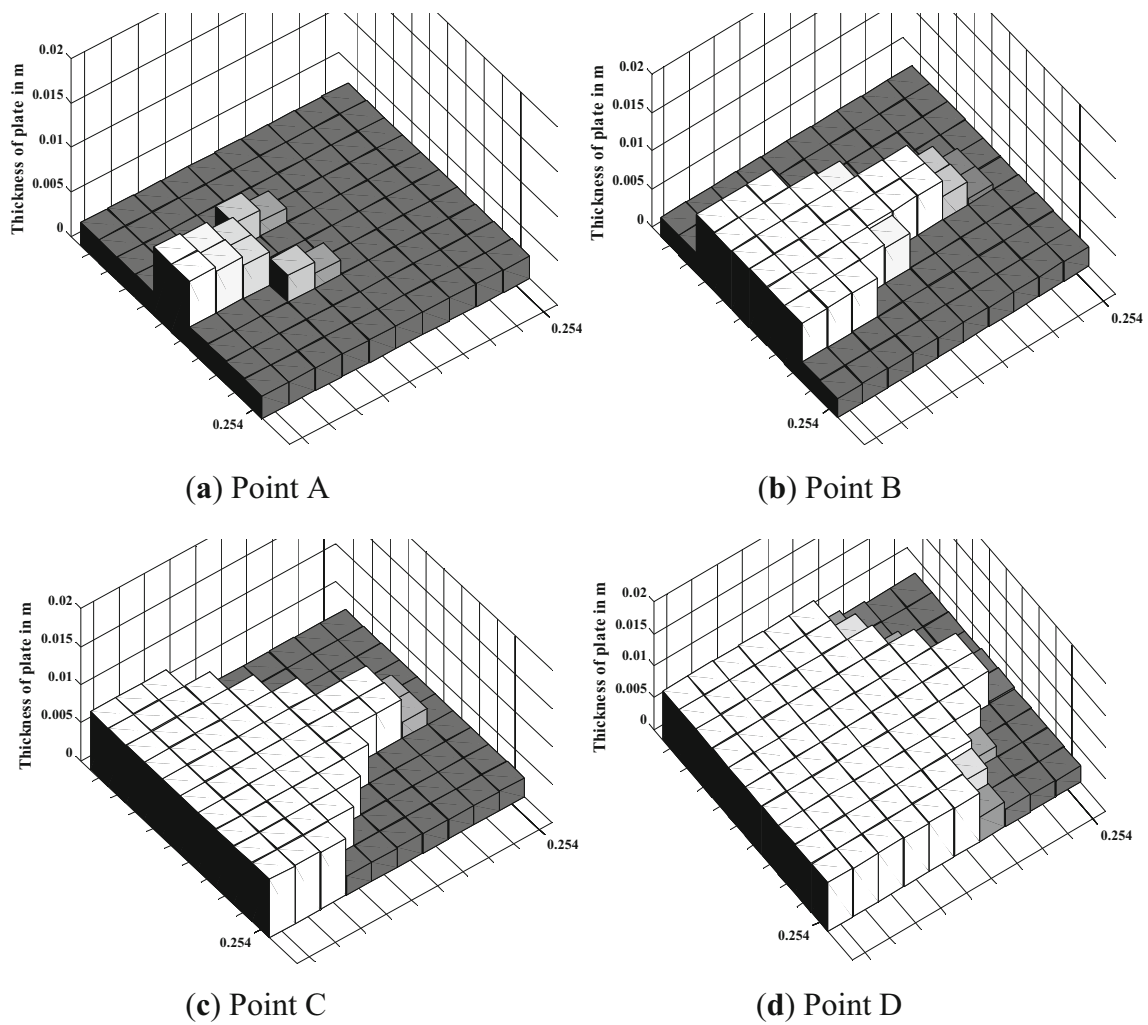


Figure 15. Thickness distributions of cantilever (CFFF) plate located at points A, B, C and D on the Pareto front obtained at the 4th iteration of adaptive weighted sum method.

