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Optimization of imperfect economic manufacturing models with a power demand rate dependent production rate

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Abstract. The constant demand rate is the most common assumption of the basic economic production quantity model, which is not very frequent in practice. In real world situations, demand usually varies with time. With regard to the widespread necessity of power demand pattern, demand is supposed to follow a power law. Another unrealistic assumption is perfect quality of all items. This paper presents a production system with defective items to determine the optimal replenishment quantity, cycle length and backordered size with a power demand rate dependent production rate. We assume that a manufacturer may be faced with three different cases regarding to the date that defective items are drawn from inventory. The set-up, backordering, inspection, and production costs, as well as holding cost of both perfect and imperfect items are accounted in the inventory system. An algorithm is offered to optimize total inventory cost and then numerical analyses are presented to demonstrate the applicability of the proposed models. Finally, some sensitivity analyses and managerial insights are provided.

Keywords. Production modeling; inventory control; power demand pattern; defective items; imperfect manufacturing; backordered.

1. Introduction

Harris [1] first presented the economic order quantity (EOQ) to minimize the total inventory cost. Since the EOQ model consists of some assumptions, by relaxing the assumption that all orders are obtained together, the economic manufacturing quantity (EMQ) model is introduced by Taft [2]. The basic EOQ and EPQ models suppose that the demand rate is constant; however, it is not realistic in practice, and generally customer's demand varies with time. Therefore, many researchers are interested in studying inventory systems when demand depends on time. For example, Silver and Meal [3] proposed an approximation to find an optimum lot size when demand varies with time. Donaldson [4] presented an inventory system, whose demand has a linear time-varying trend and then proposed an approach to obtain the optimal solutions of it. Ritchie [5] considered an inventory system in which demand increases linearly. Bose *et al* [6] investigated an EOQ model with a demand that changes with time positively and linearly, considering shortages and deterioration. Teng [7] proposed a method to obtain optimal inventory policies, considering a linear trend in deterministic demand. There was little published work on inventory systems with decreasing trend in demand before the paper presented by Zhao *et al* [8]. They presented an analytic algorithm to solve such problems. Lo *et al* [9] introduced an optimum policy for the problem of inventory management, where demand changes linearly and used a model called the "two-equation model".

Yang *et al* [10] proposed a parametric eclectic model with demand decreasing non-linearly. Omar and Yeo [11] considered a manufacturing system for a situation that new products are manufactured using one kind of raw material, used products are repaired and demand is assumed to vary with time continuously. Maihami and Kamalabadi [12] adopted a demand function dependent on both time and price for an inventory model with decaying items. Pando *et al* [13] studied a model with the holding cost non-linearly dependent on both quantity and time and the stock level dependent demand rate.

There are different ways to take out items during the cycle length to supply demand. They are referred to as demand patterns. Various types of these patterns are discussed by researchers. When the demand rate is constant during the scheduling period, its pattern is known as

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uniform. However, it is not suitable for many practical situations. In real world situations, there are some other ways to take out products from the stock. Demand of customers can be dependent on time, price, stock level, etc. Since time is one of the most common inputs of demand functions in practice, there are some patterns considering it as an input like linear, quadratic, exponential, non-linear and power. Demand used in this study is supposed to follow a power law, because of the wide-ranging uses and being more applicable than the others. This pattern can be used for the situations that a high percentage of demand happens at the beginning of the cycle, because of approaching to the expiry date or demanding for the fresh products (e.g., prepared food, breads, fresh meat, fruits, yogurts and vegetables), or at the end of the scheduling period, because of becoming scarce or daily use (e.g., sugar, tea, coffee and oil). Also, it consists of the situation that demand occurs uniformly. Several papers have been published in this category.

Naddor [14] studied a power demand pattern in an order-level system. Lee and Wu [15] analyzed an EOQ model with backordering, power demand for decaying item. Then, Dye [16] completed Lee and Wu's model by proposing a model, in which backordering rate changes pro rata with time. Singh *et al* [17] proposed an inventory system considering partial backordering and power demand pattern for perishable items. Abdul-Jalbar *et al* [18] investigated a one-warehouse *N*-retailer problem, in which the demand pattern is power and backorders are allowed. Rajeswari and Vanjikkodi [19] developed a deterministic EOQ model considering constant deterioration, partially backlogged shortages and power demand pattern. Sicilia *et al* [20] analyzed inventory systems, in which deterministic demand changes with time and follows a power pattern. They discussed several scenarios: the inventory systems with and without shortages, the systems with full backlogging or entirely lost sales. Sicilia *et al* [21, 22] extended a lot-sizing system with a power demand pattern for deteriorating items.

Sicilia *et al* [23] investigated an EOQ system, in which deterioration occurs with a constant rate and deterministic demand pattern is power. Sicilia *et al* [24] developed an EPQ system, in which demand follows a time-dependent power law and production rate changes pro rata with the demand rate, allowing for backlogged shortage. San-José *et al* [25] optimized an inventory system with power demand pattern and partial backlogging. Keshavarzfard *et al* [26] developed an inventory-pricing model for multiple products in which the production rate is proportional to power demand rate.

One of the conventional assumptions in the EPQ model is perfect quality of all produced items. Resulting from process deterioration, defective raw materials or other reasons, producing items with imperfect quality is unavoidable. In the last years, several studies have been done to deal with production of defective items. Shih [27] discussed the impact of imperfect items on the replenishment size and the objective function. Schwaller [28] incorporated inspection costs in the EOQ system and assumed that a known proportion of an incoming lot is defective. Salameh and Jaber [29] considered that defective products are sold as one batch after finishing 100% inspection process. Hayek and Salameh [30] developed a manufacturing system with a random defective production rate, in which all defective items can be reworked. They derived an optimal production policy, in which backordered products are allowed. Goyal *et al* [31] presented an easy procedure to find the production policy of a vendor and buyer system with defective items. They assumed that a specified portion of imperfect products are produced during the replenishment process. Chiu [32] investigated an EPQ system for a situation that imperfect products are produced with a random rate and assumed that reworking of them starts immediately after production time, however a fraction of imperfect products are scrapped. Jamal *et al* [33] adopted two policies to find the optimum lot quantity in a production model considering rework. Ojha *et al* [34] studied a manufacturing process that manufactures imperfect products with a constant rate. They assumed that products can be delivered just after checking quality of entire batch and imperfect products have to be reworked. Also three scenarios were investigated by them. Cárdenas-Barrón [35] presented an extension of [33] by adding planned backorders. Taleizadeh *et al* [36] studied a production system with limited capacity and allowing for backorders, in which manufacturing imperfect items follows either a normal or a uniform probability distribution. Taleizadeh *et al* [37] modeled an inventory system with rework process, multiple products and single machine, to find the optimal lot size. Ouyang *et al* [38] discussed a situation that management invests capital to improve quality. Also, they considered defective products and inspection policy in their model. Furthermore, Taleizadeh *et al* [39] analyzed a manufacturing system considering defective items, rework process and multiple products. Taleizadeh *et al* [40] presented a production system with one machine, multiple products, and interruption in process, scrap, rework and backordering.

Jaber *et al* [41] extended [29] to a situation that replacing defective products is impossible due to the distance of the supplier. They modeled two different cases to deal with this condition. Taleizadeh *et al* [42] and Taleizadeh and Noori [43] suggested an inventory system for a three-layer supply chain considering defective items. They assumed three scenarios. First, all imperfect products are disposed. Second, imperfect products are reworked and sold as perfect items. Third, scenario consists of selling imperfect products as a batch with a lower price than price of perfect items. Treviño-Garza *et al* [44] obtained the optimal value for the replenishment quantity models using two solution procedures. They considered a system of both vendor and buyer and assumed that the imperfect items are produced as well.

Table 1. Codes and explanations to categorize papers.

Terms	Code	Explanations	Codes
Model Type	(MT)	EOQ/EPQ/Both of them	O/P/B
Consumption Rate	(CR)	Known/Random	K/R
Demand Input	(DI)	Time, Inventory, Price, Nothing	Ti, I, P, No
Demand Form	(DF)	Constant, Exponential, Power, Linear, Non-Linear, None	C, E, Po, Li, NL, N
Production Form	(PF)	Constant, Power, Other	C, Po, O
Products	(P)	Single/Multiple	Si/Mu
Imperfect Items	(II)	Yes/No	Y/No
Imperfect Type	(IT)	Reworked, Repaired Out, Scrapped, Returned, Sold, Neither	R, RO, S, Re, So, Ne
Holding Cost of Imperfect items	(HI)	Yes/No	Y/No
Deterioration	(D)	Yes/No	Y/No
Shortage	(Sh)	Backlogged, Lost Sales, Partial Backlogged, Neither	B, L, PB, Ne
Interruption	(In)	Yes/No	Y/No

Taleizadeh and Wee [45] proposed a production system by assuming one machine, multiple products, manufacturing limitations, defective items, rework, and partial backlogging. Tai [46] analyzed an inventory system, in which a different screening process is considered for each single quality characteristic. Each screening process has independent screening rate and defective percentage. Taleizadeh *et al* [47] worked on an EPQ model with multiple shipments and rework of imperfect products to find the number of shipments, replenishment size and the price. Hsu and Hsu [48] studied optimal replenishment size models with defective products by assuming three scenarios according to the time of selling imperfect products. Taleizadeh and Moshtagh [49] worked on imperfect production processes, quality dependent return and lot sales in a closed loop supply chain. Table 1 shows some terms and their codes and explanations to categorize all reviewed papers. Then in table 2, a categorization of those papers is provided.

This work differs from the existing papers in some directions. With regard to the literature review until now no research is done on the jointly considering inventory systems with power demand rate dependent production rate, backlogging and defective items. In the real world, producing defective items is part of the production process. As regards it is not included in models with a power demand pattern. Also, such problems do not involve the costs (e.g., inspection and production) that are impartible parts of a production process. Actually inspection is a process itself and so that the cost associated with it should be considered in the model. As well when the cost of a production process for each item is not assumed, the results of modeling a system may be unreal. In practice, holding of defective items has cost; however, it is rarely applied in the existing studies. Therefore, firstly we extend model presented in [24] by allowing for defective items. Secondly, we consider three different situations for the proposed model regarding to the date that imperfect products are drawn from the stock. Thirdly, our model consists of production and inspection costs as well as holding cost of imperfect items. The arrangement of the rest of this work is as follows. Problem definition is available in section 2. Moreover, three developed models and the related procedure to solve them, are presented in sections 3 and 4, respectively. Then in section 5, an example is investigated. Section 6 consists of some sensitivity analyses and managerial insights. Finally, conclusions are provided in section 7.

