On the Radical Center of Four Spheres*

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We provide two families of vector formulas that determine the radical center of four spheres. As some applications, we show formulas for points in similar situations to those of the Monge point, provide new geometric results for radical centers, and we give a new proof of a conjecture by Victor Thébault from 1953.

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Introduction

In this article, we provide two families of vector formulas that determine the radical center of four spheres. As some applications, we show formulas for points in similar situations to those of the Monge point, provide new geometric results for radical centers, and we give a new proof of a conjecture by Victor Thébault from 1953 [1, 2].

Throughout this article, the authors use the following terms and notations. The Euclidean vector connecting an initial point P with a terminal point Q (in the three-dimensional Euclidean affine space $\mathbb{E}^3$) is denoted by $\overrightarrow{PQ} = -P + Q$. Zero vector in $\mathbb{E}^3$ is denoted by $\overrightarrow{0}$. The notation $\overrightarrow{AB} \cdot \overrightarrow{CD}$ denotes the dot product of two Euclidean vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$. The notation $\overrightarrow{AB} \times \overrightarrow{CD}$ denotes the cross product of two Euclidean vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

To avoid ambiguities, the following definitions are given: Let $\mathcal{T}$ be a tetrahedron in the Euclidean space $\mathbb{E}^3$. A sphere that touches four faces of the tetrahedron $\mathcal{T}$ is called the *insphere* of $\mathcal{T}$. Every tetrahedron has a special point known as the *Monge point*, which is the intersection of the six planes that are perpendicular to a given edge and pass through the midpoint of the opposite edge.

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The power of a point P with respect to a sphere ω with center O and radius R is the number $Pow(\omega, P) = OP^2 - R^2$.

Let ω_1, ω_2, ω_3, and ω_4 be spheres with non-coplanar centers in $\mathbb{E}^3$. There exists a unique point that has the same power with respect to each of these spheres. This point is called the *radical center* of the spheres $\omega_1, \omega_2, \omega_3$, and ω_4. Readers can consult the references for these topics [3–8].

1. Results

Theorem 1. *Let ω_a, ω_b, ω_c, ω_d be four spheres with radii r_a, r_b, r_c, r_d respectively, and whose four centers A, B, C, D form a tetrahedron $\mathcal{T} \equiv ABCD \subset \mathbb{E}^3$, where $\mathbb{E}^3$ is the Euclidean affine space. Let R be the radical center of these four spheres. Let O be any point in space. Then*

$$\begin{aligned}\overrightarrow{OR} = \frac{1}{2\overrightarrow{AB}\cdot\left(\overrightarrow{BC}\times\overrightarrow{CD}\right)} &\left(\left(\lambda_a\overrightarrow{OA} + \mu_b\overrightarrow{OB}\right)\cdot\overrightarrow{AB}\left(\overrightarrow{BC}\times\overrightarrow{CD}\right)\right. \\ &+ \left(\lambda_b\overrightarrow{OB} + \mu_c\overrightarrow{OC}\right)\cdot\overrightarrow{BC}\left(\overrightarrow{CD}\times\overrightarrow{AB}\right) \\ &\left.+ \left(\lambda_c\overrightarrow{OC} + \mu_d\overrightarrow{OD}\right)\cdot\overrightarrow{CD}\left(\overrightarrow{AB}\times\overrightarrow{BC}\right)\right), \qquad (1)\end{aligned}$$

where

$$\lambda_a = \frac{r_b^2-r_a^2+\left\|\overrightarrow{AB}\right\|^2}{\left\|\overrightarrow{AB}\right\|^2}, \mu_b = \frac{r_a^2-r_b^2+\left\|\overrightarrow{AB}\right\|^2}{\left\|\overrightarrow{AB}\right\|^2}, \lambda_b = \frac{r_c^2-r_b^2+\left\|\overrightarrow{BC}\right\|^2}{\left\|\overrightarrow{BC}\right\|^2},$$
$$\mu_c = \frac{r_b^2-r_c^2+\left\|\overrightarrow{BC}\right\|^2}{\left\|\overrightarrow{BC}\right\|^2}, \lambda_c = \frac{r_d^2-r_c^2+\left\|\overrightarrow{CD}\right\|^2}{\left\|\overrightarrow{CD}\right\|^2}, \mu_d = \frac{r_c^2-r_d^2+\left\|\overrightarrow{CD}\right\|^2}{\left\|\overrightarrow{CD}\right\|^2}.$$

Note that $2\overrightarrow{AB}\cdot\left(\overrightarrow{BC}\times\overrightarrow{CD}\right) = 12V$, where V is the signed volume of the tetrahedron $\mathcal{T} \equiv ABCD$.

Proof. Let σ_{ab} be the radical plane of the pair of spheres $\{\omega_a, \omega_b\}$. Similarly, let σ_{bc} be the radical plane of the pair of spheres of spheres $\{\omega_b, \omega_c\}$, and let σ_{cd} be the radical plane of the pair of spheres of spheres $\{\omega_c, \omega_d\}$. Let $M_{ab} = \sigma_{ab} \cap AB$, $M_{bc} = \sigma_{bc} \cap BC$

and $M_{cd} = \sigma_{cd} \cap CD$ be the intersection points. Because of the perpendicularity of the radical planes, we have: $\overrightarrow{RM}_{ab} \cdot \overrightarrow{AB} = 0$, $\overrightarrow{RM}_{bc} \cdot \overrightarrow{BC} = 0$, and $\overrightarrow{RM}_{cd} \cdot \overrightarrow{CD} = 0$.

Since $\overrightarrow{OR} + \overrightarrow{RM}_{ab} = \overrightarrow{OM}_{ab}$, $\overrightarrow{OR} + \overrightarrow{RM}_{bc} = \overrightarrow{OM}_{bc}$ and $\overrightarrow{OR} + \overrightarrow{RM}_{cd} = \overrightarrow{OM}_{cd}$, then: $\overrightarrow{OR} \cdot \overrightarrow{AB} = \overrightarrow{OM}_{ab} \cdot \overrightarrow{AB}$, $\overrightarrow{OR} \cdot \overrightarrow{BC} = \overrightarrow{OM}_{bc} \cdot \overrightarrow{BC}$ and $\overrightarrow{OR} \cdot \overrightarrow{CD} = \overrightarrow{OM}_{cd} \cdot \overrightarrow{CD}$.

Let us take $\mathcal{S} = \left\{O; \overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3\right\}$ an orthonormal reference system on $\mathbb{E}^3$, and let us consider the components of the vectors $\overrightarrow{AB} = \left(\alpha_x, \alpha_y, \alpha_z\right)$, $\overrightarrow{BC} = \left(\beta_x, \beta_y, \beta_z\right)$, $\overrightarrow{CD} = \left(\gamma_x, \gamma_y, \gamma_z\right)$, $\overrightarrow{OR} = \left(\delta_x, \delta_y, \delta_z\right)$ in the base $\left\{\overrightarrow{e}_i\right\}$.Thus, we have the following linear system

$$\begin{cases} \alpha_x\delta_x + \alpha_y\delta_y + \alpha_z\delta_z = \rho_{ab} \\ \beta_x\delta_x + \beta_y\delta_y + \beta_z\delta_z = \rho_{bc} \\ \gamma_x\delta_x + \gamma_y\delta_y + \gamma_z\delta_z = \rho_{cd} \end{cases},$$

where $\rho_{ab} = \overrightarrow{OM}_{ab} \cdot \overrightarrow{AB}$, $\rho_{bc} = \overrightarrow{OM}_{bc} \cdot \overrightarrow{BC}$, $\rho_{cd} = \overrightarrow{OM}_{cd} \cdot \overrightarrow{CD}$.

