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# Stagnation point flow and heat transfer for a viscoelastic fluid impinging on a quiescent fluid

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**Abstract.** A theoretical study is made in the region near the stagnation point when a lighter incompressible viscoelastic fluids impinges orthogonally on the surface of another quiescent heavier incompressible viscous fluid. Similarity solutions of the momentum balance equations for both fluids are equalized at the interface. It is noted that an exact boundary layer solution is obtained for the lower lighter fluid. The velocity of the lower fluid is independent of lateral interface velocity but the velocity of the upper viscoelastic fluid increases with increasing lateral interface velocity. It is observed that lateral interface velocity increases with increasing viscoelastic parameter for fixed values of density and viscosity ratio of the two fluids. The convective heat transfer is investigated base on the similarity solutions for the temperature distribution of the two fluids. The interface temperature increases with increasing viscoelastic parameter of the upper viscoelastic fluid.

**Keywords.** Stagnation-point flow; viscoelastic fluid; quiescent fluid; heat transfer.

## 1. Introduction

The flow of an incompressible fluid past a rigid or moving surface has several engineering applications within, for instance, polymer processing. A great deal of literature is available on the two-dimensional stagnation-point flow over a solid plate. On the other hand, there is free stagnation point flow or line interior to a homogeneous fluid domain or at the interface between two immiscible fluids. They can be steady or unsteady two-dimensional or three-dimensional viscous fluid flow. The classical problems of two-dimensional stagnation point flow of a viscous fluid over a rigid surface were examined by Hiemenz [1] and axisymmetric three-dimensional stagnation point flow over rigid surface was studied by Homann [2]. Hadamard [3] investigated a problem of forward and reverse two-fluid stagnation point flows with limited Reynolds number. This type of flow occurs at the front and rear of a liquid sphere of one fluid in uniform translation through a different quiescent immiscible fluid. Recently, numerous applications of viscoelastic fluids in several industrial manufacturing processes regenerated the interest among researchers to investigate the stagnation point flow of viscoelastic fluid flow over a rigid or stretching sheet. The steady flow of a second-order fluid (viscoelastic) past a stretching sheet was analysed by Rajagopal *et al* [4]. Mahapatra and Gupta [5] considered the steady two-dimensional stagnation-point flow of an incompressible viscoelastic fluid over a flat deformable surface. The heat transfer in the steady laminar flow of an incompressible viscoelastic fluid past a semi-infinite stretching sheet was investigated by Sarma and Nageswara Rao [6]. The steady orthogonal stagnation-point flow of an incompressible viscous fluid on the surface of another heavier incompressible viscous quiescent fluid was investigated by Wang [7]. In this problem, although the pressure is not equalled at the surface, the interface is considered approximately horizontal by gravity.

Later, Wang [8] studied two dynamic stagnation flows that are formed on a flat interface. This problem is associated with transpiration cooling, extirpation cooling, etc. Liu [9] extended Wang's work [8] to investigate the two-dimensional impingement of a light fluid on the surface of a heavier fluid at an arbitrary angle of incidence with no surface distortion. Wang [10] solved for the spatially developing boundary layers produced by uniform shear flow of one lighter fluid over a second heavier quiescent fluid. Tilley and Weidman [11] gave the solution for the impingement of two viscous, immiscible oblique stagnation flows forming a flat interface. They found the response of a quiescent lower fluid to an imposed oblique stagnation-point flow of the upper fluid. However, Wang [10] studied the special case of normal stagnation point flow. An example of such a flow is provided by the continuous spreading of split oil on water. The similarity solution was

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analysed for the unsteady stagnation point flow on the surface of quiescent fluid with or without magnetic field by Surma *et al* [12]. The effect of magnetic field on the stagnation-point flow of an incompressible viscous electrically conducting fluid on the surface of another electrically conducting quiescent fluid was investigated by Reza and Gupta [13]

The objective of this paper is to extend the work studied by Wang [7] to the case when a lighter incompressible viscoelastic fluid impinges orthogonally on the surface of another quiescent heavier incompressible viscous fluid. The boundary conditions are applied at the interface layer of the two fluids, which is assumed to be flat. This condition can be accomplished for small $x$ (i.e., region near the stagnation-point) or large density difference $(\rho_2 \gg \rho_1)$ or when surface tension is large. The velocity distributions in both fluids are found by matching the velocities and tangential stresses in the two fluids at the interface. The energy equations in both the fluids are solved by matching the temperature and heat flux at the interface of the fluids to analyse the temperature distribution. The results on heat transfer, with particular reference to determining the interface temperature, play an important role in controlling the heat transfer in the problems involving powering of viscoelastic fluids on a substrate.

The phenomenon of the frequent accidental spilling of crude oil [14] (non-Newtonian fluid) on the surface of water (viscous fluid) has motivated us to study this problem. In general, crude oils have different rheological properties based on dilution. For example, crude oil [15, 16] has viscoelastic prosperities. The oil spreads more or less on the water surface by balancing the gravity and surface tension. It is interesting to note that spilling of crude oil model on water surface can describe the spreading and vaporization of pools of liquid spilled. The vapour diffusion should be considered, which motivates us to study the heat transfer for this problem to investigate the phenomenon of controlling the vaporization process during spill or after spill. Aims of this work are to study the velocity profile of upper viscoelastic fluid and interface temperature.