2. Problem definition

We consider a manufacturing factory with production and inspection stages. The demand of the product has a power pattern in each inventory cycle. It is supposed that the production rate changes pro rata with the demand rate. Due to many reasons a fraction of the produced lot is assumed to be imperfect. Such products are discovered in the inspection stage. Management of the factory desires to determine the minimum inventory costs of the system, and satisfy the customer demand simultaneously. Behavior of the inventory is studied for three cases, dependent on when defective items are drawn from the inventory. In case I, we investigate the situation that imperfect items are scrapped or sold at the time that they are identified. So that in this case, the holding cost of defective items is zero. In order to reduce some costs (e.g., holding cost), it seems to be better to scrap or sell imperfect products as soon as possible; however in practice, selling or scrapping items day-to-day may be infeasible. However, in some industries (e.g., pharmaceutical companies) it is inescapable. In cases II and III, imperfect products are held in the stock and sold when the replenishment and scheduling periods are finished, respectively. The cycle length and the reorder point are two decision variables of the system. A minimizing approach is applied to specify the optimum replenishment policies of the inventory system. Figure 1 shows the system of processing a lot size.

We apply the following notations in our model.

T: Cycle length or scheduling period (time).
s: Reorder point per lot (units).
Q: Production lot size (units).
t': Production cycle length (time).
d: Total demand during the scheduling period (units).

Table 2. Categorization of reviewed papers.

No.	Paper referred	MT	CR	DF	DI	PF	P	II	IT	HI	D	Sh	In
1	Harris [1]	O	K	C	No	–	Si	No	–	–	No	Ne	–
2	Taft [2]	P	K	C	No	C	Si	No	–	–	No	Ne	No
3	Silver and Meal [3]	O	K	–	Ti	–	Si	No	–	–	No	Ne	–
4	Donaldson [4]	O	K	Li	Ti	–	Si	No	–	–	No	Ne	–
5	Ritchie [5]	O	K	Li	Ti	–	Si	No	–	–	No	Ne	–
6	Bose *et al* [6]	O	K	Li	Ti	–	Si	No	–	–	Y	B	–
7	Teng [7]	O	K	Li	Ti	–	Si	No	–	–	No	B	–
8	Zhao *et al* [8]	O	K	Li	Ti	–	Si	No	–	–	No	Ne	–
9	Lo *et al* [9]	O	K	Li	Ti	–	Si	No	–	–	No	Ne	–
10	Yang *et al* [10]	O	K	NL	Ti	–	Si	No	–	–	No	Ne	–
11	Omar and Yeo [11]	P,O	K	N	Ti	C	Si	No	–	–	No	Ne	No
12	Maihami and Kamalabadi [12]	O	K	Li, E	P, Ti	–	Si	No	–	–	Y	PB	–
13	Pando *et al* [13]	O	K	N	I, Ti	–	Si	No	–	–	No	Ne	–
14	Naddor [14]	O	K	Po	Ti	–	Si	No	–	–	No	Ne	–
15	Lee and Wu [15]	O	K /R	Po	Ti	–	Si	No	–	–	Y	B	–
16	Dye [16]	O	K /R	Po	Ti	–	Si	No	–	–	Y	B	
17	Singh *et al* [17]	O	K	Po	Ti	–	Si	No	–	–	Y	PB	–
18	Abdul-Jalbar *et al* [18]	O	K	Po	Ti	–	Si	No	–	–	No	B	
19	Rajeswari and Vanjikkodi [19]	O	K	Po	Ti	–	Si	No	–	–	Y	PB	–
20	Sicilia *et al* [20]	O	K	Po	Ti	–	Si	No	–	–	No	B, LS, Ne	–
21	Sicilia *et al* [21]	O	K	Po	Ti	–	Si	No	–	–	Y	–	–
22	Sicilia *et al* [22]	P	K	Po	Ti	C	Si	No	–	–	Y	Ne	No
23	Sicilia *et al* [23]	O	K	Po	Ti	–	Si	No	–	–	Y	B	–
24	Sicilia *et al* [24]	P	K	Po	Ti	Po	Si	No	–	–	No	B	No
25	San-José *et al* [25]	O	K	Po	Ti	Po	Si	No	–	–	No	PB	No
26	Keshavarzfard *et al* [26]	P	K	Li, Po	P, Ti	Po	Mu	No	–	–	No	B	No
27	Shih [27]	O	K	C	No	–	Si	Y	–	No	No	Ne	–
28	Schwaller [28]	O	K	C	No	–	Si	Y	–	No	No	B	–
29	Salameh and Jaber [29]	O,P	K	C	No	–	Si	Y	–	No	No	Ne	No
30	Hayek and Salameh [30]	P	K	C	No	C	Si	Y	R	No	No	B	No
31	Goyal *et al* [31]	P	K	C	No	C	Si	Y	So	No	No	Ne	No
32	Chiu [32]	P	K	C	No	C	Si	Y	R/S	No	No	B	No
33	Jamal et al [33]	P	K	C	No	C	Si	Y	R	No	No	Ne	No
34	Ojha *et al* [34]	P	K	C	No	C	Si	Y	R	No	No	Ne	No
35	Cárdenas-Barrón [35]	P	K	C	No	C	Si	Y	R	No	No	B	No
36	Taleizadeh *et al* [36]	P	K	C	No	C	Mu	Y	S	No	No	B	No
37	Taleizadeh *et al* [37]	P	K	C	No	C	Mu	Y	R	No	No	Ne	No
38	Ouyang *et al* [38]	P	K	C	No	C	Si	Y	–	No	No	Ne	No
39	Taleizadeh *et al* [39]	P	K	C	No	C	Mu	Y	R	No	No	PB	No
40	Taleizadeh *et al* [40]	P	K	C	No	C	Mu	Y	R/S	No	No	B	Y
41	Jaber *et al* [41]	O	K	C	No	–	Si	Y	RO/So	No	No	Ne	–
42	Taleizadeh *et al* [42]	P	K	C	No	C	Si	Y	S/R/So	No	No	–	No
43	Treviño-Garza *et al* [44]	P	K	C	No	C	Si	Y	–	No	No	Ne	No
44	Taleizadeh and Wee [45]	P	K	C	No	C	Mu	Y	R	No	No	PB	No
45	Tai [46]	O	K	C	No	–	Si	Y	Re	No	No	B	–
46	Taleizadeh *et al* [47]	P	K	C	No	C	Si	Y	R/S	No	No	Ne	No
47	Hsu and Hsu [48]	P	K	C	No	C	Si	Y	S/So	Y	No	B	No
48	Taleizadeh and Moshtagh [49]	B	K	C	No	C	Si	Y	R	Y	No	LS	No
49	This Paper	P	K	Po	Ti	Po	Si	Y	S/So	Y	No	B	No

r: Average demand ($r = d/T$) (units).
C_h: Carrying cost (\$/unit/unit time).
C_b: Cost of backordering (\$/unit/unit time).
C_i: Cost of inspecting (\$/unit).
C_p: Cost of producing (\$/unit).
C_o: Setup cost per cycle (\$/replenishment).
λ: Defective rate.
$CD(t)$: Demand up to time t ($0 \le t \le T$).
$D(t)$: Demand rate at time t ($0 \le t \le T$).
$P(t)$: Production rate at time t ($0 \le t \le T$).

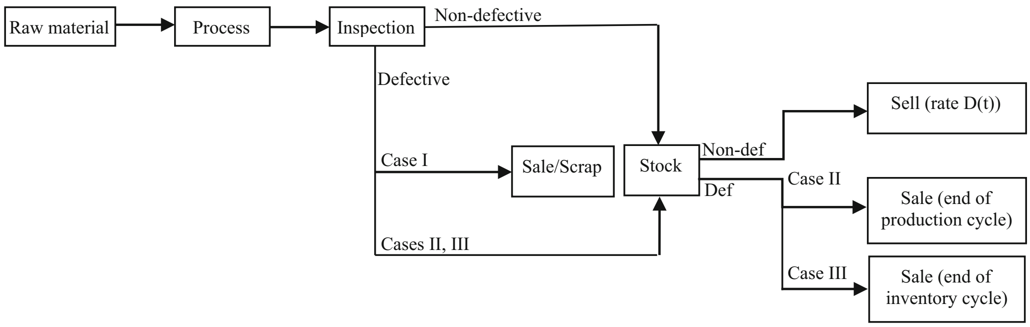

Figure 1. Processing a lot size.

$I(t)$: Net stock level at time t $(0 \le t \le T)$.
$ID(t)$: Stock level of imperfect products at time t.
$I_h(s, T)$: Average number of items holding in stock (units).
$I_b(s, T)$: Average number of backordered items (units).
$I_d(T)$: Average amount of defective items carried in inventory (units).
$CH(s, T)$: Cost of carrying products (\$/unit time).
$CB(s, T)$: Backordering cost (\$/unit time).
$CO(s, T)$: Setup cost (\$/unit time).
$CI(s, T)$: Inspection cost (\$/unit time).
$CP(s, T)$: Production cost (\$/unit time).
$CD(T)$: Holding cost of defective items (\$/unit time).
$TC_j(s, T)$: Total cost for case j (j = *I*, *II*, *III*) (\$/unit time).

We keep the main assumptions given in [24].

(1) Infinite-horizon is assumed.
(2) The amount of demand throughout the inventory cycle T is considered to be d, and average demand rate is $r = d/t$ units per cycle.
(3) The inventory system consists of a single item.
(4) The demand rate is less than the production rate.
(5) The production rate $P(t)$ is proportional to demand rate at any time t $(0 \le t \le t')$ and is defined by $P(t) = \alpha D(t)$ with $\alpha > 1$.
(6) We suppose that producing defective items is unavoidable and the fraction of defective items or defective rate is denoted by λ, which is a constant value.
(7) The produced items of perfect quality are added to inventory with rate $(1 - \lambda)P(t) - D(t)$, during the production cycle.
(8) To warrant that the consumer demand is totally covered by the products of perfect quality, it is assumed that $\alpha(1 - \lambda) - 1 > 0$ or $1 - \frac{1}{\alpha} > \lambda$.
(9) Shortages are allowed and fully backordered.
(10) To warrant that there is enough replenishment capacity to meet the demand, we assume that inspection occurs immediately after producing an item.
(11) However, the average demand per cycle d is deterministic, the number of items withdrawn from stock is dependent on the time at which they are removed. Therefore, we suppose that the cumulative demand $CD(t)$ up to time t $(0 \le t \le T)$ follows a power pattern and is given by $CD(t) = d\left(\frac{t}{T}\right)^{1/n}$, Where d is the demand quantity during the inventory cycle and n is the demand pattern index, with $0 < n < \infty$.