Using Cramer's rule, the system solution is

$$\delta_x = \frac{\begin{vmatrix} \rho_{ab} & \alpha_y & \alpha_z \\ \rho_{bc} & \beta_y & \beta_z \\ \rho_{cd} & \gamma_y & \gamma_z \end{vmatrix}}{\begin{vmatrix} \alpha_x & \alpha_y & \alpha_z \\ \beta_x & \beta_y & \beta_z \\ \gamma_x & \gamma_y & \gamma_z \end{vmatrix}} =$$

$$\frac{\rho_{ab}\begin{vmatrix} \beta_y & \beta_z \\ \gamma_y & \gamma_z \end{vmatrix} - \rho_{bc}\begin{vmatrix} \alpha_y & \alpha_z \\ \gamma_y & \gamma_z \end{vmatrix} + \rho_{cd}\begin{vmatrix} \alpha_y & \alpha_z \\ \beta_y & \beta_z \end{vmatrix}}{\det_{\overrightarrow{e}_i}\left(\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}\right)},$$

$$\delta_y = \frac{-\rho_{ab}\begin{vmatrix} \beta_x & \beta_z \\ \gamma_x & \gamma_z \end{vmatrix} + \rho_{bc}\begin{vmatrix} \alpha_x & \alpha_z \\ \gamma_x & \gamma_z \end{vmatrix} - \rho_{cd}\begin{vmatrix} \alpha_x & \alpha_z \\ \beta_x & \beta_z \end{vmatrix}}{\det_{\overrightarrow{e}_i}\left(\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}\right)},$$

$$\delta_z = \frac{\rho_{ab}\begin{vmatrix} \beta_x & \beta_y \\ \gamma_x & \gamma_y \end{vmatrix} - \rho_{bc}\begin{vmatrix} \alpha_x & \alpha_y \\ \gamma_x & \gamma_y \end{vmatrix} + \rho_{cd}\begin{vmatrix} \alpha_x & \alpha_y \\ \beta_x & \beta_y \end{vmatrix}}{\det_{\overrightarrow{e}_i}\left(\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}\right)}.$$

Consequently, we have:

$$\overrightarrow{OR} = \frac{\rho_{ab}\left(\overrightarrow{BC}\times\overrightarrow{CD}\right)+\rho_{bc}\left(\overrightarrow{CD}\times\overrightarrow{AB}\right)+\rho_{cd}\left(\overrightarrow{AB}\times\overrightarrow{BC}\right)}{\overrightarrow{AB}\cdot\left(\overrightarrow{BC}\times\overrightarrow{CD}\right)} =$$

$$\frac{\overrightarrow{OM}_{ab}\cdot\overrightarrow{AB}\left(\overrightarrow{BC}\times\overrightarrow{CD}\right)+\overrightarrow{OM}_{bc}\cdot\overrightarrow{BC}\left(\overrightarrow{CD}\times\overrightarrow{AB}\right)+\overrightarrow{OM}_{cd}\cdot\overrightarrow{CD}\left(\overrightarrow{AB}\times\overrightarrow{BC}\right)}{\overrightarrow{AB}\cdot\left(\overrightarrow{BC}\times\overrightarrow{CD}\right)}.$$

Now, if we take the coordinates of $A = \left(a_x, a_y, a_z\right)$ and $B = \left(b_x, b_y, b_z\right)$ in $\mathcal{S}$, the equation of the radical plane σ_{ab} of the pair of spheres $\{\omega_a, \omega_b\}$ is

$$\begin{aligned} &a_x^2 + a_y^2 + a_z^2 - b_x^2 - b_y^2 - b_z^2 - 2xa_x - 2ya_y - 2za_z \\ &+2xb_x + 2yb_y + 2zb_z - r_a^2 + r_b^2 = 0. \end{aligned}$$

Also, $M_{ab} = \lambda\left(a_x, a_y, a_z\right) + \mu\left(b_x, b_y, b_z\right)$ with $\lambda + \mu = 1$. Since $M_{ab} \in \sigma_{ab}$, if we solve the system

$$\begin{cases} a_x^2 + a_y^2 + a_z^2 - b_x^2 - b_y^2 - b_z^2 - 2\left(\lambda a_x + \mu b_x\right) a_x \\ -2\left(\lambda a_y + \mu b_y\right) a_y - 2\left(\lambda a_z + \mu b_z\right) a_z \\ +2\left(\lambda a_x + \mu b_x\right) b_x + 2\left(\lambda a_y + \mu b_y\right) b_y \\ +2\left(\lambda a_z + \mu b_z\right) b_z - r_a^2 + r_b^2 = 0 \\ \lambda + \mu = 1 \end{cases},$$

we obtain:

$$\begin{cases} \lambda = \frac{1}{2}\frac{a_x^2+a_y^2+a_z^2+b_x^2+b_y^2+b_z^2-2a_xb_x-2a_yb_y-2a_zb_z-r_a^2+r_b^2}{a_x^2+a_y^2+a_z^2+b_x^2+b_y^2+b_z^2-2a_xb_x-2a_yb_y-2a_zb_z} = \\ \frac{1}{2}\frac{r_b^2-r_a^2+\left\|\overrightarrow{OA}\right\|^2+\left\|\overrightarrow{OB}\right\|^2-2\overrightarrow{OA}\cdot\overrightarrow{OB}}{\left\|\overrightarrow{OA}\right\|^2+\left\|\overrightarrow{OB}\right\|^2-2\overrightarrow{OA}\cdot\overrightarrow{OB}} = \frac{1}{2}\frac{r_b^2-r_a^2+\left\|\overrightarrow{AB}\right\|^2}{\left\|\overrightarrow{AB}\right\|^2} \\ \mu = \frac{1}{2}\frac{a_x^2+a_y^2+a_z^2+b_x^2+b_y^2+b_z^2-2a_xb_x-2a_yb_y-2a_zb_z+r_a^2-r_b^2}{a_x^2+a_y^2+a_z^2+b_x^2+b_y^2+b_z^2-2a_xb_x-2a_yb_y-2a_zb_z} = \\ \frac{1}{2}\frac{r_a^2-r_b^2+\left\|\overrightarrow{OA}\right\|^2+\left\|\overrightarrow{OB}\right\|^2-2\overrightarrow{OA}\cdot\overrightarrow{OB}}{\left\|\overrightarrow{OA}\right\|^2+\left\|\overrightarrow{OB}\right\|^2-2\overrightarrow{OA}\cdot\overrightarrow{OB}} = \frac{1}{2}\frac{r_a^2-r_b^2+\left\|\overrightarrow{AB}\right\|^2}{\left\|\overrightarrow{AB}\right\|^2} \end{cases}.$$

Therefore

$$\overrightarrow{OM}_{ab} = \lambda\overrightarrow{OA} + \mu\overrightarrow{OB} =$$

$$\frac{1}{2}\frac{r_b^2-r_a^2+\left\|\overrightarrow{AB}\right\|^2}{\left\|\overrightarrow{AB}\right\|^2}\overrightarrow{OA} + \frac{1}{2}\frac{r_a^2-r_b^2+\left\|\overrightarrow{AB}\right\|^2}{\left\|\overrightarrow{AB}\right\|^2}\overrightarrow{OB}.$$

If we repeat the calculation twice cyclically (i.e., changing A, B by B, C and A, B by C, D), we get

$$\overrightarrow{OM}_{bc} = \frac{1}{2}\frac{r_c^2-r_b^2+\left\|\overrightarrow{BC}\right\|^2}{\left\|\overrightarrow{BC}\right\|^2}\overrightarrow{OB} + \frac{1}{2}\frac{r_b^2-r_c^2+\left\|\overrightarrow{BC}\right\|^2}{\left\|\overrightarrow{BC}\right\|^2}\overrightarrow{OC},$$
$$\overrightarrow{OM}_{cd} = \frac{1}{2}\frac{r_d^2-r_c^2+\left\|\overrightarrow{CD}\right\|^2}{\left\|\overrightarrow{CD}\right\|^2}\overrightarrow{OC} + \frac{1}{2}\frac{r_c^2-r_d^2+\left\|\overrightarrow{CD}\right\|^2}{\left\|\overrightarrow{CD}\right\|^2}\overrightarrow{OD}.$$

This completes the proof. □

We show the following numerical example to illustrate the theorem.