## 2. Flow analysis

Consider a viscoelastic, incompressible liquid of density $\rho_1$, viscosity $\mu_1$ that impinges orthogonally on the surface of another quiescent, heavier incompressible viscous Newtonian fluid of density $\rho_2$, viscosity $\mu_2$. Figure 1 shows a sketch of the physical problem. Let the upper light fluid be denoted by the subscript 1 and the lower heavier fluid be denoted by the subscript 2. Let $(x, y_1)$ denote the cartesian coordinates for the upper fluid with $x = 0$ as the symmetry plane and $x$-axis is taken along the interface between the two fluids. The coordinate system for the lower fluid is $(x, y_2)$ as shown in the same figure. It is noted that $z$-axis is normal to the $(x, y_1)$ plane.

The constitutive equation for an incompressible viscoelastic fluid followed by Walters' liquid B model is [17, 18]

$$\tau_{ik} = -p\delta_{ik} + \tau'_{ik} \tag{1}$$

where

$$\tau'_{ik}(x,t) = 2\int_{-\infty}^{t} \Psi(t-t')\frac{\partial x^i}{\partial x'^m}\frac{\partial x^k}{\partial x'^r} \times e^{(1)mr}(x',t')dt'. \tag{2}$$

Further,

$$\Psi(t-t') = \int_0^{\infty} \frac{N(\tau)}{\tau} e^{-(t-t')\tau} d\tau, \tag{3}$$

$N(\tau)$ being the distribution function of relaxation time $\tau$. In these equations, $\tau_{ik}$ is the stress tensor, $p$ an arbitrary pressure, $\delta_{ik}$ is the metric tensor of a convected coordinate system $x^i, x'^i (= x'^i(x,t,t'))$ is the position at time $t'$ of the element, i.e., instantaneously at the point $x^i$ at time $t$, and $e^{(1)}_{ik}$ is the rate-of-strain tensor.

The boundary layer approximation is considered to study the stagnation-point flow of incompressible viscoelastic fluid (Walters' liquid B model) on the surface of another quiescent, heavier incompressible viscous Newtonian fluid. The behaviours of boundary layer flows of viscoelstic fluid are mobile and not highly elastic. They have only a very short (in fact, infinitesimal) part of the history of the deformation gradient that has an influence on the stress. These fluids do not exhibit the phenomenon of stress relaxation, which means that with the instantaneous cessation of all local motion, the stress becomes pure pressure [19].

The equation of state (2) can then be written in the simplified form as

$$\tau'^{ik} = 2\mu e^{(1)ik} - 2k_0 \frac{D}{Dt} e^{(1)ik} \tag{4}$$

where $\mu = \int_0^\infty N(\tau)d\tau$ is the limiting viscosity at small rates of shear, $k_0 = \int_0^\infty \tau N(\tau)d\tau$ and terms involving $\int_0^\infty \tau^n N(\tau)d\tau \quad (n \geq 2)$ have been neglected. Furthermore, $D/Dt$ denotes convected differentiation of a tensor quantity in relation to the material in motion as defined by Oldroyd [20]. For a contravariant tensor $b^{ik}$,

$$\frac{Db^{ik}}{Dt} = \frac{\partial b^{ik}}{\partial t} + v^m \frac{\partial b^{ik}}{\partial x^m} - \frac{\partial v^k}{\partial x^m} b^{im} - \frac{\partial v^i}{\partial x^m} b^{mk} \tag{5}$$

where $v^i$ is the velocity vector and $k_0$ is the elastic constant of the fluid.

The momentum balance equations for steady two-dimensional flow of the upper viscoelastic fluids are