The demand rate at time t $(0 \le t \le T)$, follows a time-power pattern too and is the derivative of the function $CD(t)$, that is $D(t) = \frac{rt^{(1-n)/n}}{nT^{(1-n)/n}}$, with $0 \le t < T$. The nature of this demand pattern is completely defined by n. If the demand pattern index is $n = 1$, then demand is uniform (has a constant rate) and the inventory decreases linearly. When a great portion of demand happens mainly at the beginning of the cycle, then demand follows a pattern law with index $n > 1$. But if a larger percentage of demand occurs at the end of the scheduling period, then the demand of the inventory system is defined by a power pattern index $n < 1$. Also by using this kind of demand function, it is supposed that the demand is dependent on both time and the length of the scheduling period. The length of the inventory cycle or scheduling period is a fraction of the unit time. For example, assume that the unit time is a year. If $T = \frac{1}{2}, \frac{1}{3}, \ldots$ then a year consists of 2, 3, … inventory cycles, respectively.

The decision variables are cycle length T and reorder point s.

3. Mathematical models

This section extends the model proposed by [24] in some directions. Let $I(t)$ be the net stock level at time $t (0 \le t \le T)$. Inventory cycle starts with s units net stock at time 0. Also production period starts at time t = 0 and continues until $t = t'$. We suppose that the demand rate is less than the production rate in interval $0 \le t \le t'$. With regard to that in real world situations the production process is usually imperfect, a certain fraction λ of defective items is supposed to be produced in each production period. Thus, the production rate of non-defective items can be obtained by$(1-\lambda)P(t)$. Therefore, inventory is accumulated during the production period $[0, t']$, at a rate $(1-\lambda)P(t) - D(t)$.

Under these conditions, the following differential equations govern the system:

$$\frac{dI(t)}{dt} = (1-\lambda)P(t) - D(t) = ((1-\lambda)\alpha - 1)D(t) = ((1-\lambda)\alpha - 1)\frac{rt^{(1-n)/n}}{nT^{(1-n)/n}}, \quad 0 \le t \le t' \tag{1}$$

$$\frac{dI(t)}{dt} = -D(t) = -\frac{rt^{(1-n)/n}}{nT^{(1-n)/n}}, \quad t' \le t \le T \tag{2}$$

With regard to boundary conditions $I(0) = I(T) = s$, the above differential equations are solved and the solutions are as follows:

$$I(t) = s + ((1-\lambda)\alpha - 1)rT\left(\frac{t}{T}\right)^{1/n}, \quad 0 \le t \le t' \tag{3}$$

$$I(t) = s + rT - rT\left(\frac{t}{T}\right)^{1/n}, \quad t' \le t \le T \tag{4}$$

The net inventory level at t', $I(t')$, specified by both Eqs. (3) and (4) must be equal. So that t' will be found:

$$t' = \frac{T}{(1-\lambda)^n \alpha^n} \tag{5}$$

When $\lambda = 0$, Eq. (5) reduces to $t' = \frac{T}{\alpha^n}$ (given in [24]), that is less than $t' = \frac{T}{(1-\lambda)^n\alpha^n}$. It is correct because when defective items are produced, system needs more production time to meet the demand. Therefore, function $I(t)$ is increasing on $[0, t')$ and decreasing on $(t', T]$. Also $I(t)$ is a continuous and T-periodic function on interval $[0, \infty)$. The total demand on interval $[0, T)$ is computed by:

$$\int_0^T D(t)dt = \int_0^T \frac{r}{n}\left(\frac{t}{T}\right)^{\frac{1}{n}-1} dt = rT \tag{6}$$

A production size Q must be added to stock at the end of each cycle as follows:

$$Q = \int_0^{t'} P(t)dt = \alpha\int_0^{t'} \frac{r}{n}\left(\frac{t}{T}\right)^{\frac{1}{n}-1} dt = \frac{rT}{1-\lambda} \tag{7}$$

When the system does not consist of defective items ($\lambda = 0$), the replenishment size is less than the situation with defective items and it needs to produce more items to meet the demand.

Since total demand is rT by producing $\frac{rT}{1-\lambda}$ units in production time, fraction λ of this lot size, given by $\frac{\lambda rT}{1-\lambda}$, are defective items and fraction $1-\lambda$, given by rT, are non-defective items. Therefore, rT units of the lot size are of good quality and it can totally fill the demand of the cycle.

When the production quantity is entirely added to stock, the maximum stock level is obtained and calculated by:

$$I(t') = s + ((1-\lambda)\alpha - 1)rT\left(\frac{t'}{T}\right)^{1/n} = s + \frac{(1-\lambda)\alpha - 1}{(1-\lambda)\alpha}rT \tag{8}$$

3.1 *The average inventory level and the average shortage*

With regard to the reorder point s and the maximum inventory level given in Eq. (8), three different behaviors of system may occur.

(1) If $s \ge 0$, there are no shortages and the system only includes inventories.
(2) If $(I(t') \ge 0$ and $s \le 0)$ or $\frac{-((1-\lambda)\alpha-1)}{(1-\lambda)\alpha}rT \le s \le 0$, the system includes some inventories and some shortages too.
(3) If $I(t') \le 0$ or $s \le \frac{-((1-\lambda)\alpha-1)}{(1-\lambda)\alpha}rT$, only shortages occur.

If $s \ge 0$, then only inventories are contained. The average quantity of inventory is as follows:

$$I_h(s,T) = \frac{1}{T}\int_0^{t'}\left[s + ((1-\lambda)\alpha - 1)\frac{rT}{T^{1/n}}t^{1/n}\right]dt + \frac{1}{T}\int_{t'}^{T}\left[s + rT - \frac{rT}{T^{1/n}}t^{1/n}\right]dt = s + \frac{((1-\lambda)^n\alpha^n - 1)rT}{(n+1)(1-\lambda)^n\alpha^n} \tag{9}$$

Also, there are no shortages here, $I_w(s,T) = 0$.

If $s \le \frac{-((1-\lambda)\alpha-1)}{(1-\lambda)\alpha}rT$, only shortages exist. The average quantity of shortage:

$$I_b(s,T) = \frac{-1}{T}\int_0^{t'}\left[s + ((1-\lambda)\alpha - 1)\frac{rT}{T^{1/n}}t^{1/n}\right]dt - \frac{1}{T}\int_{t'}^{T}\left[s + rT - \frac{rT}{T^{1/n}}t^{1/n}\right]dt = \frac{(1-(1-\lambda)^n\alpha^n)rT}{(n+1)(1-\lambda)^n\alpha^n} - s \tag{10}$$

And there are no inventories carried, $I_h(s,T) = 0$.

Eventually, if $\frac{-((1-\lambda)\alpha-1)}{(1-\lambda)\alpha}rT \le s \le 0$, both backorders and inventories occur. Suppose that at times t_1 and t_2 within the production period and the period without production, respectively, the stock level reaches zero. Since $I(t_1) = I(t_2) = 0$, from Eqs. (3) and (4) we obtain t_1 and t_2 according to decision variables s and T:

$$t_1 = \frac{(-s)^n T}{((1-\lambda)\alpha - 1)^n r^n T^n} \tag{11}$$

$$t_2 = \frac{(s + rT)^n T}{r^n T^n} \tag{12}$$

In this situation, the average inventory level is as follows:

$$\begin{aligned} I_h(s,T) &= \frac{1}{T}\int_{t_1}^{t'} \left[s + ((1-\lambda)\alpha - 1)\frac{rT}{T^{1/n}} t^{1/n} \right] dt \\ &+ \frac{1}{T}\int_{t'}^{t_2} \left[s + rT - \frac{rT}{T^{1/n}} t^{1/n} \right] dt \\ &= \frac{(s+rT)^{n+1}}{(n+1) r^n T^n} - \frac{rT}{(n+1)(1-\lambda)^n \alpha^n} \\ &+ \frac{(-s)^{n+1}}{(n+1)((1-\lambda)\alpha - 1)^n r^n T^n} \end{aligned} \tag{13}$$

And the average shortage is:

$$\begin{aligned} I_b(s,T) &= \frac{1}{T}\int_{0}^{t_1} \left[s + ((1-\lambda)\alpha - 1)\frac{rT}{T^{1/n}} t^{1/n} \right] dt \\ &+ \frac{1}{T}\int_{t_2}^{T} \left[s + rT - \frac{rT}{T^{1/n}} t^{1/n} \right] dt \\ &= \frac{(s+rT)^{n+1}}{(n+1) r^n T^n} - \frac{rT}{(n+1)} \\ &+ \frac{(-s)^{n+1}}{(n+1)((1-\lambda)\alpha - 1)^n r^n T^n} - s \end{aligned} \tag{14}$$

Also, the average number of production runs is $\frac{1}{T}$.

3.2 *Costs and optimal inventory policies*

We consider three different cases of inventory systems and find the optimal inventory policy for them.

Case I In this situation, at the time when a defective item is recognized, it may be sold with a discount or may be scrapped. These defective items are not assumed to be in inventory (see figure 2). Now, we model the elements of cost function in the proposed model. Notice that the unit of time can be for example "year". The production cost is as follows:

$$CP(T) = C_p \frac{Q}{T} = C_p \frac{r}{1-\lambda} \tag{15}$$

The inspection cost is:

$$CI(T) = C_i \frac{Q}{T} = C_i \frac{r}{1-\lambda} \tag{16}$$

The set-up cost is:

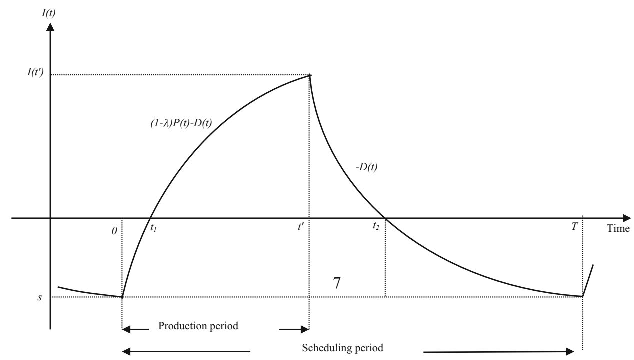

Figure 2. Net inventory level for the EPQ model with a power demand rate dependent production rate and defective items (case I).

$$CO(T) = C_o\left(\frac{1}{T}\right) = \frac{C_o}{T} \tag{17}$$

The holding cost is given by $CH(s,T) = C_h I_h(s,T)$. With regard to three situations mentioned before, three holding costs may occur. First if $s \geq 0$, from Eq. (9) the holding cost is as follows:

$$CH(s,T) = C_h\left[s + \frac{((1-\lambda)^n\alpha^n - 1)rT}{(n+1)(1-\lambda)^n\alpha^n} \right] \tag{18}$$

If $\frac{-((1-\lambda)\alpha-1)}{(1-\lambda)\alpha} rT \leq s \leq 0$, from Eq. (13) we have:

$$\begin{aligned} CH(s,T) = C_h \Bigg[&\frac{(s+rT)^{n+1}}{(n+1) r^n T^n} - \frac{rT}{(n+1)(1-\lambda)^n \alpha^n} \\ &+ \frac{(-s)^{n+1}}{(n+1)((1-\lambda)\alpha - 1)^n r^n T^n} \Bigg] \end{aligned} \tag{19}$$