Example 1. *Let ω_a, ω_b, ω_c, ω_d be four spheres with radii $r_a = 1$, $r_b = 2$, $r_c = 3$, $r_d = 3$ respectively, and whose four centers $A = (0,0,0)$, $B = (3,3,3)$, $C = (1,2,3)$, $D = (-3,5,1)$. We want to calculate the radical center using the formulas of Theorem 1:*

Classically (without using the new result of Theorem 1, which we consider to be useful in modern geometric computational problems) to find the radical center requires solving the following system:

$$\begin{cases} \left(x^2+y^2+z^2-1^2\right) \\ -\left((x-3)^2+(y-3)^2+(z-3)^2-2^2\right)=0 \\ \left(x^2+y^2+z^2-1^2\right) \\ -\left((x-1)^2+(y-2)^2+(z-3)^2-3^2\right)=0 \\ \left(x^2+y^2+z^2-1^2\right) \\ -\left((x+3)^2+(y-5)^2+(z-1)^2-3^2\right)=0 \end{cases}.$$

The solution of this system gives the coordinates of the radical center $R = \left(\frac{53}{24}, \frac{55}{12}, -\frac{67}{24}\right)$.

On the other hand, with Theorem 1 it is not required to solve the system; in a systematic way with dot products and cross products, the calculation is as follows.

We take $O = (0,0,0)$:

$\lambda_a = \frac{4-1+\|(3,3,3)\|^2}{\|(3,3,3)\|^2} = \frac{10}{9}$, $\mu_b = \frac{1-4+\|(3,3,3)\|^2}{\|(3,3,3)\|^2} = \frac{8}{9}$,

$\lambda_b = \frac{9-4+\|(1,2,3)-(3,3,3)\|^2}{\|(1,2,3)-(3,3,3)\|^2} = 2, \mu_c = \frac{4-9+\|(1,2,3)-(3,3,3)\|^2}{\|(1,2,3)-(3,3,3)\|^2} = 0,$

$\lambda_c = \frac{9-9+\|(-3,5,1)-(1,2,3)\|^2}{\|(-3,5,1)-(1,2,3)\|^2} = 1,$

$\mu_d = \frac{9-9+\|(-3,5,1)-(1,2,3)\|^2}{\|(-3,5,1)-(1,2,3)\|^2} = 1.$

$2\overrightarrow{AB} \cdot \left(\overrightarrow{BC} \times \overrightarrow{CD}\right) = 2(3,3,3) \cdot (((1,2,3) - (3,3,3))$

$\times ((-3,5,1) - (1,2,3))) = -72.$

$\left(\lambda_a \overrightarrow{OA} + \mu_b \overrightarrow{OB}\right) \cdot \overrightarrow{AB} = \left(\frac{10}{9}(0,0,0) + \frac{8}{9}(3,3,3)\right)$

$\cdot(3,3,3) = 24,$

$\left(\lambda_b \overrightarrow{OB} + \mu_c \overrightarrow{OC}\right) \cdot \overrightarrow{BC} = (2(3,3,3) + 0(1,2,3)) \cdot ((1,2,3)$

$-(3,3,3)) = -18,$

$\left(\lambda_c \overrightarrow{OC} + \mu_d \overrightarrow{OD}\right) \cdot \overrightarrow{CD} = (1(1,2,3) + 1(-3,5,1)) \cdot ((-3,5,1)$

$-(1,2,3)) = 21.$

$\left(\overrightarrow{BC} \times \overrightarrow{CD}\right) = (((1,2,3) - (3,3,3)) \times ((-3,5,1)$

$-(1,2,3))) = (2,-4,-10),$

$\left(\overrightarrow{CD} \times \overrightarrow{AB}\right) = (((-3,5,1) - (1,2,3)) \times (3,3,3)) =$

$(15,6,-21),$

$\left(\overrightarrow{AB} \times \overrightarrow{BC}\right) = (((3,3,3)) \times ((1,2,3) - (3,3,3))) =$

$(3,-6,3).$

Finally $\overrightarrow{OR} = \frac{24(2,-4,-10)-18(15,6,-21)+21(3,-6,3)}{-72} =$

$\left(\frac{53}{24}, \frac{55}{12}, -\frac{67}{24}\right).$

Next, the authors provide yet another formula to obtain the radical center.

Theorem 2. *Using the same hypothesis as for Theorem 1, it is also true that*

$$\begin{aligned}\overrightarrow{OR} = \frac{1}{2\overrightarrow{AB}\cdot\left(\overrightarrow{BC}\times\overrightarrow{CD}\right)} &\left(\left(\lambda_c\overrightarrow{OC} + \mu_d\overrightarrow{OD}\right)\cdot\overrightarrow{AB}\left(\overrightarrow{BC}\times\overrightarrow{CD}\right)\right.\\ &+ \left(\lambda_d\overrightarrow{OD} + \mu_a\overrightarrow{OA}\right)\cdot\overrightarrow{BC}\left(\overrightarrow{CD}\times\overrightarrow{AB}\right)\\ &\left.+ \left(\lambda_a\overrightarrow{OA} + \mu_b\overrightarrow{OB}\right)\cdot\overrightarrow{CD}\left(\overrightarrow{AB}\times\overrightarrow{BC}\right)\right), \end{aligned} \tag{2}$$

where

$$\lambda_c = \frac{r_b^2 - r_a^2 + \left\|\overrightarrow{OA}\right\|^2 - \left\|\overrightarrow{OB}\right\|^2 + 2\overrightarrow{OB}\cdot\overrightarrow{OD} - 2\overrightarrow{OA}\cdot\overrightarrow{OD}}{\overrightarrow{OA}\cdot\overrightarrow{OC} - \overrightarrow{OA}\cdot\overrightarrow{OD} + \overrightarrow{OB}\cdot\overrightarrow{OD} - \overrightarrow{OB}\cdot\overrightarrow{OC}},$$

$$\mu_d = \frac{r_a^2 - r_b^2 - \left\|\overrightarrow{OA}\right\|^2 + \left\|\overrightarrow{OB}\right\|^2 - 2\overrightarrow{OB}\cdot\overrightarrow{OC} + 2\overrightarrow{OA}\cdot\overrightarrow{OC}}{\overrightarrow{OA}\cdot\overrightarrow{OC} - \overrightarrow{OA}\cdot\overrightarrow{OD} + \overrightarrow{OB}\cdot\overrightarrow{OD} - \overrightarrow{OB}\cdot\overrightarrow{OC}},$$