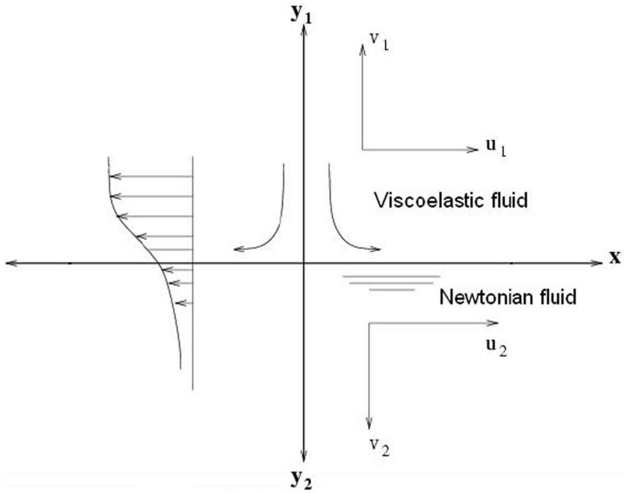


**Figure 1.** Physical sketch of the problem.

$$
\begin{aligned}
u_1\frac{\partial u_1}{\partial x}+v_1\frac{\partial u_1}{\partial y_1} &= -\frac{1}{\rho_1}\frac{\partial p_1}{\partial x}+\nu_1\left(2\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial y_1^2}+\frac{\partial^2 v_1}{\partial x\partial y_1}\right)\\
&-\frac{k_0}{\rho_1}\left[2\frac{\partial}{\partial x}\left(u_1\frac{\partial^2 u_1}{\partial x^2}+v_1\frac{\partial^2 u_1}{\partial x\partial y_1}-2\left(\frac{\partial u_1}{\partial x}\right)^2-\frac{\partial u_1}{\partial y_1}\left(\frac{\partial u_1}{\partial y_1}+\frac{\partial v_1}{\partial x}\right)\right)\right.\\
&+\frac{\partial}{\partial y}\left(u_1\frac{\partial^2 u_1}{\partial x\partial y_1}+u_1\frac{\partial^2 v_1}{\partial x^2}+v_1\frac{\partial^2 u_1}{\partial y_1^2}+v_1\frac{\partial^2 v_1}{\partial x\partial y_1}\right.\\
&\left.\left.-2\left(\frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial x}+\frac{\partial u_1}{\partial y_1}\frac{\partial v_1}{\partial y_1}\right)\right)\right]
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
u_1\frac{\partial v_1}{\partial x}+v_1\frac{\partial v_1}{\partial y_1} &= -\frac{1}{\rho_1}\frac{\partial p_1}{\partial y_1}+\nu_1\left(2\frac{\partial^2 v_1}{\partial y_1^2}+\frac{\partial^2 v_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial x\partial y_1}\right)\\
&-\frac{k_0}{\rho_1}\left[\frac{\partial}{\partial x}\left(u_1\frac{\partial^2 v_1}{\partial x^2}+u_1\frac{\partial^2 u_1}{\partial x\partial y_1}+v_1\frac{\partial^2 v_1}{\partial x\partial y_1}+v_1\frac{\partial^2 u_1}{\partial y_1^2}\right.\right.\\
&\left.-2\left(\frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial x}+\frac{\partial u_1}{\partial y_1}\frac{\partial v_1}{\partial y}\right)\right)\\
&+2\frac{\partial}{\partial y}\left(u_1\frac{\partial^2 v_1}{\partial x\partial y_1}+v_1\frac{\partial^2 v_1}{\partial y_1^2}-2\left(\frac{\partial v_1}{\partial y_1}\right)^2\right.\\
&\left.\left.-\frac{\partial v_1}{\partial x}\left(\frac{\partial u_1}{\partial y}+\frac{\partial v_1}{\partial x}\right)\right)\right]
\end{aligned}
\tag{7}
$$

where $\nu_1=\frac{\mu_1}{\rho_1}$. Here $\rho_1$, $\mu_1$ and $k_0$ represent density, viscosity and viscoelastic parameter of the upper fluid, respectively. The equation of continuity for the upper fluid is

$$\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y_1}=0. \tag{8}$$

Similarly, the momentum equations in the lower immiscible viscous fluid are

$$u_2\frac{\partial u_2}{\partial x}+v_2\frac{\partial u_2}{\partial y_2}=-\frac{1}{\rho_2}\frac{\partial p_2}{\partial x}+\nu_2\left(\frac{\partial^2 u_2}{\partial x_2^2}+\frac{\partial^2 u_2}{\partial y_2^2}\right) \tag{9}$$

$$u_2\frac{\partial v_2}{\partial x}+v_2\frac{\partial v_2}{\partial y_2}=-\frac{1}{\rho_2}\frac{\partial p_2}{\partial y_2}+\nu_2\left(\frac{\partial^2 v_2}{\partial x_2^2}+\frac{\partial^2 v_2}{\partial y_2^2}\right) \tag{10}$$

where $\nu_2$ is the kinematic viscosity of lower fluid. The equation of continuity for the lower fluid is

$$\frac{\partial u_2}{\partial x}+\frac{\partial v_2}{\partial y_2}=0. \tag{11}$$

Eliminating $p$ from Eqs. (6) and (7)

$$
\begin{aligned}
&u_1\frac{\partial^2 v_1}{\partial x^2}+v_1\frac{\partial^2 v_1}{\partial x\partial y}-u_1\frac{\partial^2 u_1}{\partial x\partial y}+v_1\frac{\partial^2 u_1}{\partial y^2}\\
&=\nu\left(\frac{\partial^3 v_1}{\partial x^3}+\frac{\partial^3 v_1}{\partial x\partial y^2}-\frac{\partial^3 u_1}{\partial x^2\partial y}-\frac{\partial^3 u_1}{\partial y^3}\right)\\
&\quad-\frac{k_0}{\rho_1}\left[\frac{\partial}{\partial x^2}\left(u_1\frac{\partial^2 v_1}{\partial x^2}+u_1\frac{\partial^2 u_1}{\partial x\partial y_1}-2\left(\frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial x}+\frac{\partial u_1}{\partial y_1}\frac{\partial v_1}{\partial y}\right)\right.\right.\\
&\quad\left.+v_1\frac{\partial^2 v_1}{\partial x\partial y_1}+v_1\frac{\partial^2 u_1}{\partial y_1^2}\right)\\
&\quad+2\frac{\partial}{\partial x\partial y_1}\left(u_1\frac{\partial^2 v_1}{\partial x\partial y_1}+v_1\frac{\partial^2 v_1}{\partial y_1^2}+\left(\frac{\partial u_1}{\partial y_1}\right)^2-\left(\frac{\partial v_1}{\partial x_1}\right)^2\right.\\
&\quad\left.-u_1\frac{\partial^2 u_1}{\partial x^2}-v_1\frac{\partial^2 u_1}{\partial x\partial y}\right)-\frac{\partial^2}{\partial y^2}\\
&\quad\left(u_1\frac{\partial^2 u_1}{\partial x\partial y_1}-2\left(\frac{\partial u_1}{\partial y_1}\frac{\partial v_1}{\partial x}+\frac{\partial u_1}{\partial y_1}\frac{\partial v_1}{\partial y_1}\right)\right.\\
&\quad\left.\left.+u_1\frac{\partial^2 v_1}{\partial x^2}+v_1\frac{\partial^2 u}{\partial y_1^2}+v_1\frac{\partial v_1}{\partial x\partial y_1}\right)\right].
\end{aligned}
\tag{12}
$$