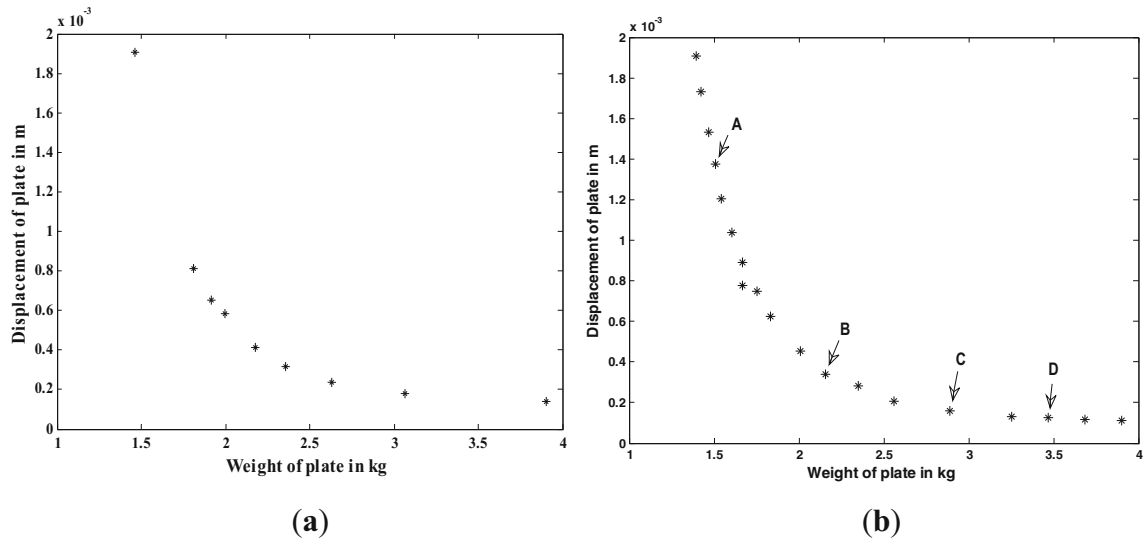


Figure 16. Pareto fronts for simply supported (SSSS) plate using (a) weighted sum and (b) adaptive weighted sum methods at 4th iteration.