$$\lambda_d = \frac{r_c^2 - r_b^2 + \left\|\overrightarrow{OB}\right\|^2 - \left\|\overrightarrow{OC}\right\|^2 + 2\overrightarrow{OC}\cdot\overrightarrow{OA} - 2\overrightarrow{OB}\cdot\overrightarrow{OA}}{\overrightarrow{OB}\cdot\overrightarrow{OD} - \overrightarrow{OB}\cdot\overrightarrow{OA} + \overrightarrow{OC}\cdot\overrightarrow{OA} - \overrightarrow{OC}\cdot\overrightarrow{OD}},$$

$$\mu_a = \frac{r_b^2 - r_c^2 - \left\|\overrightarrow{OB}\right\|^2 + \left\|\overrightarrow{OC}\right\|^2 - 2\overrightarrow{OC}\cdot\overrightarrow{OD} + 2\overrightarrow{OB}\cdot\overrightarrow{OD}}{\overrightarrow{OB}\cdot\overrightarrow{OD} - \overrightarrow{OB}\cdot\overrightarrow{OA} + \overrightarrow{OC}\cdot\overrightarrow{OA} - \overrightarrow{OC}\cdot\overrightarrow{OD}},$$

$$\lambda_a = \frac{r_d^2 - r_c^2 + \left\|\overrightarrow{OC}\right\|^2 - \left\|\overrightarrow{OD}\right\|^2 + 2\overrightarrow{OD}\cdot\overrightarrow{OB} - 2\overrightarrow{OC}\cdot\overrightarrow{OB}}{\overrightarrow{OC}\cdot\overrightarrow{OA} - \overrightarrow{OC}\cdot\overrightarrow{OB} + \overrightarrow{OD}\cdot\overrightarrow{OB} - \overrightarrow{OD}\cdot\overrightarrow{OA}},$$

and

$$\mu_b = \frac{r_c^2 - r_d^2 - \left\|\overrightarrow{OC}\right\|^2 + \left\|\overrightarrow{OD}\right\|^2 - 2\overrightarrow{OD}\cdot\overrightarrow{OA} + 2\overrightarrow{OC}\cdot\overrightarrow{OA}}{\overrightarrow{OC}\cdot\overrightarrow{OA} - \overrightarrow{OC}\cdot\overrightarrow{OB} + \overrightarrow{OD}\cdot\overrightarrow{OB} - \overrightarrow{OD}\cdot\overrightarrow{OA}}.$$

Proof. Let σ_{ab} be the radical plane of the pair of spheres $\{\omega_a, \omega_b\}$. Similarly, let σ_{bc} be the radical plane of the pair of spheres of spheres $\{\omega_b, \omega_c\}$, and let σ_{cd} be the radical plane of the pair of spheres of spheres $\{\omega_c, \omega_d\}$. Let $M_{ab} = \sigma_{ab} \cap CD$, $M_{bc} = \sigma_{bc} \cap DA$ and $M_{cd} = \sigma_{cd} \cap AB$ be the intersection points. Because of the perpendicularity of the radical planes, we have

$$\overrightarrow{RM}_{ab}\cdot\overrightarrow{AB} = 0,\ \overrightarrow{RM}_{bc}\cdot\overrightarrow{BC} = 0,$$

and

$$\overrightarrow{RM}_{cd}\cdot\overrightarrow{CD} = 0.$$

Since

$$\overrightarrow{OR} + \overrightarrow{RM}_{ab} = \overrightarrow{OM}_{ab},\ \overrightarrow{OR} + \overrightarrow{RM}_{bc} = \overrightarrow{OM}_{bc}$$

and

$$\overrightarrow{OR} + \overrightarrow{RM}_{cd} = \overrightarrow{OM}_{cd},$$

then

$$\overrightarrow{OR}\cdot\overrightarrow{AB}=\overrightarrow{OM}_{ab}\cdot\overrightarrow{AB},\ \overrightarrow{OR}\cdot\overrightarrow{BC}=\overrightarrow{OM}_{bc}\cdot\overrightarrow{BC}$$

and

$$\overrightarrow{OR}\cdot\overrightarrow{CD}=\overrightarrow{OM}_{cd}\cdot\overrightarrow{CD}.$$

Let us take $\mathcal{S}=\left\{O;\vec{e}_1,\vec{e}_2,\vec{e}_3\right\}$ an orthonormal reference system on $\mathbb{E}^3$, and let us consider the components of the vectors $\overrightarrow{AB}=\left(\alpha_x,\alpha_y,\alpha_z\right)$, $\overrightarrow{BC}=\left(\beta_x,\beta_y,\beta_z\right)$, $\overrightarrow{CD}=\left(\gamma_x,\gamma_y,\gamma_z\right)$, $\overrightarrow{OR}=\left(\delta_x,\delta_y,\delta_z\right)$ in the base $\left\{\vec{e}_i\right\}$. Thus, we have the following linear system

$$\begin{cases}\alpha_x\delta_x+\alpha_y\delta_y+\alpha_z\delta_z=\rho_{ab}\\ \beta_x\delta_x+\beta_y\delta_y+\beta_z\delta_z=\rho_{bc}\\ \gamma_x\delta_x+\gamma_y\delta_y+\gamma_z\delta_z=\rho_{cd}\end{cases},$$

where $\rho_{ab}=\overrightarrow{OM}_{ab}\cdot\overrightarrow{AB}$, $\rho_{bc}=\overrightarrow{OM}_{bc}\cdot\overrightarrow{BC}$, $\rho_{cd}=\overrightarrow{OM}_{cd}\cdot\overrightarrow{CD}$.

Using the same reasoning as in the proof of the previous Theorem 1 we have $\overrightarrow{OR}$ equal to

$$\frac{\overrightarrow{OM}_{ab}\cdot\overrightarrow{AB}\left(\overrightarrow{BC}\times\overrightarrow{CD}\right)+\overrightarrow{OM}_{bc}\cdot\overrightarrow{BC}\left(\overrightarrow{CD}\times\overrightarrow{AB}\right)+\overrightarrow{OM}_{cd}\cdot\overrightarrow{CD}\left(\overrightarrow{AB}\times\overrightarrow{BC}\right)}{\overrightarrow{AB}\cdot\left(\overrightarrow{BC}\times\overrightarrow{CD}\right)}.$$

Now, if we take the coordinates of $C=\left(c_x,c_y,c_z\right)$ and $D=\left(d_x,d_y,d_z\right)$ in $\mathcal{S}$, the equation of the radical plane σ_{ab} of the pair of spheres $\{\omega_a,\omega_b\}$ is

$$\begin{aligned}&a_x^2+a_y^2+a_z^2-b_x^2-b_y^2-b_z^2-2xa_x-2ya_y-2za_z\\&+2xb_x+2yb_y+2zb_z-r_a^2+r_b^2=0.\end{aligned}$$