The boundary conditions for this problem are

$$u_1\to U_1(x),\ \ v_1\to V_1(y_1)\ \text{as}\ y_1\to\infty \tag{13}$$

$$u_1(x,0)=u_2(x,0) \tag{14}$$

$$u_2\to U_2(x),\ \ v_2\to V_2(y_2)\ \text{as}\ y_2\to\infty. \tag{15}$$

The potential, stagnation point flow of the lighter fluid (upper fluid) is described by

$$U_1(x)=ax,\quad V_1(y_1)=-ay_1\ \text{as}\ y_1\to\infty \tag{16}$$

where $a>0$ is a constant. Since the lower (heavier) fluid is at rest at infinity, it should be stated that the horizontal velocity tends to zero and the vertical velocity tends to a constant since the fluid spreads out near the interface and must be replenished. Hence, we must have

$$U_2(x)=0,\quad V_2(y_2)=\text{constant as}\ y_2\to\infty. \tag{17}$$

To apply the boundary conditions at the interface of two fluids, it is considered that the interface is flat. This assumption is made for small $x$ (i.e., region near the stagnation point) or large density differences $(\rho_1 \ll \rho_2)$ or when surface tension is large.

For the upper lighter fluid, we consider

$$u_1 = axf'(\eta), \quad v_1 = -\sqrt{v_1 a} f(\eta), \quad \eta = \frac{y_1}{\sqrt{(v_1/a)}} \tag{18}$$

where $v_1$ is the kinematic viscosity and a prime denote derivative with respect to $\eta$. Clearly, the equation of continuity is satisfied with $u_1$ and $v_1$ given by Eq. (18).

Similarly, for the lower Newtonian fluid (heavier fluid), we consider

$$u_2 = a\beta x h'(\xi), \quad v_2 = -\sqrt{v_2 a\beta} h(\xi), \quad \xi = \frac{y_2}{\sqrt{(v_2/a\beta)}} \tag{19}$$

where the constant $\beta$ is interpreted as the lateral motion of the interface. The value of $\beta$ will be found out and apparently its values range from zero (at a solid boundary) to one (at stress-free boundary). It is found that the equation of continuity for lower fluid is satisfied for the given $u_2$ and $v_2$.

Using (18), the momentum equation for upper fluid (12) reduces to

$$f'^2 - ff'' = 1 + f''' - k(2f'f''' - f''^2 - ff^{iv}) \tag{20}$$

where $k$ is the positive dimensionless viscoelastic parameter for upper fluid given by $k = \frac{k_0 a}{v_1}$. The boundary conditions for upper fluid become

$$f(0) = 0, \quad f'(0) = \beta, \quad f'(\infty) = 1. \tag{21}$$

It may be seen from (20) that the presence of elasticity in the fluid yields a fourth order differential equation, whereas in the viscous case $k = 0$, the order of the equation is three. It would thus appear that an additional boundary condition has to be imposed to obtain the solution. However, implicit in the derivation of (20) is the neglect of the terms of order $k^2$. Therefore we seek a solution of (20) in the form

$$f = f_0(\eta) + kf_1(\eta) + O(k^2) \tag{22}$$

valid for sufficiently small $k$.

Substituting (22) in (20) and equating coefficients of $k^0$ and $k$, we get

$$f_0'^2 - f_0 f_0'' = 1 + f_0''' \tag{23}$$

$$f_1''' + f_0 f_1'' - 2f_0' f_1' + f_0'' f_1 = 2f_0' f_0''' - f_0''^2 - f_0 f_0^{iv} \tag{24}$$

and the boundary conditions become

$$f_0(0) = 0, \quad f_0'(0) = \beta, \quad f_0'(\infty) = 1 \tag{25}$$

$$f_1(0) = 0, \quad f_0'(0) = 0, \quad f_1'(\infty) = 0. \tag{26}$$

Since the flow decays to zero as $y_2 \to \infty$, using (19), the Navier–Stokes equations (9) and (10) for the lower immiscible fluid reduce to

$$h''' + hh'' - h'^2 = 0. \tag{27}$$

It is noticed that the velocities must be equal at the interface; the boundary conditions for the lower fluid are

$$h(0) = 0, \quad h'(0) = 1, \quad h'(\infty) = 0. \tag{28}$$

The function $h(\xi)$ is independent of $\beta$. The solution of (27) subject to the the boundary conditions (28) is given by

$$h(\xi) = 1 - e^{-\xi}. \tag{29}$$

The value of the vertical velocity of the lower Newtonian fluid (heavier fluid) at infinity is derived using Eqs. (17), (19) and (29) as

$$u_2(y_2) = -\sqrt{v_2 \beta a} \quad \text{as} \quad y_2 \to \infty. \tag{30}$$

It is observed that it is not an arbitrary constant. It depends on the physical parameters $v_2$, $\beta$ and $a$. This is physically plausible as the vertical velocity of the lower Newtonian fluid (heavier fluid) depends on kinematic viscosity ($v_2$, lateral motion of the interface ($\beta$) and the straining motion of the upper fluid ($a$).