And if $s \leq \frac{-((1-\lambda)\alpha-1)}{(1-\lambda)\alpha} rT$, we have $CH(s,T) = 0$, because there are no inventories in the system. Finally, the shortage cost is calculated by $CB(s,T) = C_b I_b(s,T)$. If $s \geq 0$, $CB(s,T) = 0$, because there are no shortages. For $\frac{-((1-\lambda)\alpha-1)}{(1-\lambda)\alpha} rT \leq s \leq 0$, from Eq. (14) the shortage cost is as follows:

$$\begin{aligned} CB(s,T) = C_b \Bigg[&\frac{(s+rT)^{n+1}}{(n+1) r^n T^n} - \frac{rT}{(n+1)} \\ &+ \frac{(-s)^{n+1}}{(n+1)((1-\lambda)\alpha - 1)^n r^n T^n} - s \Bigg] \end{aligned} \tag{20}$$

Finally, if $s \leq \frac{-((1-\lambda)\alpha-1)}{(1-\lambda)\alpha} rT$, from Eq. (10), the shortage cost is given by

$$CB(s,T) = C_b\left[\frac{(1-(1-\lambda)^n\alpha^n)rT}{(n+1)(1-\lambda)^n\alpha^n} - s \right] \tag{21}$$

The total cost is the sum of all five costs. That is:

$$\begin{aligned} TC_I(s,T) = &CP(T) + CI(T) + CO(T) + CH(s,T) \\ &+ CB(s,T) \end{aligned} \tag{22}$$

Therefore, the total cost in three possible situations can be found. First if $s \geq 0$, the total cost per unit time is given by:

$$TC_I(s,T) = C_p\frac{r}{1-\lambda} + C_i\frac{r}{1-\lambda} + \frac{C_o}{T} + C_h\left[s + \frac{((1-\lambda)^n\alpha^n - 1)rT}{(n+1)(1-\lambda)^n\alpha^n}\right] \quad (23)$$

If $\frac{-((1-\lambda)\alpha-1)}{(1-\lambda)\alpha}rT \leq s \leq 0$, the total cost is calculated by:

$$\begin{aligned} TC_I(s,T) &= C_p\frac{r}{1-\lambda} + C_i\frac{r}{1-\lambda} + \frac{C_o}{T} \\ &+ C_h\left[\frac{(s+rT)^{n+1}}{(n+1)r^nT^n} - \frac{rT}{(n+1)(1-\lambda)^n\alpha^n} + \frac{(-s)^{n+1}}{(n+1)((1-\lambda)\alpha-1)^n r^nT^n}\right] \\ &+ C_b\left[\frac{(s+rT)^{n+1}}{(n+1)r^nT^n} - \frac{rT}{(n+1)} + \frac{(-s)^{n+1}}{(n+1)((1-\lambda)\alpha-1)^n r^nT^n} - s\right] \end{aligned} \quad (24)$$

And for $s \leq \frac{-((1-\lambda)\alpha-1)}{(1-\lambda)\alpha}rT$, the total cost is as follows:

$$TC_I(s,T) = C_p\frac{r}{1-\lambda} + C_i\frac{r}{1-\lambda} + \frac{C_o}{T} + C_b\left[\frac{(1-(1-\lambda)^n\alpha^n)rT}{(n+1)(1-\lambda)^n\alpha^n} - s\right] \quad (25)$$

To find the minimum of the function $TC_I(s,T)$, we consider three different regions of s. As the cost $TC_I(0,T)$ is always less than the cost $TC_I(s,T)$, the minimum cost cannot be in the region $s \geq 0$. Also, since$TC_I(s,T)$ is always greater than$TC_I\left(\frac{-((1-\lambda)\alpha-1)}{(1-\lambda)\alpha}rT, T\right)$, then minimum cost cannot be at $s \leq \frac{-((1-\lambda)\alpha-1)}{(1-\lambda)\alpha}rT$. So that, the optimal cost can be found at $\frac{-((1-\lambda)\alpha-1)}{(1-\lambda)\alpha}rT \leq s \leq 0$. From partial derivatives of objective function (22) with respect to decision variables, we have:

$$\frac{\partial TC_I(s,T)}{\partial s} = (C_h + C_b)\left[\frac{(s+rT)^n}{r^nT^n} - \frac{(-s)^n}{((1-\lambda)\alpha-1)^n r^nT^n}\right] - C_b \quad (26)$$

$$\begin{aligned} \frac{\partial TC_I(s,T)}{\partial T} &= (C_h + C_b)\left[\frac{(s+rT)^n(rT-ns)}{(n+1)r^nT^{n+1}} - \frac{n(-s)^{n+1}}{(n+1)((1-\lambda)\alpha-1)^n r^nT^{n+1}}\right] \\ &- \frac{C_hr}{(n+1)(1-\lambda)^n\alpha^n} - \frac{C_br}{(n+1)} - \frac{C_o}{T^2} \end{aligned} \quad (27)$$

Equaling these derivatives to zero the optimal solution (s^*, T^*) can be calculated. Thus, we have:

$$(C_h + C_b)\left[\frac{(s+rT)^n}{r^nT^n} - \frac{(-s)^n}{((1-\lambda)\alpha-1)^n r^nT^n}\right] - C_b = 0 \quad (28)$$

$$\begin{aligned} &(C_h + C_b)\left[\frac{(s+rT)^n(rT-ns)}{(n+1)r^nT^{n+1}} - \frac{n(-s)^{n+1}}{(n+1)((1-\lambda)\alpha-1)^n r^nT^{n+1}}\right] \\ &- \frac{C_hr}{(n+1)(1-\lambda)^n\alpha^n} - \frac{C_br}{(n+1)} - \frac{C_o}{T^2} = 0 \end{aligned} \quad (29)$$

Let x be a new variable defined by $x = \frac{-s}{rT}$. Then the region $\frac{-((1-\lambda)\alpha-1)}{(1-\lambda)\alpha}rT \leq s \leq 0$ is equivalent to $0 \leq x \leq \frac{((1-\lambda)\alpha-1)}{(1-\lambda)\alpha}$. Also, Eqs. (28) and (29) are respectively equivalent to:

$$(1-x)^n - \frac{x^n}{((1-\lambda)\alpha-1)^n} - \frac{C_b}{C_h + C_b} = 0 \quad (30)$$

$$\begin{aligned} &\frac{(C_h + C_b)(1-x)^n(1+nx)r}{(n+1)} - \frac{n(C_h+C_b)rx^{n+1}}{(n+1)((1-\lambda)\alpha-1)^n} \\ &- \frac{C_hr}{(n+1)(1-\lambda)^n\alpha^n} - \frac{C_br}{(n+1)} - \frac{C_o}{T^2} \\ &= 0 \end{aligned} \quad (31)$$

Proposition 1 *Equal* $(1-x)^n - \frac{x^n}{((1-\lambda)\alpha-1)^n} - \frac{C_b}{C_b+C_h} = 0$ *has a unique solution* x^**on the interval* $\left(0, \frac{((1-\lambda)\alpha-1)}{(1-\lambda)\alpha}\right)$.

Proof Please see "Appendix A".

Use one of the numerical methods (e.g., Newton–Raphson) to find the solution x^* (see, i.e., [50]). From Eq. (30), we have:

$$\frac{x^n}{((1-\lambda)\alpha-1)^n} = (1-x)^n - \frac{C_b}{C_h + C_b} \quad (32)$$

Now, by replacement of Eq. (32) in Eq. (31), we have:

$$\begin{aligned} &\frac{(C_h + C_b)(1-x)^n r}{(n+1)} + \frac{nC_brx}{(n+1)} - \frac{C_hr}{(n+1)(1-\lambda)^n\alpha^n} \\ &- \frac{C_br}{(n+1)} - \frac{C_o}{T^2} \\ &= 0 \end{aligned} \quad (33)$$

Finally, we can obtain the best inventory cycle length T^*, by replacement the optimal solution of Eq. (30) in Eq. (33). That cycle length is calculated by:

$$T^* = \sqrt{\frac{C_o}{r\left[\frac{(C_h+C_b)(1-x^*)^n}{(n+1)} + \frac{nC_bx^*}{(n+1)} - \frac{C_h}{(n+1)(1-\lambda)^n\alpha^n} - \frac{C_b}{(n+1)}\right]}} \quad (34)$$

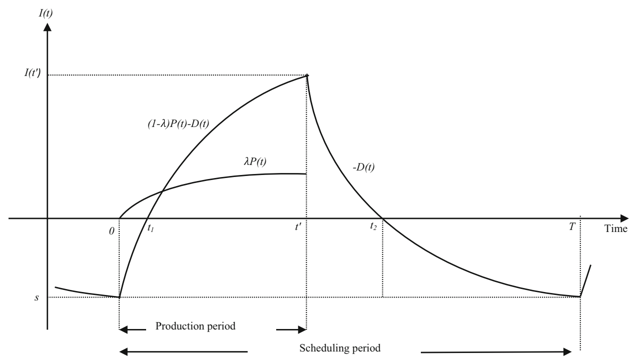

Figure 3. Net inventory level for the EPQ with a power demand rate dependent production rate and defective items (Case II).

The optimal shortage level is $s^* = -x^* rT^*$. Also as $Q = \frac{rT}{1-\lambda}$ then the economic production quantity Q^* is as follows:

$$Q^* = \frac{1}{1-\lambda}\sqrt{\frac{C_o r}{\left[\frac{(C_h+C_b)(1-x^*)^n}{(n+1)} + \frac{nC_b x^*}{(n+1)} - \frac{C_h}{(n+1)(1-\lambda)^n\alpha^n} - \frac{C_b}{(n+1)}\right]}} \tag{35}$$

Proposition 2 *The total cost function $TC_I(s,T)$ is strictly convex.*

Proof See "Appendix A".

Case II In this case, the items with imperfect quality are preserved in inventory and sold in each cycle, when the replenishment period is finished (see figure 3).