Also $M_{ab}=\lambda\left(c_x,c_y,c_z\right)+\mu\left(d_x,d_y,d_z\right)$ with $\lambda+\mu=1$. Since $M_{ab}\in\sigma_{ab}$, if we solve the system

$$\begin{cases}a_x^2+a_y^2+a_z^2-b_x^2-b_y^2-b_z^2-2\left(\lambda c_x+\mu d_x\right)a_x\\ -2\left(\lambda c_y+\mu d_y\right)a_y-2\left(\lambda c_z+\mu d_z\right)a_z+2\left(\lambda c_x+\mu d_x\right)b_x\\ +2\left(\lambda c_y+\mu d_y\right)b_y+2\left(\lambda c_z+\mu d_z\right)b_z-r_a^2+r_b^2=0\\ \lambda+\mu=1\end{cases},$$

we obtain

$$\Delta = d_x b_x - c_x b_x + d_y b_y - c_y b_y - d_y a_y + c_y a_y$$
$$-d_z a_z + c_z a_z + d_z b_z - c_z b_z - a_x d_x + a_x c_x,$$

$$\Phi = r_b^2 - r_a^2 + a_x^2 + a_y^2 + a_z^2 - b_x^2 - b_y^2 - b_z^2$$
$$+2b_x d_x + 2b_y d_y + 2b_z d_z - 2a_x d_x - 2a_y d_y - 2a_z d_z,$$

$$\Psi = -r_a^2 + r_b^2 + a_x^2 + a_y^2 + a_z^2 - b_x^2 - b_y^2 - b_z^2$$
$$+2c_x b_x + 2c_y b_y + 2c_z b_z - 2a_x c_x - 2a_y c_y - 2a_z c_z,$$

$$\begin{cases} \lambda = \frac{\Phi}{2\Delta} = \frac{1}{2}\frac{r_b^2 - r_a^2 + \|\overrightarrow{OA}\|^2 - \|\overrightarrow{OB}\|^2 + 2\overrightarrow{OB}\cdot\overrightarrow{OD} - 2\overrightarrow{OA}\cdot\overrightarrow{OD}}{\overrightarrow{OA}\cdot\overrightarrow{OC} - \overrightarrow{OA}\cdot\overrightarrow{OD} + \overrightarrow{OB}\cdot\overrightarrow{OD} - \overrightarrow{OB}\cdot\overrightarrow{OC}} \\ \mu = -\frac{\Psi}{2\Delta} = \frac{1}{2}\frac{r_a^2 - r_b^2 - \|\overrightarrow{OA}\|^2 + \|\overrightarrow{OB}\|^2 - 2\overrightarrow{OB}\cdot\overrightarrow{OC} + 2\overrightarrow{OA}\cdot\overrightarrow{OC}}{\overrightarrow{OA}\cdot\overrightarrow{OC} - \overrightarrow{OA}\cdot\overrightarrow{OD} + \overrightarrow{OB}\cdot\overrightarrow{OD} - \overrightarrow{OB}\cdot\overrightarrow{OC}} \end{cases}.$$

Therefore

$$\overrightarrow{OM}_{ab} = \lambda\overrightarrow{OC} + \mu\overrightarrow{OD} =$$
$$\frac{1}{2}\frac{r_b^2 - r_a^2 + \|\overrightarrow{OA}\|^2 - \|\overrightarrow{OB}\|^2 + 2\overrightarrow{OB}\cdot\overrightarrow{OD} - 2\overrightarrow{OA}\cdot\overrightarrow{OD}}{\overrightarrow{OA}\cdot\overrightarrow{OC} - \overrightarrow{OA}\cdot\overrightarrow{OD} + \overrightarrow{OB}\cdot\overrightarrow{OD} - \overrightarrow{OB}\cdot\overrightarrow{OC}}\overrightarrow{OC}$$
$$+ \frac{1}{2}\frac{r_a^2 - r_b^2 - \|\overrightarrow{OA}\|^2 + \|\overrightarrow{OB}\|^2 - 2\overrightarrow{OB}\cdot\overrightarrow{OC} + 2\overrightarrow{OA}\cdot\overrightarrow{OC}}{\overrightarrow{OA}\cdot\overrightarrow{OC} - \overrightarrow{OA}\cdot\overrightarrow{OD} + \overrightarrow{OB}\cdot\overrightarrow{OD} - \overrightarrow{OB}\cdot\overrightarrow{OC}}\overrightarrow{OD}.$$

If we repeat the calculation twice cyclically (i.e., changing A, B by B, C and A, B by C, D), we get

$$\overrightarrow{OM}_{bc} = \frac{1}{2}\frac{r_c^2 - r_b^2 + \|\overrightarrow{OB}\|^2 - \|\overrightarrow{OC}\|^2 + 2\overrightarrow{OC}\cdot\overrightarrow{OA} - 2\overrightarrow{OB}\cdot\overrightarrow{OA}}{\overrightarrow{OB}\cdot\overrightarrow{OD} - \overrightarrow{OB}\cdot\overrightarrow{OA} + \overrightarrow{OC}\cdot\overrightarrow{OA} - \overrightarrow{OC}\cdot\overrightarrow{OD}}\overrightarrow{OD}$$
$$+ \frac{1}{2}\frac{r_b^2 - r_c^2 - \|\overrightarrow{OB}\|^2 + \|\overrightarrow{OC}\|^2 - 2\overrightarrow{OC}\cdot\overrightarrow{OD} + 2\overrightarrow{OB}\cdot\overrightarrow{OD}}{\overrightarrow{OB}\cdot\overrightarrow{OD} - \overrightarrow{OB}\cdot\overrightarrow{OA} + \overrightarrow{OC}\cdot\overrightarrow{OA} - \overrightarrow{OC}\cdot\overrightarrow{OD}}\overrightarrow{OA},$$
$$\overrightarrow{OM}_{cd} = \frac{1}{2}\frac{r_d^2 - r_c^2 + \|\overrightarrow{OC}\|^2 - \|\overrightarrow{OD}\|^2 + 2\overrightarrow{OD}\cdot\overrightarrow{OB} - 2\overrightarrow{OC}\cdot\overrightarrow{OB}}{\overrightarrow{OC}\cdot\overrightarrow{OA} - \overrightarrow{OC}\cdot\overrightarrow{OB} + \overrightarrow{OD}\cdot\overrightarrow{OB} - \overrightarrow{OD}\cdot\overrightarrow{OA}}\overrightarrow{OA}$$
$$+ \frac{1}{2}\frac{r_c^2 - r_d^2 - \|\overrightarrow{OC}\|^2 + \|\overrightarrow{OD}\|^2 - 2\overrightarrow{OD}\cdot\overrightarrow{OA} + 2\overrightarrow{OC}\cdot\overrightarrow{OA}}{\overrightarrow{OC}\cdot\overrightarrow{OA} - \overrightarrow{OC}\cdot\overrightarrow{OB} + \overrightarrow{OD}\cdot\overrightarrow{OB} - \overrightarrow{OD}\cdot\overrightarrow{OA}}\overrightarrow{OB}.$$

This completes the proof. □

To illustrate Theorem 2, we recalculate the previous numerical example.