Further, the tangential stresses of the upper and lower fluid are continuous at the interface. It gives

$$\rho_1 v_1 \frac{\partial u_1}{\partial y_1}(0) = -\rho_2 v_2 \frac{\partial u_2}{\partial y_2}(0). \tag{31}$$

This yields

$$\frac{f''(0)}{-\beta^{3/2} h''(0)} = \frac{\rho_2}{\rho_1}\left(\frac{v_2}{v_1}\right)^{1/2} \equiv R(\text{say}). \tag{32}$$

Using (29), we get from (32) that

$$f''(0) = R\beta^{3/2}. \tag{33}$$

This equation is used to determine $\beta$, which depends on the visco-elastic parameter $k$.

## 3. Heat transfer

Heat transfer is important, particularly when there is forced convective heat transfer. Suppose the temperature at $y_1 \to \infty$ is constant temperature, $T_{1\infty}$ (say), and the temperature at $y_2 \to \infty$ is also constant temperature, $T_{2\infty}$(say).

The energy equations for both fluids without considering the viscous dissipation are given by

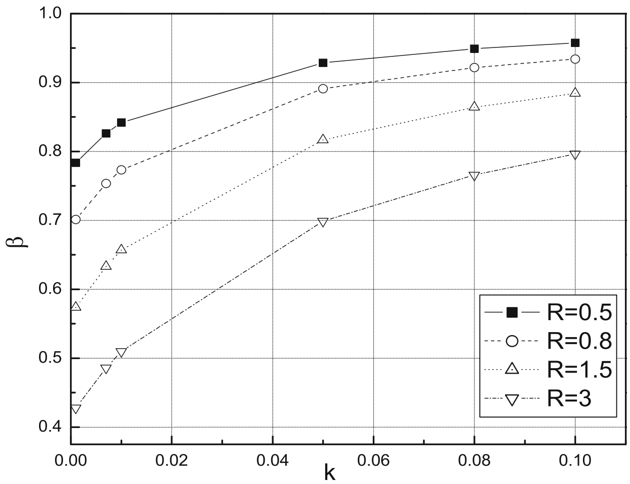


**Figure 2.** The values of $\beta$ with viscoelastic parameter $k$ for several values of $R$.

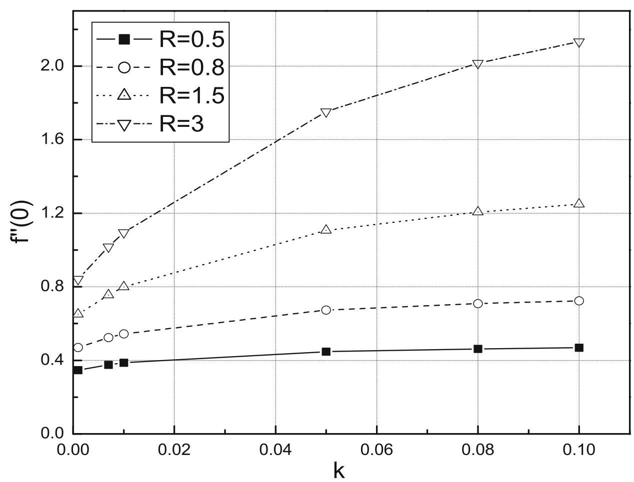


**Figure 3.** Variation of $f''(0)$ with viscoelastic parameter $k$ for several values of $R$.

$$u_i\frac{\partial T_i}{\partial x}+v_i\frac{\partial T_i}{\partial y_i}=\lambda_i\left(\frac{\partial^2 T_i}{\partial x^2}+\frac{\partial^2 T_i}{\partial y_i^2}\right) \tag{34}$$

where $T_i$ and $\lambda_i$ for $i=1,2$ denote the temperature and thermal diffusivity of upper and lower fluid, respectively.

Considering the similarity solution of the temperature distribution in the upper viscoelastic fluid and lower Newtonian fluid, the dimensionless temperatures $\theta_1(\eta)$ and $\theta_2(\xi)$ are taken as

$$\theta_1(\eta)=\frac{T_1(x,y_1)-T_{1\infty}}{T_0-T_{1\infty}},\quad \theta_2(\xi)=\frac{T_1(x,y_2)-T_{1\infty}}{T_0-T_{2\infty}} \tag{35}$$

where $T_0$ is unknown constant temperature of the interface. Using Eq. (35), the energy equations (34) reduce to

$$\theta_1''(\eta)+P_1 f(\eta)\theta_1'(\eta)=0 \tag{36}$$

$$\theta_2''(\xi)+P_2 h(\xi)\theta_2'(\xi)=0 \tag{37}$$

where $P_1=\frac{\nu_1}{\lambda_1}$ and $P_2=\frac{\nu_2}{\lambda_2}$ are the Prandtl number for upper viscoelastic fluid and for lower fluid, respectively. The boundary conditions are

$$\theta_1(0)=1\quad \theta_1(\infty)=0 \tag{38}$$

$$\theta_2(0)=1\qquad \theta_2(\infty)=0. \tag{39}$$

Using the boundary condition (39) and (29) and integrating (37), we obtain

$$\theta_2(\eta)=1+C_1\int_0^{\xi}e^{-(e^{-s}+s)}ds \tag{40}$$

where

$$C_1=1\Big/\left(\int_0^{\xi}e^{-(e^{-s}+s)}ds\right). \tag{41}$$

In this problem, heat flux is continuous at the interface. To determine the interface temperature, we can write (see Landau and Lifshitz [21])

$$\lambda_1\frac{\partial T_1}{\partial y_1}(0)=-\lambda_2\frac{\partial T_2}{\partial y_2}(0). \tag{42}$$