The difference between Case I and Case II is that in situation II all of the imperfect products are held in inventory until finishing the replenishment cycle. In addition to five different costs explained in the previous situation, Case II consists of the holding cost of defective items. In this situation during the period $[0, t']$, inventory of defective items would have risen at a rate $\lambda P(t)$. So that, the differential equation is given by:

$$\frac{dID(t)}{dt} = \lambda P(t) = \lambda\alpha D(t) = \frac{\lambda\alpha r t^{(1-n)/n}}{nT^{(1-n)/n}}, \quad 0 \le t \le t' \tag{36}$$

When $ID(t)$ is the inventory level of defective items and $ID(0) = 0$. The solution of the above differential equation is:

$$ID(t) = \lambda\alpha rT\left(\frac{t}{T}\right)^{1/n}, \quad 0 \le t \le t' \tag{37}$$

Suppose that $I_d(T)$ be the average amount of defective items carried in inventory. It can be calculated by:

$$I_d(T) = \frac{1}{T}\int_0^{t'}\left[\frac{\lambda\alpha rT}{T^{1/n}} t^{1/n}\right]dt = \frac{\lambda nrT}{(n+1)(1-\lambda)^{n+1}\alpha^n} \tag{38}$$

And, the cost of carrying defective items is:

$$CD(T) = C_h I_d(T) = \frac{C_h\lambda nrT}{(n+1)(1-\lambda)^{n+1}\alpha^n} \tag{39}$$

By adding $CD(T)$ to the total cost of the previous case, the total cost of this situation will be found by:

$$TC_{II}(s,T) = TC_I(s,T) + CD(T) \tag{40}$$

Similar to Case I, the minimum inventory cost, can be found in the region $\frac{-((1-\lambda)\alpha-1)}{(1-\lambda)\alpha} rT \le s \le 0$. So, we have:

$$\begin{aligned} TC_{II}(s,T) = {} & C_p\frac{r}{1-\lambda} + C_i\frac{r}{1-\lambda} + \frac{C_o}{T} \\ & + C_h\left[\frac{(s+rT)^{n+1}}{(n+1)r^nT^n} - \frac{rT}{(n+1)(1-\lambda)^n\alpha^n}\right. \\ & \left. + \frac{(-s)^{n+1}}{(n+1)((1-\lambda)\alpha-1)^n r^nT^n}\right] \\ & + C_b\left[\frac{(s+rT)^{n+1}}{(n+1)r^nT^n} - \frac{rT}{(n+1)}\right. \\ & \left. + \frac{(-s)^{n+1}}{(n+1)((1-\lambda)\alpha-1)^n r^nT^n} - s\right] \\ & + \frac{C_h\lambda nrT}{(n+1)(1-\lambda)^{n+1}\alpha^n} \end{aligned} \tag{41}$$

Equaling partial derivatives of the total cost function (41) to zero, we have:

$$\begin{aligned} \frac{\partial TC_{II}(s,T)}{\partial s} = {} & (C_h + C_b)\left[\frac{(s+rT)^n}{r^nT^n} - \frac{(-s)^n}{((1-\lambda)\alpha-1)^n r^nT^n}\right] \\ & - C_b \\ = {} & 0 \end{aligned} \tag{42}$$

$$\begin{aligned} \frac{\partial TC_{II}(s,T)}{\partial T} = {} & (C_h + C_b)\left[\frac{(s+rT)^n(rT-ns)}{(n+1)r^nT^{n+1}}\right. \\ & \left. - \frac{n(-s)^{n+1}}{(n+1)((1-\lambda)\alpha-1)^n r^nT^{n+1}}\right] \\ & - \frac{C_h r}{(n+1)(1-\lambda)^n\alpha^n} - \frac{C_b r}{(n+1)} \\ & - \frac{C_o}{T^2} + \frac{C_h\lambda nr}{(n+1)(1-\lambda)^{n+1}\alpha^n} = 0 \end{aligned} \tag{43}$$

Defining $x = \frac{-s}{rT}$, the region $\frac{-((1-\lambda)\alpha-1)}{(1-\lambda)\alpha} rT \le s \le 0$ is equivalent to $0 \le x \le \frac{((1-\lambda)\alpha-1)}{(1-\lambda)\alpha}$. Also, Eqs. (42) and (43) are respectively equivalent to:

$$(1-x)^n - \frac{x^n}{((1-\lambda)\alpha-1)^n} - \frac{C_b}{C_b + C_h} = 0 \tag{44}$$

$$
\begin{aligned}
&\frac{(C_h+C_b)(1-x)^n(1+nx)}{(n+1)}-\frac{n(C_h+C_b)rx^{n+1}}{(n+1)((1-\lambda)\alpha-1)^n}\\
&-\frac{C_h r}{(n+1)(1-\lambda)^n\alpha^n}-\frac{C_b r}{(n+1)}-\frac{C_o}{T^2}\\
&+\frac{C_h\lambda nr}{(n+1)(1-\lambda)^{n+1}\alpha^n}\\
&=0
\end{aligned}
\tag{45}
$$

Equation (44) is exactly same as Eq. (30). So solution x^* within $\left(0, \frac{((1-\lambda)\alpha-1)}{(1-\lambda)\alpha}\right)$ is unique. Now, by replacement of Eq. (32) in Eq. (45), we have:

$$
\begin{aligned}
&\frac{(C_h+C_b)r(1-x)^n}{(n+1)}+\frac{nC_b rx}{(n+1)}-\frac{C_h r}{(n+1)(1-\lambda)^n\alpha^n}\\
&-\frac{C_b r}{(n+1)}-\frac{C_o}{T^2}+\frac{C_h\lambda nr}{(n+1)(1-\lambda)^{n+1}\alpha^n}\\
&=0
\end{aligned}
\tag{46}
$$

Finally, we can obtain the optimal inventory policy (s^*, T^*), by replacement the optimal solution of Eq. (44) in Eq. (46). The optimal scheduling period is calculated by:

$$
T^* = \sqrt{\frac{C_o}{r\left[\frac{(C_h+C_b)(1-x^*)^n}{(n+1)}+\frac{nC_b x^*}{(n+1)}-\frac{C_h}{(n+1)(1-\lambda)^n\alpha^n}-\frac{C_b}{(n+1)}+\frac{C_h\lambda n}{(n+1)(1-\lambda)^{n+1}\alpha^n}\right]}}
\tag{47}
$$

The optimal shortage level is $s^* = -x^* rT^*$. Also, Q^* is as follows:

$$
Q^* = \frac{1}{1-\lambda}\sqrt{\frac{C_o r}{\left[\frac{(C_h+C_b)(1-x^*)^n}{(n+1)}+\frac{nC_b x^*}{(n+1)}-\frac{C_h}{(n+1)(1-\lambda)^n\alpha^n}-\frac{C_b}{(n+1)}+\frac{C_h\lambda n}{(n+1)(1-\lambda)^{n+1}\alpha^n}\right]}}
\tag{48}
$$

Using the second order derivatives of $TC_{II}(s,T)$ respect to s and T, we have the same formula and Hessian as in case I. Thus, $TC_{II}(s,T)$ is strictly convex (see proposition 2).

Case III The behavior of this inventory system is similar to Case II, the only difference is that the imperfect products are sold when each inventory cycle is finished (see figure 4). In this situation, producing defective items starts just after $t = 0$ and continues up to $t = t'$, when the inventory level of defective items attains a maximum level $ID(t')$. With respect to that the fraction λ of all $Q = \frac{rT}{1-\lambda}$ produced items are defective, the total number of defective items is as follows:

$$
ID(t') = \frac{\lambda rT}{1-\lambda}
\tag{49}
$$

And after reaching that, this value doesn't change until T. So, $I_d(T)$ is given by:

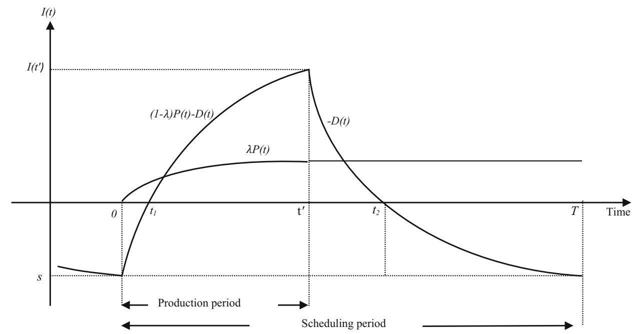

Figure 4. Net inventory level for the EPQ with a power demand rate dependent production rate and defective items (Case III).

$$
\begin{aligned}
I_d(T) &= \frac{1}{T}\left(\int_0^{t'}\left[\frac{\lambda\alpha rT}{T^{1/n}}t^{1/n}\right]dt+\frac{\lambda rT}{1-\lambda}(T-t')\right)\\
&=\frac{\lambda nrT}{(n+1)(1-\lambda)^{n+1}\alpha^n}+\frac{\lambda rT}{1-\lambda}\left(T-\frac{T}{(1-\lambda)^n\alpha^n}\right)\\
&=\frac{\lambda rT}{(1-\lambda)^{n+1}\alpha^n}\left((1-\lambda)^n\alpha^n-\frac{1}{n+1}\right)
\end{aligned}
\tag{50}
$$

And, the cost of inventory of defective items is:

$$
CD(T) = C_h I_d(T) = \frac{C_h\lambda rT}{(1-\lambda)^{n+1}\alpha^n}\left((1-\lambda)^n\alpha^n-\frac{1}{n+1}\right)
\tag{51}
$$

By adding $CD(T)$ to the total cost of the previous case, the total cost of this situation will be found by:

$$
TC_{III}(s,T) = TC_I(s,T) + CD(T)
\tag{52}
$$

So that, we have:

$$
\begin{aligned}
&TC_{III}(s,T)\\
&=C_p\frac{r}{1-\lambda}+C_i\frac{r}{1-\lambda}+\frac{C_o}{T}+C_h\left[\frac{(s+rT)^{n+1}}{(n+1)r^nT^n}-\frac{rT}{(n+1)(1-\lambda)^n\alpha^n}\right.\\
&\left.+\frac{(-s)^{n+1}}{(n+1)((1-\lambda)\alpha-1)^n r^nT^n}\right]+C_b\left[\frac{(s+rT)^{n+1}}{(n+1)r^nT^n}-\frac{rT}{(n+1)}\right.\\
&\left.+\frac{(-s)^{n+1}}{(n+1)((1-\lambda)\alpha-1)^n r^nT^n}-s\right]\\
&+\frac{C_h\lambda rT}{(1-\lambda)^{n+1}\alpha^n}\left((1-\lambda)^n\alpha^n-\frac{1}{n+1}\right)
\end{aligned}
\tag{53}
$$

Equaling partial derivatives of the total cost function (52) to zero, we have:

$$
\begin{aligned}
\frac{\partial TC_{III}(s,T)}{\partial s} &= (C_h+C_b)\left[\frac{(s+rT)^n}{r^nT^n}-\frac{(-s)^n}{((1-\lambda)\alpha-1)^n r^nT^n}\right]\\
&-C_b\\
&=0
\end{aligned}
\tag{54}
$$

$$\frac{\partial TC_{III}(s,T)}{\partial T} = (C_h + C_b)\left[\frac{(s+rT)^n(rT-ns)}{(n+1)r^nT^{n+1}} - \frac{n(-s)^{n+1}}{(n+1)((1-\lambda)\alpha-1)^n r^n T^{n+1}}\right] - \frac{C_h r}{(n+1)(1-\lambda)^n\alpha^n} - \frac{C_b r}{(n+1)} - \frac{C_o}{T^2} + \frac{C_h\lambda r}{(1-\lambda)^{n+1}\alpha^n}\left((1-\lambda)^n\alpha^n - \frac{1}{n+1}\right) = 0 \tag{55}$$