Example 2. *Let ω_a, ω_b, ω_c, ω_d be four spheres with radii $r_a = 1$, $r_b = 2$, $r_c = 3$, $r_d = 3$ respectively, and whose four centers $A = (0,0,0)$, $B = (3,3,3)$, $C = (1,2,3)$, $D = (-3,5,1)$. We want to calculate the radical center using the formulas of Theorem 3:*

We have:

$$\frac{4-1+\|(0,0,0)\|^2-\|(3,3,3)\|^2+2(3,3,3)\cdot(-3,5,1)-2(0,0,0)\cdot(-3,5,1)}{(0,0,0)\cdot(1,2,3)-(0,0,0)\cdot(-3,5,1)+(3,3,3)\cdot(-3,5,1)-(3,3,3)\cdot(1,2,3)} =$$

$\lambda_c = \frac{2}{3}$,

$$\frac{1-4-\|(0,0,0)\|^2+\|(3,3,3)\|^2-2(3,3,3)\cdot(1,2,3)+2(0,0,0)\cdot(1,2,3)}{(0,0,0)\cdot(1,2,3)-(0,0,0)\cdot(-3,5,1)+(3,3,3)\cdot(-3,5,1)-(3,3,3)\cdot(1,2,3)} =$$

$\mu_d = \frac{4}{3}$,

$$\frac{9-4+\|(3,3,3)\|^2-\|(1,2,3)\|^2+2(1,2,3)\cdot(0,0,0)-2(3,3,3)\cdot(0,0,0)}{(3,3,3)\cdot(-3,5,1)-(3,3,3)\cdot(0,0,0)+(1,2,3)\cdot(0,0,0)-(1,2,3)\cdot(-3,5,1)} =$$

$\lambda_d = -18$,

$$\frac{4-9-\|(3,3,3)\|^2+\|(1,2,3)\|^2-2(1,2,3)\cdot(-3,5,1)+2(3,3,3)\cdot(-3,5,1)}{(3,3,3)\cdot(-3,5,1)-(3,3,3)\cdot(0,0,0)+(1,2,3)\cdot(0,0,0)-(1,2,3)\cdot(-3,5,1)} =$$

$\mu_a = 20$,

$$\frac{9-9+\|(1,2,3)\|^2-\|(-3,5,1)\|^2+2(-3,5,1)\cdot(3,3,3)-2(1,2,3)\cdot(3,3,3)}{(1,2,3)\cdot(0,0,0)-(1,2,3)\cdot(3,3,3)+(-3,5,1)\cdot(3,3,3)-(-3,5,1)\cdot(0,0,0)} =$$

$\lambda_a = \frac{13}{3}$,

$$\frac{9-9-\|(1,2,3)\|^2+\|(-3,5,1)\|^2-2(-3,5,1)\cdot(0,0,0)+2(1,2,3)\cdot(0,0,0)}{(1,2,3)\cdot(0,0,0)-(1,2,3)\cdot(3,3,3)+(-3,5,1)\cdot(3,3,3)-(-3,5,1)\cdot(0,0,0)} =$$

$\mu_b = -\frac{7}{3}$.

$\left(\lambda_c\overrightarrow{OC} + \mu_d\overrightarrow{OD}\right)\cdot\overrightarrow{AB} = \left(\frac{2}{3}(1,2,3) + \frac{4}{3}(-3,5,1)\right)$

$\cdot(3,3,3) = 24$,

$\left(\lambda_d\overrightarrow{OD} + \mu_a\overrightarrow{OA}\right)\cdot\overrightarrow{BC} = (-18(-3,5,1) + 200(0,0,0))$

$\cdot((1,2,3) - (3,3,3)) = -18$,

$\left(\lambda_a\overrightarrow{OA} + \mu_b\overrightarrow{OB}\right)\cdot\overrightarrow{CD} = \left(\frac{13}{3}(0,0,0) - \frac{7}{3}(3,3,3)\right)$

$\cdot((-3,5,1) - (1,2,3)) = 21$.

$\left(\overrightarrow{BC}\times\overrightarrow{CD}\right) = (((1,2,3) - (3,3,3)) \times ((-3,5,1)$

$-(1,2,3))) = (2,-4,-10)$,

$\left(\overrightarrow{CD}\times\overrightarrow{AB}\right) = (((-3,5,1) - (1,2,3)) \times (3,3,3)) =$

$(15, 6, -21)$,

$\left(\overrightarrow{AB} \times \overrightarrow{BC}\right) = (((3,3,3)) \times ((1,2,3) - (3,3,3))) =$

$(3, -6, 3)$.

Finally $\overrightarrow{OR} = \frac{24(2,-4,-10)-18(15,6,-21)+21(3,-6,3)}{-72} =$

$\left(\frac{53}{24}, \frac{55}{12}, -\frac{67}{24}\right)$.

Remark 1: Using the same kind of demonstration as before, we can prove the well-known vector formula for the Monge point M of the tetrahedron $\mathcal{T} \equiv ABCD$:

$$\begin{aligned}\overrightarrow{OM} = \frac{1}{2\overrightarrow{AB}\cdot\left(\overrightarrow{BC}\times\overrightarrow{CD}\right)} &\left(\left(\overrightarrow{OC} + \overrightarrow{OD}\right) \cdot \overrightarrow{AB}\left(\overrightarrow{BC} \times \overrightarrow{CD}\right)\right. \\ &+ \left(\overrightarrow{OD} + \overrightarrow{OA}\right) \cdot \overrightarrow{BC}\left(\overrightarrow{CD} \times \overrightarrow{AB}\right) \\ &+ \left.\left(\overrightarrow{OA} + \overrightarrow{OB}\right) \cdot \overrightarrow{CD}\left(\overrightarrow{AB} \times \overrightarrow{BC}\right)\right). \qquad (3)\end{aligned}$$

Proof of remark 1. First, let us remember that the Monge point M of the tetrahedron $\mathcal{T} \equiv ABCD$ is the intersection of the six planes that are perpendicular to a given edge and pass through the midpoint of the opposite edge.

The Monge point M of the tetrahedron $\mathcal{T} \equiv ABCD$ is the intersection of the six planes that are perpendicular to a given edge and pass through the midpoint of the opposite edge.

Therefore, we can use the same proof of Theorem 2 above, but instead of using the planes called σ, now we use the six planes which define the Monge point, and we add the following formulas $\overrightarrow{OM}_{ab} = \lambda\overrightarrow{OC} + \mu\overrightarrow{OD} = \frac{1}{2}\overrightarrow{OC} + \frac{1}{2}\mu\overrightarrow{OD}$, $\overrightarrow{OM}_{bc} = \frac{1}{2}\overrightarrow{OD} + \frac{1}{2}\overrightarrow{OA}$, $\overrightarrow{OM}_{cd} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OB}$.

This completes the proof. □

Remark 2: Let P be the intersection of three orthogonal planes to the edges AB, BC and CD of a tetrahedron $\mathcal{T} \equiv ABCD$, which planes cut across the edges C^*D^*, D^*A^* and A^*B^*, respectively, of another tetrahedron $\mathcal{T}^* \equiv A^*B^*C^*D^*$. As a result of the proofs of Theorem 1 and Theorem 2 above, the following equation is true:

$$\overrightarrow{OP} = \frac{1}{\overrightarrow{AB}\cdot\left(\overrightarrow{BC}\times\overrightarrow{CD}\right)}\left(\left(\lambda_c\overrightarrow{OC^*} + \mu_d\overrightarrow{OD^*}\right)\cdot\overrightarrow{AB}\left(\overrightarrow{BC}\times\overrightarrow{CD}\right)\right.$$
$$+\left(\lambda_d\overrightarrow{OD^*} + \mu_a\overrightarrow{OA^*}\right)\cdot\overrightarrow{BC}\left(\overrightarrow{CD}\times\overrightarrow{AB}\right)$$
$$\left.+\left(\lambda_a\overrightarrow{OA^*} + \mu_b\overrightarrow{OB^*}\right)\cdot\overrightarrow{CD}\left(\overrightarrow{AB}\times\overrightarrow{BC}\right)\right), \tag{4}$$

where $\lambda_c + \mu_d = \lambda_d + \mu_a = \lambda_a + \mu_b = 1$.