It is assumed the there is no heat source or contact resistance on the interface. Since $\lambda_1=\kappa_1\rho_1(c_p)_1$ and $\lambda_2=\kappa_2\rho_2(c_p)_2$ are the specific heat (at constant pressure) of the upper and lower fluid, respectively, the dimensionless interface temperature can be found from (43) using (35):

$$\widehat{T}=\left(\frac{1}{1+A}\right)+T_r\left(\frac{A}{1+A}\right). \tag{43}$$

Here

$$\widehat{T}=\frac{T_0}{T_{2\infty}},\,T_r=\frac{T_{1\infty}}{T_{2\infty}}, \tag{44}$$

and

$$A=\beta^{-1/2}\frac{\rho_2}{\rho_1}\frac{1}{R}\frac{P_2}{P_1}\frac{\theta_1'(0)}{\theta_2'(0)}. \tag{45}$$

## 4. Numerical results and discussion

Equations (23) and (24) subject to the boundary conditions (25) and (26) are solved numerically to analyse the $f'(\eta)$ and $f(\eta)$ by a finite-difference method for several values of $R$ and viscoelastic parameter $k$.

To discretize Eqs. (23) and (24), we used a central-difference scheme as follows:

$$(V_\eta)_i = \frac{V_{i+1} - V_{i-1}}{2\delta\eta} + \mathrm{O}\Big((\delta\eta)^2\Big), \quad (46)$$

$$(V_{\eta\eta})_i = \frac{V_{i+1} - 2V_i + V_{i-1}}{(\delta\eta)^2} + \mathrm{O}\Big((\delta\eta)^2\Big), \quad (47)$$

where $V$ stands for $G$ or $g$, $i$ is the grid-index in the $\eta$-direction with

$$\eta_i = i\delta\eta, \quad i = 0, 1, 2, ...,$$

$\delta\eta$ being the increment along $\eta$-axis. We use Newton's linearization method to linearize the discretized equations as follows. We assumed that the values of the dependent variables at the $k$th iteration are known. Then the values of these variables at the next iteration are obtained from the following equation:

$$V_i^{k+1} = V_i^k + (\varDelta V)_i^k, \quad (48)$$

where $V$ stands for $f_0'$ or $f_1'$ and $(\Delta V)_i^k$ represents the error at the $k$th iteration, $i = 0, 1, 2, ...$ Using (48) in (23) and (24) and dropping terms quadratic in $(\triangle V)_i^k$, we get a system of linear algebraic equations for $(\triangle V)_i^k$. The resulting system of tri-diagonal equations is solve by the Thomas Algorithm [22]. In view of the asymptotic analysis representing the exponential decay of the relevant flow variables at large distances from the interface, it is found that the boundary conditions at infinity are effectively satisfied at $\eta \sim 2$.

Figure 2 depicts the variation of R with $\beta$ for several values of viscoelastic parameter of the upper fluid when density of the two fluids $\rho_1/\rho_2$ is considered constant. It is interesting to note that the lateral velocity $\beta$ increases with increasing visco-elastic parameter $k$ for fixed value of $R$. Again, it is observed from this figure that the values of $\beta$ increase for fixed values of viscoelastic parameter $k$ and constant values of $\frac{\rho_1}{\rho_2}$ with increasing viscosity ratio $\frac{\nu_1}{\nu_2}$ ( i.e., as R decreases). The variation of $f''(0)$ with viscoelastic parameter for several values of R is displayed in figure 3. It is seen that for a given value of $R$, $f''(0)$ increases monotonically with increase in the viscoelastic parameter $k$. It is observed that the values of $f''(0)$ increase with increasing viscoelastic parameter $R$ for fixed values of $k$. It is very interesting to know the effect of $R$ on shear stress at the interface, which controls the lateral motion of the interface. Figure 4 shows the variation of $f'(\eta)$ with $\eta$ for several values of $\beta$ for fixed value of viscoelastic parameter $k = 0.005$ of the upper fluid. On the other hand, figure 5 displays the $f'(\eta)$ with $\eta$ for several values of viscoelastic parameter $k$ for a fixed value of $\beta = 0.5$. It is observed that $f'(\eta)$ increases with increasing value of lateral velocity $\beta$ for fixed values of $k$ since the lateral motion at the interface increases on increasing the parameter $\beta$. Here, the momentum that is diffused away from the interface leads to increasing velocity of upper fluid due to increasing $\beta$. It is observed that $\beta$ increases with increasing viscolesatic