Again defining $x = \frac{-s}{rT}$, the region $\frac{-((1-\lambda)\alpha-1)}{(1-\lambda)\alpha} rT \le s \le 0$ is equivalent to $0 \le x \le \frac{((1-\lambda)\alpha-1)}{(1-\lambda)\alpha}$. Also, Eqs. (54) and (55) are respectively equivalent to:

$$(1-x)^n - \frac{x^n}{((1-\lambda)\alpha-1)^n} - \frac{C_b}{C_b + C_h} = 0 \tag{56}$$

$$\frac{(C_h + C_b)(1-x)^n(1+nx)}{(n+1)} - \frac{n(C_h + C_b)rx^{n+1}}{(n+1)((1-\lambda)\alpha-1)^n} - \frac{C_h r}{(n+1)(1-\lambda)^n\alpha^n} - \frac{C_b r}{(n+1)} - \frac{C_o}{T^2} + \frac{C_h\lambda r}{(1-\lambda)^{n+1}\alpha^n}\left((1-\lambda)^n\alpha^n - \frac{1}{n+1}\right) = 0 \tag{57}$$

Now, by replacement of Eq. (32) in Eq. (57), we have:

$$\frac{(C_h + C_b)r(1-x)^n}{(n+1)} + \frac{nC_b rx}{(n+1)} - \frac{C_h r}{(n+1)(1-\lambda)^n\alpha^n} - \frac{C_b r}{(n+1)} - \frac{C_o}{T^2} + \frac{C_h\lambda r}{(1-\lambda)^{n+1}\alpha^n}\left((1-\lambda)^n\alpha^n - \frac{1}{n+1}\right) = 0 \tag{58}$$

Finally, like the previous cases, we have:

$$T^* = \sqrt{\frac{C_o}{r\left[\frac{(C_h+C_b)(1-x^*)^n}{(n+1)} + \frac{nC_b x^*}{(n+1)} - \frac{C_h}{(n+1)(1-\lambda)^n\alpha^n} - \frac{C_b}{(n+1)} + \frac{C_h\lambda}{(1-\lambda)^{n+1}\alpha^n}\left((1-\lambda)^n\alpha^n - \frac{1}{n+1}\right)\right]}} \tag{59}$$

The optimal shortage level is $s^* = -x^* rT^*$. Also, Q^* is as follows:

$$Q^* = \frac{1}{1-\lambda}\sqrt{\frac{C_o r}{\left[\frac{(C_h+C_b)(1-x^*)^n}{(n+1)} + \frac{nC_b x^*}{(n+1)} - \frac{C_h}{(n+1)(1-\lambda)^n\alpha^n} - \frac{C_b}{(n+1)} + \frac{C_h\lambda}{(1-\lambda)^{n+1}\alpha^n}\left((1-\lambda)^n\alpha^n - \frac{1}{n+1}\right)\right]}} \tag{60}$$

By taking the second order derivatives of $TC_{III}(s,T)$ with respect to s and T, we have the same formula and Hessian as in case I. Thus, the function $TC_{III}(s,T)$ is strictly convex (see proposition 2).

4. Procedure for determining the optimal values

A brief procedure is explained to obtain the optimum values for all the three proposed cases. Notice that optimal value of variable x is obtained by Eq. (30), or (44) or (56), that are the same for all three cases.

Step 1 Enter the values of parameters.
Step 2 Obtain x^* of equation $(1-x)^n - \frac{x^n}{((1-\lambda)\alpha-1)^n} - \frac{C_b}{C_b+C_h} = 0$, using a numerical method.
Step 3 Specify T^*, using Eq. (34) for Case I, Eq. (47) for Case II or Eq. (59) for case III. Use $s^* = -x^* rT^*$ to calculate optimal reorder point.
Step 4 Determine optimal lot size Q^* given by formula (35) for Case I, (48) for Case II or (60) for Case III.
Step 5 Calculate the minimum cost TC_I^* for Case I using Eq. (24), TC_{II}^* for Case II using Eq. (41) and TC_{III}^* for Case III using Eq. (53).

5. Numerical example

In order to provide input data, we consider an inventory system, in which the parametric values are defined as C_h = \$4 per unit and year, C_b = \$5 per unit, C_o = \$100 per replenishment, C_i = \$2 per unit, C_p = \$6 per unit, r = 1200 units per year, α = 1.4, λ = 0.2 and n = 2. Using Eq. (30), the following equation must be solved.

$$(1-x)^2 - \frac{x^2}{0.12^2} = \frac{5}{9} \tag{61}$$

There is a unique solution for this equation inside the interval $\left(0, \frac{((1-\lambda)\alpha-1)}{(1-\lambda)\alpha} = 0.107143\right)$, that is $x^* = 0.067286$. Now for three different cases, we have following optimal values:

Case I Using Eq. (34), the optimal cycle length is $T^* = 0.8927\,\text{year} = 325.8355\,\text{days}$. From Eq. (35) the economic lot size is $Q^* = 1339.1$ units. Using formula $s^* = -x^*rT^*$, the optimal reorder point is $s^* = -72.0826$ units. From Eq. (24), the optimal cost is $TC_I^* = \$12224$ per year.

Case II Using Eq. (47), the optimal cycle length is $T^* = 0.3620\,\text{year} = 132.13\,\text{days}$. From Eq. (48) the economic lot size is $Q^* = 542.9552$ units. Using formula $s^* = -x^*rT^*$, the optimal reorder point is $s^* = -29.2266$ units. From Eq. (41), the optimal cost is $TC_{II}^* = \$12553$ per year.

Case III Using Eq. (59), the optimal cycle length is $T^* = 0.2466\,\text{year} = 90.009\,\text{days}$. From Eq. (60) the economic lot size is $Q^* = 369.9083$ units. Using formula $s^* = -x^*rT^*$, the optimal reorder point is $s^* = -19.9117$ units. From Eq. (53), the optimal cost is $TC_{III}^* = \$12654$ per year.

Figures 5, 6 and 7 show the total cost as a function of two variables T and s for Cases I, II and III by using the above example's input parameters, respectively.

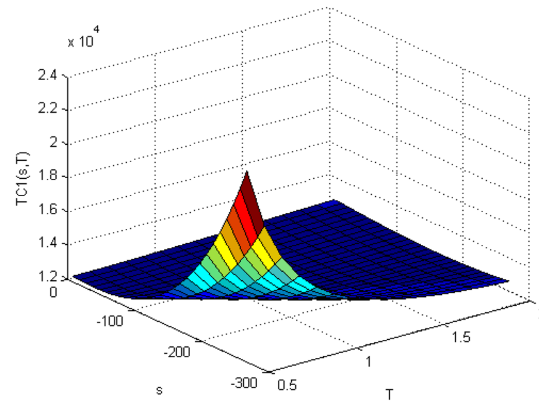

Figure 5. Total cost for Case I.

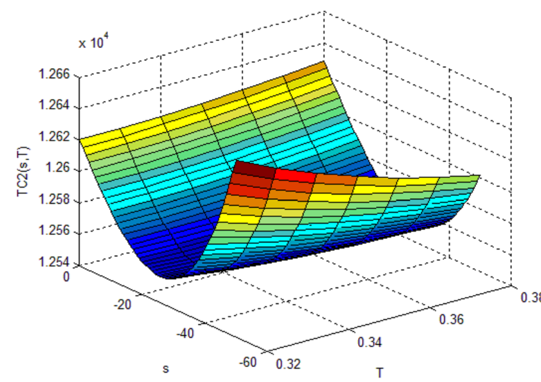

Figure 6. Total cost for Case II.

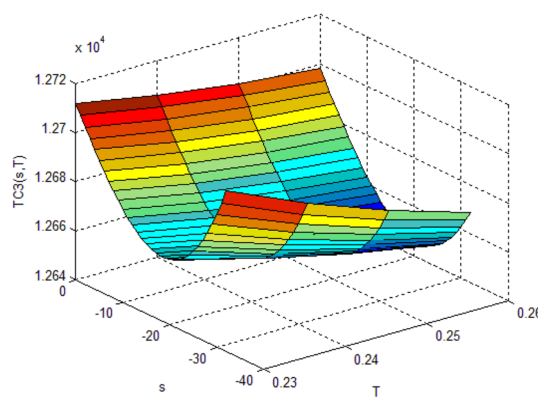

Figure 7. Total cost for Case III.

6. Numerical analyses and insights

Some numerical analyses are done to discover the effect of changes in parameters on the results. Basically, the impacts of the production rate α, the defective rate λ and the power demand index n on the variables and the total cost of the system are analyzed in this section. Some extra examples are reported in tables 3 and 4 using the input data taken from [24]. According to the following values of parameters, table 3 is provided: $n = 3$, $C_o = 100$, $r = 1200$, $C_h = 4$, $C_b = 5$, $C_i = 0$ and $C_p = 0$. Optimal policies of inventory systems considering several combinations of parameters α and λ are shown in table 3.

Also, the following values are used in table 4: $r = 1200$, $C_o = 100$, $C_h = 4$, $C_b = 5$, $C_i = 2$, $C_p = 6$ and $\alpha = 1.5$. The optimal solutions for various values of n and λ are calculated in table 4.

The graph of the minimum cost and lot size changes versus the defective rate changes for Case I is shown in figures 8 and 9, using table 4. In each figure, four different values of n (0.5, 1, 1.5 and 2) are considered.

Figures 10 and 11 show the total cost and the cycle length as functions of the defective rate for Case I, when input parameters are used from table 3. In each figure different values of α are considered ($\alpha = 1.1, 1.3, 1.5, 1.9$).

Figures 12 and 13 show changes of the total cost respect to the changes of the production rate for Cases I and II, respectively, using table 3. In each figure, four different values of λ are assumed ($\lambda = 0.02, 0.05, 0.2, 0.5$).