2. Some Applications

The vectorial results shown above will be used to prove a general result which relates the radical center of four spheres with certain elements of the tetrahedron.

Theorem 3. *In a tetrahedron $\mathcal{T} \equiv ABCD$, let A', B', C', D' be the feet of the altitudes AA', BB', CC', DD' respectively. Let ω_a, ω_b, ω_c, ω_d be four spheres having the vertices A, B, C, D as centers respectively. Let A^*, B^*, C^*, D^* be the inverses of A', B', C', D' with respect to the spheres ω_a, ω_b, ω_c, ω_d respectively. The planes which pass through the midpoints of B^*C^*, C^*A^*, A^*B^*, D^*A^*, D^*B^*, D^*C^* and are perpendicular to BC, CA, AB, DA, DB, DC respectively, are concurrent at a point P, which is the radical center R of the spheres ω_a, ω_b, ω_c, ω_d.*

By way of explanation, the segment AA' (which we call *altitude*) is the straight line segment that joins the vertex A with the point A' on the opposite side plane BCD such that the segment AA' is orthogonal to plane BCD. This point A' is called *foot of the altitude*.

By way of explanation, the segment AA' (which we call *altitude*) is the straight line segment that joins the vertex A with the point A' on the opposite side plane BCD such that the segment AA' is orthogonal to plane BCD. This point A' is called *foot of the altitude*.

Remark 3: If Theorem 3 is true, then the planes mentioned in the theorem are necessarily the radical planes of the pairs of those spheres.

Proof. Let point P_{abc} be the intersection of three orthogonal planes to the edges AB, BC and CD of the tetrahedron $\mathcal{T} \equiv ABCD$, which planes pass through the midpoints of the edges A^*B^*, B^*C^* and C^*D^*, respectively, of the tetrahedron $\mathcal{T}^* \equiv A^*B^*C^*D^*$. As a result of Remark 2:

$$\begin{aligned}\overrightarrow{OP}_{abc} = \frac{1}{2\overrightarrow{AB}\cdot\left(\overrightarrow{BC}\times\overrightarrow{CD}\right)}\Big(\left(\overrightarrow{OA^*}+\overrightarrow{OB^*}\right)\cdot\overrightarrow{AB}\left(\overrightarrow{BC}\times\overrightarrow{CD}\right)\\ +\left(\overrightarrow{OB^*}+\overrightarrow{OC^*}\right)\cdot\overrightarrow{BC}\left(\overrightarrow{CD}\times\overrightarrow{AB}\right)\\ +\left(\overrightarrow{OC^*}+\overrightarrow{OD^*}\right)\cdot\overrightarrow{CD}\left(\overrightarrow{AB}\times\overrightarrow{BC}\right)\Big),\end{aligned} \tag{5}$$

Likewise, and because the formulas are cyclical, let P_{bda} the intersection of three orthogonal planes to the edges BD, DA and AC of the tetrahedron $\mathcal{T} \equiv ABCD$, which planes pass through the midpoints of the edges B^*D^*, D^*A^* and A^*C^*, respectively, of the tetrahedron $\mathcal{T}^* \equiv A^*B^*C^*D^*$. As a result of Remark 2:

$$\begin{aligned}\overrightarrow{OP}_{bda} = \frac{1}{2\overrightarrow{BD}\cdot\left(\overrightarrow{DA}\times\overrightarrow{AC}\right)}\Big(\left(\overrightarrow{OB^*}+\overrightarrow{OD^*}\right)\cdot\overrightarrow{BD}\left(\overrightarrow{DA}\times\overrightarrow{AC}\right)\\ +\left(\overrightarrow{OD^*}+\overrightarrow{OA^*}\right)\cdot\overrightarrow{DA}\left(\overrightarrow{AC}\times\overrightarrow{BD}\right)\\ +\left(\overrightarrow{OA^*}+\overrightarrow{OC^*}\right)\cdot\overrightarrow{AC}\left(\overrightarrow{BD}\times\overrightarrow{DA}\right)\Big),\end{aligned} \tag{6}$$

Therefore, in order that $\overrightarrow{OP}_{abc} = \overrightarrow{OP}_{bda} = R$, and owing to Theorem 1, the following equations must be true:

$$\begin{aligned}\left(\overrightarrow{OA^*}+\overrightarrow{OB^*}\right)\cdot\overrightarrow{AB} &= \left(\lambda_a^{abc}\overrightarrow{OA}+\mu_b^{abc}\overrightarrow{OB}\right)\cdot\overrightarrow{AB},\\ \left(\overrightarrow{OB^*}+\overrightarrow{OC^*}\right)\cdot\overrightarrow{BC} &= \left(\lambda_b^{abc}\overrightarrow{OB}+\mu_c^{abc}\overrightarrow{OC}\right)\cdot\overrightarrow{BC},\\ \left(\overrightarrow{OC^*}+\overrightarrow{OD^*}\right)\cdot\overrightarrow{CD} &= \left(\lambda_c^{abc}\overrightarrow{OC}+\mu_d^{abc}\overrightarrow{OD}\right)\cdot\overrightarrow{CD},\end{aligned}$$

where $\lambda_a^{abc} = \frac{r_b^2-r_a^2+\left\|\overrightarrow{AB}\right\|^2}{\left\|\overrightarrow{AB}\right\|^2}$, $\mu_b^{abc} = \frac{r_a^2-r_b^2+\left\|\overrightarrow{AB}\right\|^2}{\left\|\overrightarrow{AB}\right\|^2}$,

$\lambda_b^{abc} = \frac{r_c^2-r_b^2+\left\|\overrightarrow{BC}\right\|^2}{\left\|\overrightarrow{BC}\right\|^2}$, $\mu_c^{abc} = \frac{r_b^2-r_c^2+\left\|\overrightarrow{BC}\right\|^2}{\left\|\overrightarrow{BC}\right\|^2}$,

$\lambda_c^{abc} = \frac{r_d^2-r_c^2+\left\|\overrightarrow{CD}\right\|^2}{\left\|\overrightarrow{CD}\right\|^2}$, $\mu_d^{abc} = \frac{r_c^2-r_d^2+\left\|\overrightarrow{CD}\right\|^2}{\left\|\overrightarrow{CD}\right\|^2}$.

Also, it must be true that

$$\left(\overrightarrow{OB^*}+\overrightarrow{OD^*}\right)\cdot\overrightarrow{BD}=\left(\lambda_b^{bda}\overrightarrow{OB}+\mu_d^{bda}\overrightarrow{OD}\right)\cdot\overrightarrow{BD},$$
$$\left(\overrightarrow{OD^*}+\overrightarrow{OA^*}\right)\cdot\overrightarrow{DA}=\left(\lambda_d^{bda}\overrightarrow{OD}+\mu_a^{bda}\overrightarrow{OA}\right)\cdot\overrightarrow{DA},$$
$$\left(\overrightarrow{OA^*}+\overrightarrow{OC^*}\right)\cdot\overrightarrow{AC}=\left(\lambda_a^{bda}\overrightarrow{OA}+\mu_c^{bda}\overrightarrow{OC}\right)\cdot\overrightarrow{AC},$$

where $\lambda_b^{bda}=\frac{r_d^2-r_b^2+\left\|\overrightarrow{BD}\right\|^2}{\left\|\overrightarrow{BD}\right\|^2}$, $\mu_d^{bda}=\frac{r_b^2-r_d^2+\left\|\overrightarrow{BD}\right\|^2}{\|BD\|^2}$,

$\lambda_d^{bda}=\frac{r_a^2-r_d^2+\left\|\overrightarrow{DA}\right\|^2}{\left\|\overrightarrow{DA}\right\|^2}$, $\mu_a^{bda}=\frac{r_d^2-r_a^2+\left\|\overrightarrow{DA}\right\|^2}{\left\|\overrightarrow{DA}\right\|^2}$,

$\lambda_a^{bda}=\frac{r_c^2-r_a^2+\left\|\overrightarrow{AC}\right\|^2}{\left\|\overrightarrow{AC}\right\|^2}$, $\mu_c^{bda}=\frac{r_a^2-r_c^2+\left\|\overrightarrow{AC}\right\|^2}{\left\|\overrightarrow{AC}\right\|^2}$.