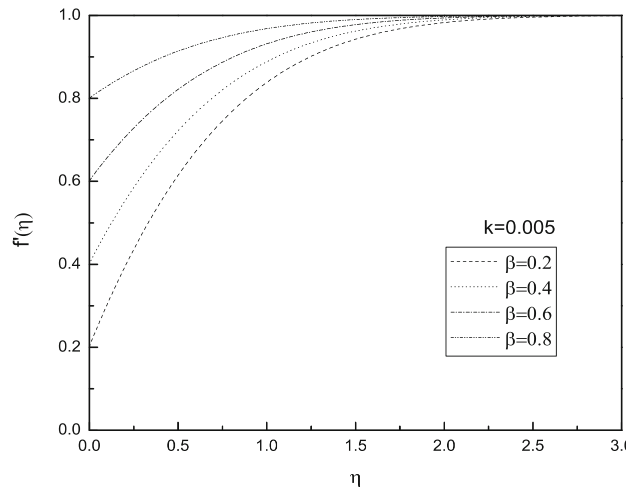


**Figure 4.** Variation of $f'(\eta)$ with $\eta$ for several values of $\beta$ and a fixed value of $k = 0.005$.

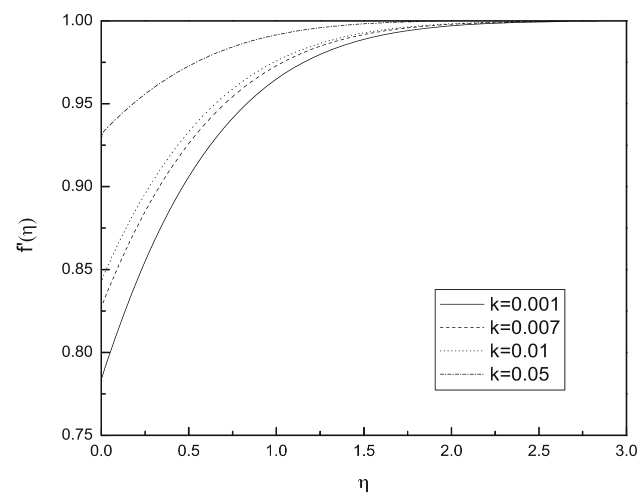


**Figure 5.** Variation of $f'(\eta)$ with $\eta$ for different values of $k$ and a fixed value of $\beta = 0.5$.

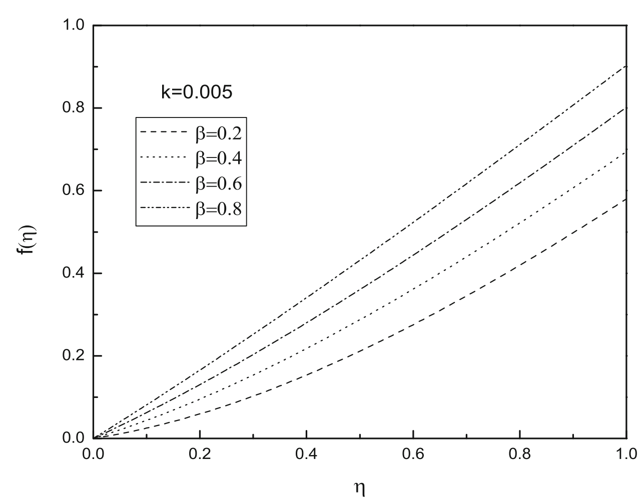


**Figure 6.** Variation of $f(\eta)$ with $\eta$ for different values of $\beta$ and a fixed value of $k = 0.005$.

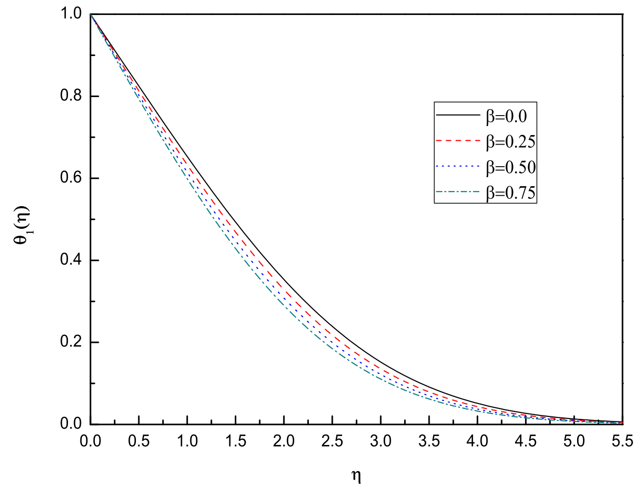


**Figure 7.** Variation of $\theta_1(\eta)$ with $\eta$ for several values of $\beta$ with $P_1 = 0.3$ when $k = 0.01$.

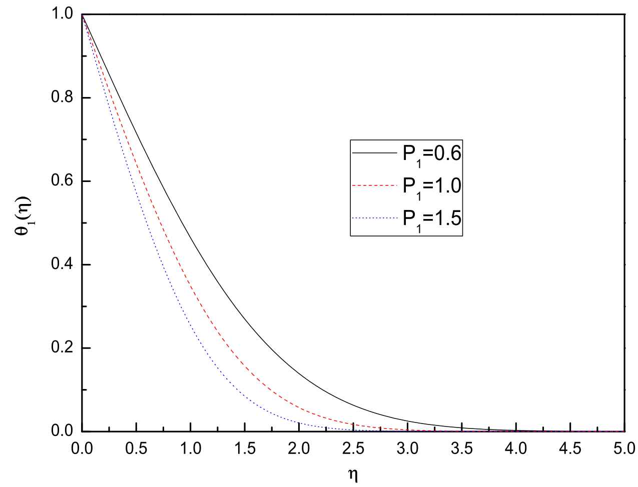


**Figure 9.** Variation of $\theta_1(\eta)$ with $\eta$ for several values of $P_1$ with $R = 3.0$ and $P_1 = 0.3$.

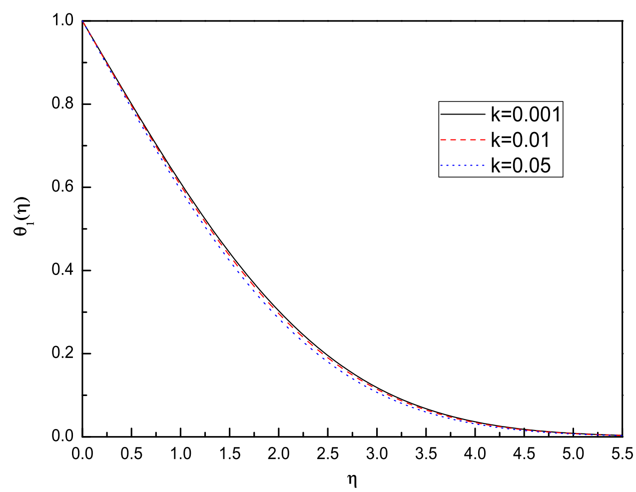


**Figure 8.** Variation of $\theta_1(\eta)$ with $\eta$ for several values of $k$ with $R = 1.5$ and $P_1 = 0.3$.

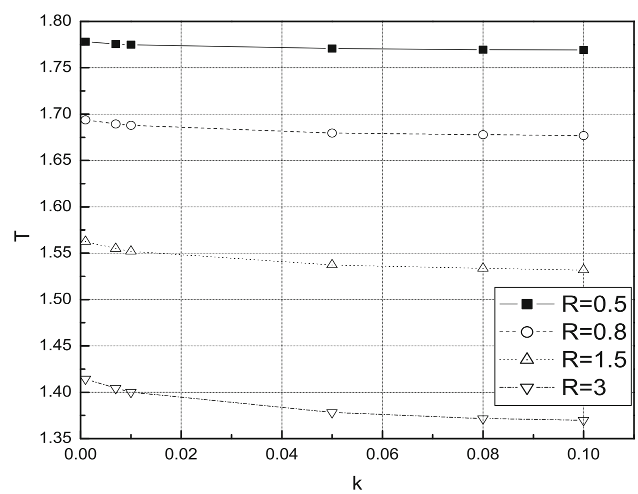


**Figure 10.** Variation of interface temperature $\widehat{T}$ with $k$ for several values $R$ when $\frac{\rho_1}{\rho_2} = 2/3$, $P_1 = 0.4$, $P_2 = 0.8$ and $\frac{(c_p)_1}{(c_p)_2} = 2$.

parameter $k$. This reflects the fact that upper fluid velocity increases with increasing $k$ (see figure 5).

The variation of $f(\eta)$ with $\eta$ for several values of $\beta$ is shown in figure 6 when viscoelastic parameter $k = 0.005$. We can interpret the observation from figure 6 that when the lateral motion at the interface increases, the vertical component of the upper fluid velocity increases due to the diffusion of momentum from the interface.

Using known values of $f(\eta)$ from (22), the numerical solution for $\theta_1(\eta)$ from (36) subject to the boundary condition (38) has been derived by a finite-difference method. Thus, the dimensionless interface temperature $\widehat{T}$ is determined using the known temperature distributions $\theta_1(\eta)$ and $\theta_2(\eta)$. Figure 7 shows the variation of $\theta_1(\eta)$ with $\eta$ for several values of $\beta$ and a fixed value of $k = 0.01$. It is noticed that the temperature of a fixed point in the upper viscoelastic fluid decreases with increasing lateral interface velocity $\beta$ for a fixed value of $k = 0.01$. and $P_1 = 0.3$. Physically this follows from the fact that conduction heat is circulated away with the fluid. Since, at the fixed point, the velocity increases with increasing lateral interface velocity ($\beta$), more heat is circulated away by the fluid than by conduction, resulting in a decrease in temperature with increase in the lateral interface velocity.