Table 3. Numerical analysis of production rate α and defective rate λ.

r = 1200, C_o = 100, C_h = 4, C_b = 5, C_i = 0, C_p = 0 and n = 3

Case	Production rate	Defective rate	x^*	T^*	Q^*	s^*	TC^*
I	α = 1.1	λ = 0.02	0.052006	1.1441	1400.9	−71.4004	174.8089
II				0.8605	1053.7	−53.7030	232.4160
III				0.6845	838.1156	−42.7153	252.0586
I		λ = 0.05	0.031776	1.5404	1945.8	−58.7381	129.8321
II				0.6931	875.4686	−26.4279	288.5662
III				0.4966	627.3088	−18.9367	320.2658
I	α = 1.3	λ = 0.02	0.129183	0.5934	726.6420	−91.9924	337.0263
II				0.5595	685.0479	−86.7266	357.4896
III				0.4965	607.9932	−76.9715	385.1548
I		λ = 0.05	0.118712	0.6386	806.6574	−90.9719	313.1822
II				0.5378	679.2764	−76.6064	371.9119
III				0.4207	531.4616	−59.9363	433.0302
I		λ = 0.1	0.096386	0.7525	1003.3	−87.0364	265.7812
II				0.4843	645.7389	−56.0162	412.9636
III				0.3378	450.3807	−39.0694	507.7303
I		λ = 0.2	0.028491	1.6407	2461.0	−56.0936	121.8980
II				0.3456	518.3996	−11.8158	578.7036
III				0.2377	356.4819	−8.1252	651.4104
I	α = 1.5	λ = 0.02	0.159193	0.4748	581.4351	−90.7092	421.1956
II				0.4629	566.8428	−88.4327	432.0385
III				0.4238	518.8829	−80.9505	462.1705
I		λ = 0.05	0.154768	0.4926	622.2047	−91.4825	406.0264
II				0.4576	577.9829	−84.9806	437.0917
III				0.3757	474.5803	−69.7774	507.7250
I		λ = 0.1	0.144553	0.5323	709.6845	−92.3283	375.7534
II				0.4404	587.2210	−76.3961	454.1160
III				0.3171	422.7598	−55.0001	579.2273
I		λ = 0.2	0.107512	0.6921	1038.1	−89.2874	288.9863
II				0.3702	555.3206	−47.7629	540.2281
III				0.2386	357.9549	−30.7876	713.8043
I	α = 1.7	λ = 0.02	0.169830	0.4266	522.3629	−86.9386	468.8269
II				0.4206	514.9709	−85.7084	475.5566
III				0.3898	477.3381	−79.4450	506.8553
I		λ = 0.05	0.167995	0.4360	550.6916	−87.8878	458.7528
II				0.4184	528.5607	−84.3558	477.9609
III				0.3516	444.1698	−70.8874	552.9552
I		λ = 0.1	0.163656	0.4560	608.0107	−89.5541	438.5886
II				0.4108	547.7270	−80.6749	486.8604
III				0.3031	404.0982	−59.5198	626.0575
I		λ = 0.2	0.146161	0.5260	789.0588	−92.2637	380.1991
II				0.3729	559.3404	−65.4030	536.3456
III				0.2357	353.5627	−41.3417	764.1718
I	α = 1.9	λ = 0.02	0.173899	0.4020	492.2310	−83.8865	497.5264
II				0.3983	487.7663	−83.1256	502.0804
III				0.3716	454.9593	−77.5346	534.0568
I		λ = 0.05	0.173057	0.4077	515.0259	−84.6724	490.5218
II				0.3972	501.7634	−82.4920	503.4872
III				0.3379	426.8690	−70.1790	580.9362
I		λ = 0.1	0.171085	0.4197	559.6163	−86.1678	476.5168
II				0.3929	523.9172	−80.6709	508.9862
III				0.2943	392.4395	−60.4265	655.9648
I		λ = 0.2	0.163007	0.4588	688.2448	−89.7510	435.8910
II				0.3698	554.6882	−72.3345	540.8441
III				0.2328	349.1342	−45.5291	799.6179

Table 4. Numerical analysis of the power demand index n and defective rate λ.

$r = 1200$, $C_o = 100$, $C_h = 4$, $C_b = 5$, $C_i = 2$, $C_p = 6$ and $\alpha = 1.5$

Case	Index of demand	Defective rate	x^*	T^*	Q^*	s^*	TC^*
I	n = 0.5	λ = 0.02	0.077131	0.4813	589.3351	−44.5469	10211
II				0.4669	571.7700	−43.2192	10224
III				0.4460	546.1191	−41.2803	10232
I		λ = 0.05	0.070871	0.4992	630.5918	−42.4561	10506
II				0.4604	581.5405	−39.1536	10540
III				0.4140	522.9936	−35.2118	10559
I		λ = 0.1	0.059985	0.5372	716.2218	−38.6663	11039
II				0.4474	596.5414	−32.2052	11114
III				0.3709	494.5434	−26.6987	11149
I		λ = 0.2	0.036315	0.6750	1012.5	−29.4137	12296
II				0.4136	620.3601	−18.0227	12484
III				0.3080	461.9538	−13.4207	12537
I	n = 1	λ = 0.02	0.142101	0.4843	593.0553	−82.5883	10209
II				0.4664	571.1548	−79.5384	10225
III				0.4367	534.7863	−74.4738	10239
I		λ = 0.05	0.132553	0.5015	633.4326	−79.7652	10504
II				0.4535	572.8004	−72.1301	10546
III				0.3921	495.3393	−62.3758	10581
I		λ = 0.1	0.115226	0.5379	717.1363	−74.3695	11039
II				0.4291	572.0771	−59.3263	11133
III				0.3373	449.7188	−46.6374	11193
I		λ = 0.2	0.074074	0.6708	1006.2	−59.6284	12298
II				0.3721	558.1562	−33.0759	12537
III				0.2652	397.7475	−23.5702	12622
I	n = 1.5	λ = 0.02	0.164830	0.4919	602.2971	−97.2911	10203
II				0.4734	579.6178	−93.6276	10218
III				0.4376	535.8040	−86.5502	10239
I		λ = 0.05	0.155271	0.5101	644.3930	−95.0528	10497
II				0.4596	580.5811	−85.6400	10540
III				0.3873	489.2493	−72.1679	10587
I		λ = 0.1	0.137308	0.5490	732.0051	−90.4591	11031
II				0.4320	576.0261	−71.1837	11130
III				0.3274	436.4867	−53.9398	11211
I		λ = 0.2	0.091643	0.6912	1036.8	−76.0152	12289
II				0.3635	545.1778	−39.9694	12550
III				0.2506	375.9564	−27.5630	12661
I	n = 2	λ = 0.02	0.170785	0.4891	598.8754	−100.2334	10205
II				0.4723	578.3360	−96.7957	10219
III				0.4339	531.3188	−88.9265	10244
I		λ = 0.05	0.162401	0.5081	641.8267	−99.0216	10499
II				0.4611	582.4845	−89.8663	10539
III				0.3831	483.8902	−74.6550	10596
I		λ = 0.1	0.145939	0.5490	732.0333	−96.1490	11031
II				0.4357	580.8940	−76.2976	11126
III				0.3223	429.7425	−56.4446	11224
I		λ = 0.2	0.100652	0.7014	1052.1	−84.7156	12258
II				0.3630	544.5268	−43.8462	12551
III				0.2439	365.8481	−29.4587	12685

Figures 14 and 15 show changes of total cost and reorder point respect to the changes of the power demand index for Case I, using table 3. In each figure, four different values of λ are assumed ($\lambda = 0.02, 0.05, 0.2, 0.5$).

Some sensitivity analyses can be expressed as follows.

- From table 3, we can observe that fixed the production rate parameter α and considering Case I, the value x^*, the total amount of backorders $-s^*$ and the minimum cost TC^* decrease as the defective rate λ increases. However, the optimal cycle length T^* and the

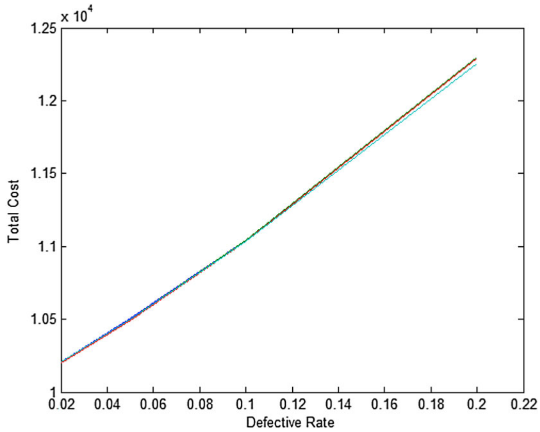

Figure 8. Changes of the total cost value with respect to the changes of the defective rate for Case I, using table 4.

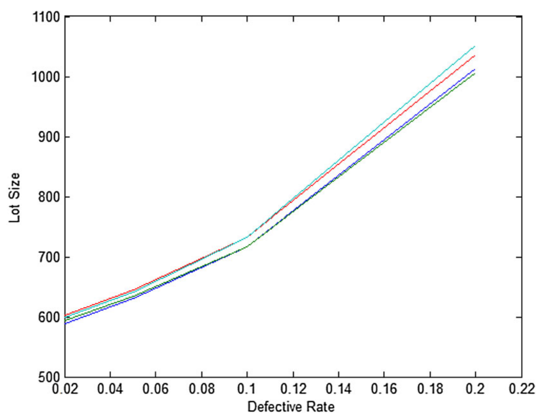

Figure 9. Changes of the lot size value with respect to the changes of the defective rate for Case I, using table 4.

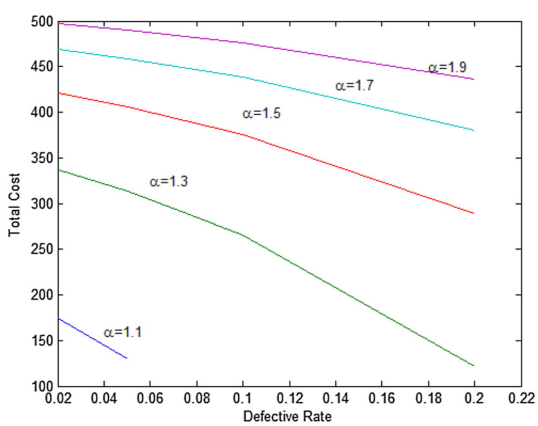

Figure 10. Changes of the total cost value with respect to the changes of the defective rate for Case I, using table 3.

economic lot size Q^* increase as the defective rate λ increases. In the same situation, but considering Case II, the minimum cost TC^* increases as the defective rate λ increases. The reason is that in Case I we did not

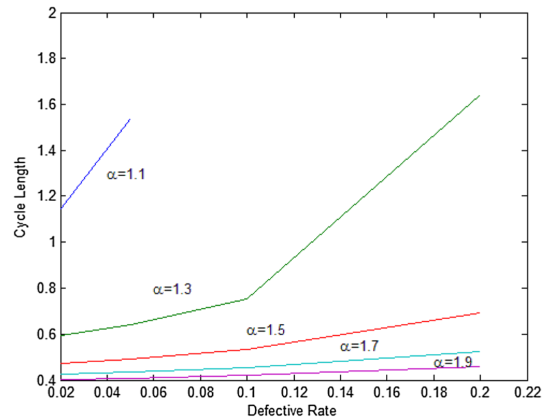

Figure 11. Changes of the cycle length value with respect to the changes of the defective rate for Case I, using table 3.