Ultimately, the above is equivalent to the radical planes of the pairs of ω spheres passing through the midpoints of the edges of the tetrahedron $\mathcal{T}^*\equiv A^*B^*C^*D^*$.

We consider a Cartesian system of coordinates such that $O=A=(0,0,0)$, $B=(1,0,0)$, $C=(\alpha,\beta,0)$, $D=(\gamma,\delta,\varepsilon)$ with $\alpha>0$, $\beta>0$ and $\varepsilon>0$. After the necessary calculations, we find that the coordinates of the feet in this system are:

$$A^{'}=\frac{\beta\varepsilon}{M_a}\left(\beta\varepsilon,(1-\alpha)\varepsilon,\delta(\alpha-1)-\beta(\gamma-1)\right),$$
$$B'=\frac{\beta\varepsilon}{M_b+\beta^2\varepsilon^2}\left(\frac{M_b}{\beta\varepsilon},\alpha\varepsilon,\beta\gamma-\alpha\delta\right),$$
$$C'=\frac{\beta\delta}{\delta^2+\varepsilon^2}\left(\frac{\alpha}{\beta\delta}\left(\delta^2+\varepsilon^2\right),\delta,\varepsilon\right),$$
$$D'=(\gamma,\delta,0),$$

where

$$M_a=\left(\alpha^2-2\alpha+1\right)\left(\delta^2+\varepsilon^2\right)+2\beta\delta(\alpha-1)+$$
$$+\beta^2\left(1+\varepsilon^2\right)+\beta\gamma(\beta\gamma-2\alpha\delta+2\delta-2\beta),$$
$$M_b=\alpha^2\left(\delta^2+\varepsilon^2\right)+\beta\gamma(\beta\gamma-2\alpha\delta).$$

Consequently, the inverse points are:

$$A^* = r_a^2 \frac{1}{\beta\varepsilon}\left(\beta\varepsilon, (1-\alpha)\varepsilon, \delta(\alpha-1) - \beta(\gamma-1)\right),$$
$$B^* = r_b^2 \left(-1, \frac{\alpha}{\beta}, \frac{\beta\gamma - \alpha\delta}{\beta\varepsilon}\right) + (1, 0, 0),$$
$$C^* = r_c^2 \left(0, -\frac{1}{\beta}, \frac{\delta}{\beta\varepsilon}\right) + (\alpha, \beta, 0),$$
$$D^* = r_d^2 \left(0, 0, -\frac{1}{\varepsilon}\right) + (\gamma, \delta, \varepsilon).$$

With a long but straightforward calculation we have:

$$\left(\overrightarrow{OA^*} + \overrightarrow{OB^*}\right)\cdot\overrightarrow{AB} = \left(\lambda_a^{abc}\overrightarrow{OA} + \mu_b^{abc}\overrightarrow{OB}\right)\cdot\overrightarrow{AB} =$$
$$r_a^2 - r_b^2 + 1,$$
$$\left(\overrightarrow{OB^*} + \overrightarrow{OC^*}\right)\cdot\overrightarrow{BC} = \left(\lambda_b^{abc}\overrightarrow{OB} + \mu_c^{abc}\overrightarrow{OC}\right)\cdot\overrightarrow{BC} =$$
$$\alpha^2 + \beta^2 + r_b^2 - r_c^2 - 1,$$
$$\left(\overrightarrow{OC^*} + \overrightarrow{OD^*}\right)\cdot\overrightarrow{CD} = \left(\lambda_c^{abc}\overrightarrow{OC} + \mu_d^{abc}\overrightarrow{OD}\right)\cdot\overrightarrow{CD} =$$
$$\delta^2 + \varepsilon^2 + \gamma^2 - \alpha^2 - \beta^2 + r_c^2 - r_d^2,$$
$$\left(\overrightarrow{OB^*} + \overrightarrow{OD^*}\right)\cdot\overrightarrow{BD} = \left(\lambda_b^{bda}\overrightarrow{OB} + \mu_d^{bda}\overrightarrow{OD}\right)\cdot\overrightarrow{BD} =$$
$$\gamma^2 + \delta^2 + \varepsilon^2 + r_b^2 - r_d^2 - 1,$$
$$\left(\overrightarrow{OD^*} + \overrightarrow{OA^*}\right)\cdot\overrightarrow{DA} = \left(\lambda_d^{bda}\overrightarrow{OD} + \mu_a^{bda}\overrightarrow{OA}\right)\cdot\overrightarrow{DA} =$$
$$-\gamma^2 - \delta^2 - \varepsilon^2 + r_d^2 - r_a^2,$$
$$\left(\overrightarrow{OA^*} + \overrightarrow{OC^*}\right)\cdot\overrightarrow{AC} = \left(\lambda_a^{bda}\overrightarrow{OA} + \mu_c^{bda}\overrightarrow{OC}\right)\cdot\overrightarrow{AC} =$$
$$\alpha^2 + \beta^2 + r_a^2 - r_c^2.$$

This completes the proof. □

Corollary 1 (Thébault's conjecture, year 1953 [1]). *In a tetrahedron $\mathcal{T} \equiv ABCD$, let A', B', C', D' be the feet of the altitudes AA', BB', CC', DD'. The planes which pass through the midpoints of $B'C'$, $C'A'$, $A'B'$, $D'A'$, $D'B'$, $D'C'$ and are perpendicular to BC, CA, AB, DA, DB, DC respectively, are concurrent at a point P,* *which is the radical center of the spheres described with the vertices A, B, C, D as centers and with the altitudes AA', BB', CC', DD' as radii.*

This conjecture was proved in [2].

Proof. Following the notation as shown in the above Theorem, we have $A' = A^*$, $B' = B^*$, $C' = C^*$, $D' = D^*$, because $A' \in \omega_a$, $B' \in \omega_b$, $C' \in \omega_c$, $D' \in \omega_d$.

This completes the proof. □

3. Conclusion

We conjecture a theorem similar to the Theorem 3 in the multidimensional space.

Conjecture 1. *Let $\mathcal{A} = A_0A_1...A_n$ be a simplex in the n-dimensional Euclidean affine space $\mathbb{E}^n$. For $i = \overline{0, n}$, let ω_i be the hypersphere centered at A_i, let A_i' be the feet of the altitude A_iA_i' (of the simplex $\mathcal{A}$), let A^* be the inverse of A_i' with respect to the hypersphere ω_i. For $i, j = \overline{0, n}$, $i \neq j$, any hyperplane that passes through the midpoint of $A_i^*A_j^*$ and is perpendicular to the line A_iA_j is concurrent at a point P, which is the radical center of spheres ω_i.*

Suggested Reading

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