Figure 8 shows the variation of $\theta_1(\eta)$with $\eta$ for several values of viscoelastic parameter $k$ and a constant value of $R = 1.5$ and Prandtl number $P_1 = 0.3$. It is noticed that the temperature at a fixed point $\eta$ decreases with increase in the value of $k$. It is also observed that the temperature decreases with increase of the Prandtl number for a fixed value of $R = 3.0$ and $k = 0.005$ (see figure 9). The variation of interface temperature $\widehat{T}$ with viscoelastic parameter is displayed in figure 10 for several values of $R$ and fixed values $\frac{\rho_1}{\rho_2} = 2/3$, $P_1 = 0.3$, $P_2 = 0.8$, $\frac{(c_p)_1}{(c_p)_2} = 2$ and $T_r = 2.0$. It is interesting to note that interface temperature decreases with an increase the viscoelastic parameter $k$ for a fixed value of $R$. It is also observed that temperature decreases with an increase of $R$ for a fixed value of the viscoelastic parameter $k$.

## 5. Conclusions

A viscoelastic fluid impinges downward on another heavier quiescent incompressible viscous fluid. The governing momentum and energy equations of this problem are reduced to a set of nonlinear ordinary differential equations using suitable similarity transformation equations. Numerical solutions of these equations are obtained by a finite-difference method for upper viscoelastic fluid. On the other hand, an analytical solution is found for the lower viscous fluid. It is noticed that for given values of the density ratio and viscosity ratio of the two fluids, the velocity of the upper viscoelastic fluid increases with increasing viscoelastic parameter. It is also interesting to note that lateral velocity $\beta$ at the interface increases with increasing viscoelastic parameter. The convective heat transfer is analysed based on the similarity solutions for the temperature distribution in the upper viscoelastic fluid and lower viscous fluid. It is found that the interface temperature increases with increasing viscoelastic parameter.

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