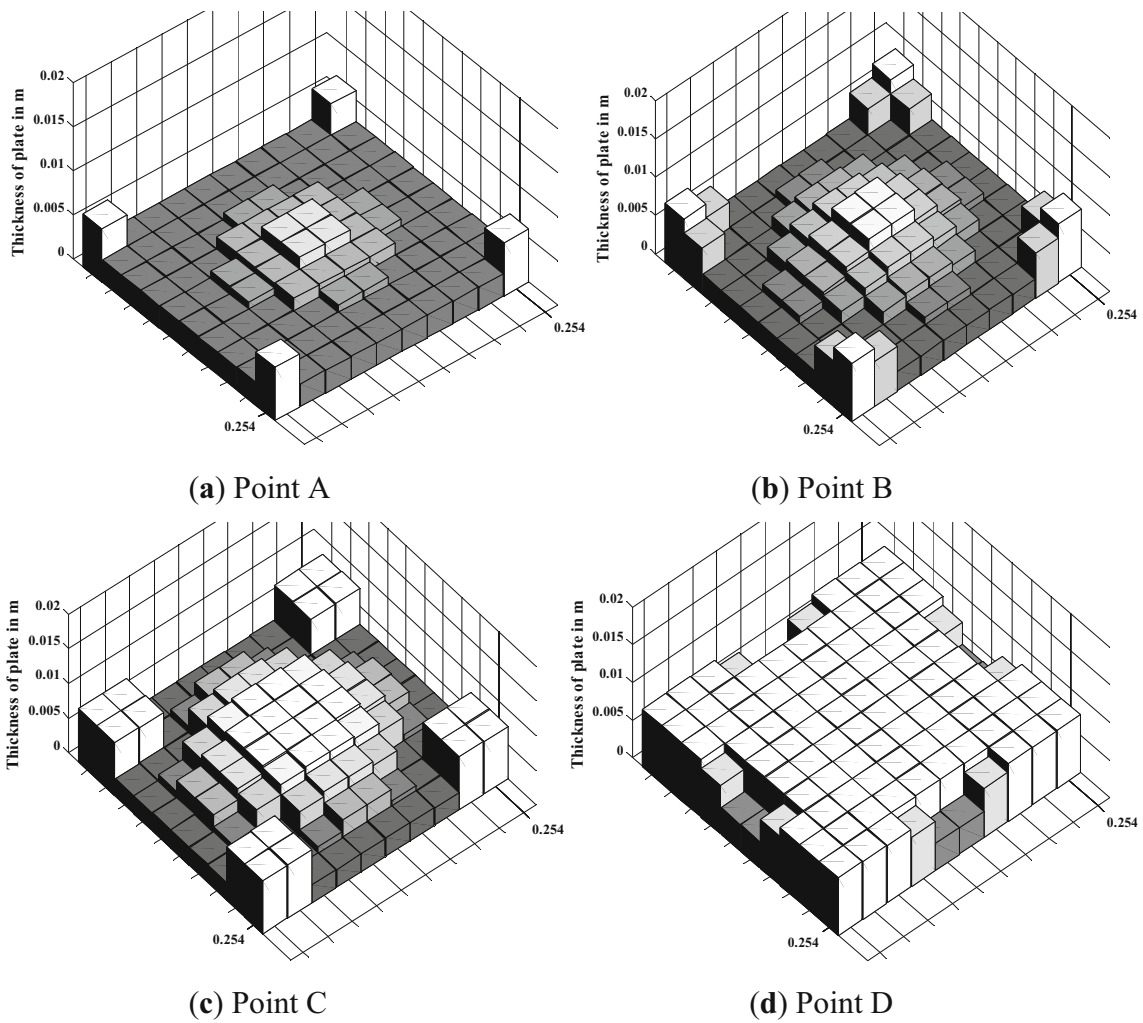


Figure 17. Thickness distributions of simply supported (SSSS) plate located at points A, B, C and D on the Pareto front obtained at the 4th iteration of adaptive weighted sum method.

5. Conclusions

In this paper, multicriteria optimization of variable thickness isotropic plates is investigated considering the minimization of weight (cost) and displacement (performance) as objectives under static and dynamic constraints. The multicriteria optimization framework is constructed by integrating the finite element method, analytical sensitivity technique along with optimization algorithms. Both weighted sum and adaptive weighted sum methods are employed to study multicriteria optimization of variable thickness isotropic plates. From the results, some of the main conclusions drawn are given here.

- (1) Maximum displacement and fundamental frequency sensitivity values for simply supported plates with varying thickness are generally high around the center and at the corners of the plate. For clamped plates, the fundamental frequency sensitivity is high near the boundary edges whereas the displacement sensitivity is high around the center of the plate. For cantilever plates, high values of displacement and fundamental frequency sensitivity are observed near the root which decrease as we move towards the tip of the plate.
- (2) The optimum weight and the thickness distribution obtained by the present single objective optimization method are in good agreement with those given in the literature. Simply supported and clamped plates show thicker elements around the centre for displacement constraint while the same plates show thicker elements near the boundary for frequency constraint. Cantilever plates show thicker elements near the root for both displacement and frequency constraints.
- (3) The Pareto optimal solutions obtained by the weighted sum method, using the predefined values of weighing factors, are mostly concentrated in the central region of the Pareto curve.
- (4) The adaptive weighted sum method gives fairly good approximation of the Pareto curve by increasing the number of optimal solutions in the sparsely populated regions based on automatic weight factor selection procedure.
- (5) The proposed multicriteria framework can be extended to study structural design optimization problems with non-convex Pareto front. The framework can also be used as an automated tool for optimizing different finite element models.

Acknowledgements

The authors would like to thank Dr. M Manjuprasad, CSIR-NAL, Bangalore and Prof. S V Dinesh, Siddaganga Institute of Technology, Tumkur for providing encouragement and support during the course of the work.

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