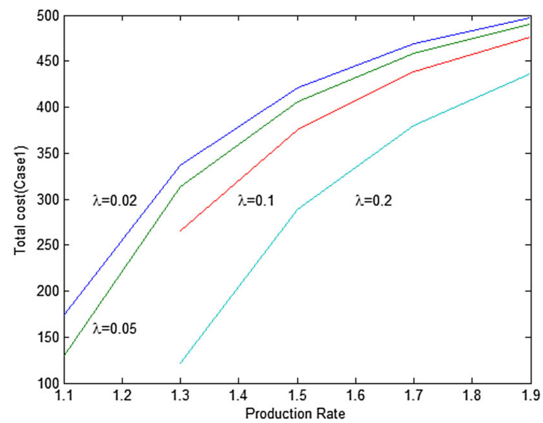

Figure 12. Changes of the total cost value with respect to the changes of the production rate for Case I, using table 3.

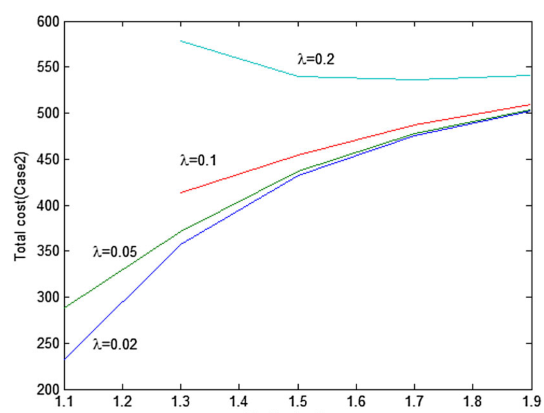

Figure 13. Changes of the total cost value with respect to the changes of the production rate for Case II, using table 3.

consider the holding cost of defective items; however, in Case 2 we consider it and because of that the results of Case II are more reasonable. Therefore, Case II can

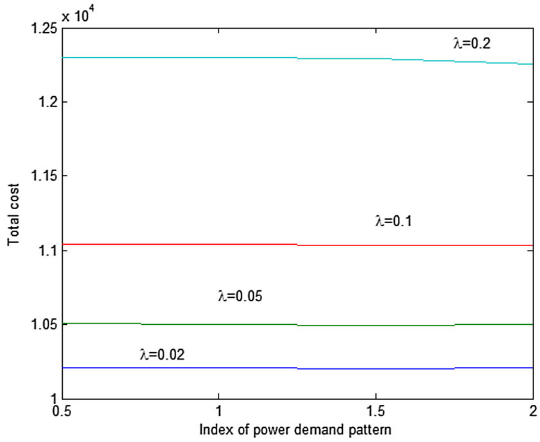

Figure 14. Changes of the total cost value with respect to the changes in the power demand index for Case I, using table 4.

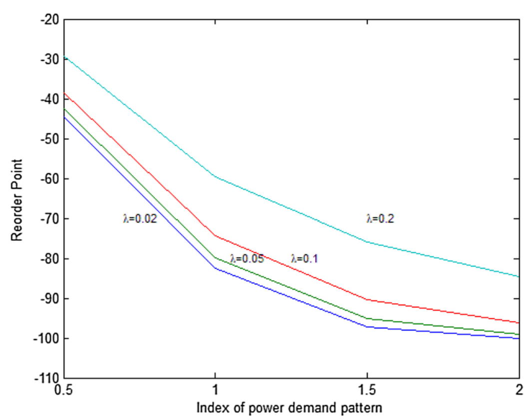

Figure 15. Changes of the reorder point value with respect to the changes in the power demand index for Case I, using table 4.

show the real world situations much better (in both cases, we do not have inspection and production costs ($C_i = C_p = 0$)). Also, fixed the defective rate λ and considering Case I, if the production rate α increases then the value x^* and the minimum cost TC^* increase. However, the optimal cycle length $T^*$and the economic lot size Q^* decrease as the production rate α increases.

- In the same table 3, fixed the production parameter α and the defective rate λ, we can observe that the optimal cycle length T^* and the economic lot size Q^* decrease from case I to case III. However, TC^* increases in this situation.
- In table 4, by fixing the power demand index n and considering Case I, if the defective rate λ increases then the cycle length T^*, the optimum lot size Q^* and the minimum cost TC^* increase. However, the total omount of backorders $-s^*$ and the value x^* decrease as the defective rate λ increases. Also, fixed the defective rate λ and considering Case I, if n increases then the value x^* increases. However, for other inventory policies, we can not find a standard pattern.
- In the same table 4, by fixing the power demand index n and the defective rate λ, we can observe that the optimal cycle length $T^*$and the optimum production quantity Q^* and the total amount of backorders $-s^*$ decrease from Case I to case III. However, TC^* increases in this situation.
- Comparing tables 3 and 4, we can see that if production and inspection costs are involved in the proposed model (in table 4, $C_i = 2$ \$/unit and $C_p = 6$ \$/unit) then fixed the other parameters the optimum cost TC^* increases as the defective rate λ increases. However, if production and inspection costs are not considered in the proposed model (in table 3, $C_i = 0$ \$/unit and $C_p = 0$ \$/unit), then fixed other parameters the optimal cost TC^* decreases as the defective rate λ increases. Additionally, it shows considering these costs can help modeling the real world situations much better. When production costs or holding cost of imperfect products are not involved, the obtained results are not reliable enough.

This research offers several managerial insights: all previous related works have focused on the EPQ model in which demand follows a power pattern without considering imperfect items or imperfect items are involved but demand is uniformly distributed. None of those models are applicable enough to be used in the real world situations. Another important feature of our models is considering the time when those defective items are removed from the inventory. One of the most valuable managerial insights of our study is that when production cost and inspection cost as well as holding cost of defective items are equal to zero the results can not be same as the real world situations, as it can be seen that in such cases when defective rate increases the optimal cost decreases (instead of increasing). But when one of production and inspection costs or holding cost of imperfect items are involved the results can model the real system accurately. Another managerial insight is that as the defective rate increases optimum cycle length, best lot quantity and total cost increase. Also case I can be suitable for very small factories that remove the imperfect items when they are discovered or the firms that produce medical items and have to remove defective products as soon as possible. Many other firms can not do the same thing and are forced to keep those items until the end of the production cycle to be reworked or the end of the inventory cycle to be scrapped or sold with a lower price.

7. Conclusions and future directions of research

In this research, we proposed EPQ models with a power demand rate dependent production rate, allowing for shortages completely backordered and defective items. Three cases are considered for the inventory system regarding to the date that defective items are drawn from inventory. In the first case, we suppose that a defective item is eliminated from inventory at the time when it is recognized. In the second and third situations, it is assumed that the items with imperfect quality are kept in stock and sold in each cycle, after ending the replenishment and the inventory cycles respectively. The optimum solutions obtained by the mathematical models are unique and easily calculated by the proposed algorithm. When imperfect items, the holding cost of defective items, inspection and production costs are not considered, the optimal inventory policies consist of the formulate obtained by [24]. One possible extension to the current paper can be considered for rework process in the inventory system. Moreover, a deterioration rate can be considered in the model. Another research can study inventory systems with assuming that shortages are lost sales or partially backorderd. Finally, one can also consider the situation where demand depends on the price of the products. Also we suggest to incorporate the concept of this work as potentials extension to the problems or models suggested by other researchers [51–53].

Appendix A

Proposition 1 *Equal* $(1-x)^n - \frac{x^n}{((1-\lambda)\alpha-1)^n} - \frac{C_b}{C_b+C_h} = 0$ *has a unique solution* x^* *on* $\left(0, \frac{((1-\lambda)\alpha-1)}{(1-\lambda)\alpha}\right)$.

Proof Suppose that f(x) is a real function on [0,1] defined by:

$$f(x) = (1-x)^n - \frac{x^n}{((1-\lambda)\alpha-1)^n} - \frac{C_b}{C_b+C_h} \quad \text{(a)}$$

f(*x*) is continuous, strictly decreasing differentiable on the interval (0,1) because (notice that according to assumption 9, $(1-\lambda)\alpha - 1 > 0$):

$$f'(x) = -n(1-x)^{n-1} - \frac{nx^{n-1}}{((1-\lambda)\alpha-1)^n} < 0 \quad \text{(b)}$$

Also, we have $f(0) = \frac{C_h}{C_b+C_h} > 0$ and $f\left(\frac{((1-\lambda)\alpha-1)}{(1-\lambda)\alpha}\right) = -\frac{C_b}{C_b+C_h} < 0$. So, using the intermediate value theory, a point x^* exists in the interval $\left(0, \frac{((1-\lambda)\alpha-1)}{(1-\lambda)\alpha}\right)$, where $y(x^*) = 0$. Finally, because of that the function is decreasing on (0,1), the point x^* is unique.

Proposition 2 *The total cost function* $TC_I(s,T)$ *is strictly convex.*

Proof Using the second order derivatives of $TC_I(s,T)$ respect to decision variables, we have:

$$\frac{\partial^2 TC_I(s,T)}{\partial s^2} = (C_h + C_b)\left[\frac{n(s+rT)^{n-1}}{r^n T^n} + \frac{n(-s)^{n-1}}{((1-\lambda)\alpha-1)^n r^n T^n}\right] \quad \text{(c)}$$

$$\frac{\partial^2 TC_I(s,T)}{\partial T^2} = (C_h + C_b)\left[\frac{ns^2(s+rT)^{n-1}}{r^n T^{n+2}} + \frac{n(-s)^{n+1}}{((1-\lambda)\alpha-1)^n r^n T^{n+2}}\right] + \frac{2C_o}{T^3} \quad \text{(d)}$$

$$\frac{\partial^2 TC_I(s,T)}{\partial s \partial T} = (C_h + C_b)\left[\frac{n(-s)(s+rT)^{n-1}}{r^n T^{n+1}} + \frac{n(-s)^n}{((1-\lambda)\alpha-1)^n r^n T^{n+1}}\right] \quad \text{(f)}$$

And the Hessian of the function $TC_I(s,T)$ is given by:

$$H(s,T) = \left[\frac{\partial^2 TC_I(s,T)}{\partial s^2}\right]\left[\frac{\partial^2 TC_I(s,T)}{\partial T^2}\right] - \left[\frac{\partial^2 TC_I(s,T)}{\partial s \partial T}\right]^2 = (C_h + C_b)\left[\frac{n(s+rT)^{n-1}}{r^n T^n} + \frac{n(-s)^{n-1}}{((1-\lambda)\alpha-1)^n r^n T^n}\right]\left[\frac{2C_o}{T^3}\right] > 0 \quad \text{(g)}$$

Eqs. (c) to (g) are positive, because in the region $\frac{-((1-\lambda)\alpha-1)}{(1-\lambda)\alpha} rT \le s \le 0$, we always have $s + rT > 0$. Therefore, the function $TC_I(s,T)$ is strictly convex.

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