Airy Functions Demystified - III*

A Fresh Look at the Relation Between Airy and Bessel Functions

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Airy and Bessel functions are one of the most popular and important special functions in various branches of physics, mathematics, and engineering. An observation to their behavior for the real argument suggest that they are related. This relation was studied earlier, but were accompanied by a number of assumptions, approximations, and sometimes even misconceptions. This motivated us to develop a fresh and transparent method to establish these relations. As the continuation of our study of the two papers published in resonance already, here we have used the general asymptotic series and the convergent series of these functions and thereby developed two new methods which throw light on the subtle interrelationships between these functions. Numerical evidences of our claims are provided for better clarity and understanding.

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Airy function, Bairy function, Bessel functions, modified Bessel functions, Macdonald function, asymptotic expansions, convergent series.

1. Introduction

Special functions like Airy functions [1], [2], [3] and Bessel functions [4] are ubiquitous in science and engineering. The aim of this paper is to establish the relation between Airy functions and the family of Bessel functions using a fresh and straightforward method. In continuation of the two papers we have written in Resonance [2], [3] and the further analysis in this direction, motivated us to study the famous relationship between Airy and Bessel functions in a nascent way. Here we show important manipulations of the asymptotic series and convergent series in a pedagogical way. These asymptotic series and convergent series for any

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order can be however derived without much difficulty [5], [6], [7]. To show the direction to the reader we have derived the asymptotic series of the family of Bessel functions for any order in the appendix as they have phenomenal applications in various areas, like Schrödinger equation in spherical and cylindrical coordinates with constant potential. Here we remark that, Dr. J. W. Nicholson has derived the relation between these functions in his papers [8], [9].

We have found some lapses in these papers which motivated us to develop a new method which is more interesting and transparent specially for students who want to delve deeply.

2. A Brief Introduction to Airy Functions

Sir George Biddell Airy, an astronomer and mathematician of the nineteenth century had introduced his very famous Airy functions ($Ai(z)$ and $Bi(z)$, $z \in \mathbb{C}$ [10], [11]) to explain the intensity profile of the rainbow [2], [3] His theory superseded Newton's and Descartes theory in certain appreciable aspects. Over the years, these Airy functions have found applications in various areas of physics, mathematics, and engineering. To name a few : The W. K. B. approximation in quantum mechanics [3], the quantum bouncing ball [3] and propagation of light and radiation of electromagnetic waves [3]. *Figutre* [1] shows the plots of Airy function ($Ai(x)$) and Bairy function ($Bi(x)$) for the real variable x [2]. We can see that as $x \to -\infty$, $Ai(x)$ and $Bi(x)$ show decreasing behavior in an oscillatory fashion while for $x \to \infty$, $Ai(x)$ shows an exponentially decreasing behavior and $Bi(x)$ shows an exponentially increasing behavior.

We can define the Airy functions in terms of a convergent series for $z \in \mathbb{C}$ as [6],

$$\begin{aligned} Ai(z) = {} & \frac{3^{-\frac{2}{3}}}{\pi} \sin\left(\frac{2\pi}{3}\right)\left(-\frac{2}{3}\right)!\left(1 + \frac{z^3}{2.3} + \frac{z^6}{2.3.5.6} + \dots\right) \\ & + \frac{3^{-\frac{1}{3}}}{\pi} \sin\left(\frac{4\pi}{3}\right)\left(-\frac{1}{3}\right)!\left(z + \frac{z^4}{3.4} + \frac{z^7}{3.4.6.7} + \dots\right), \end{aligned} \quad (1)$$

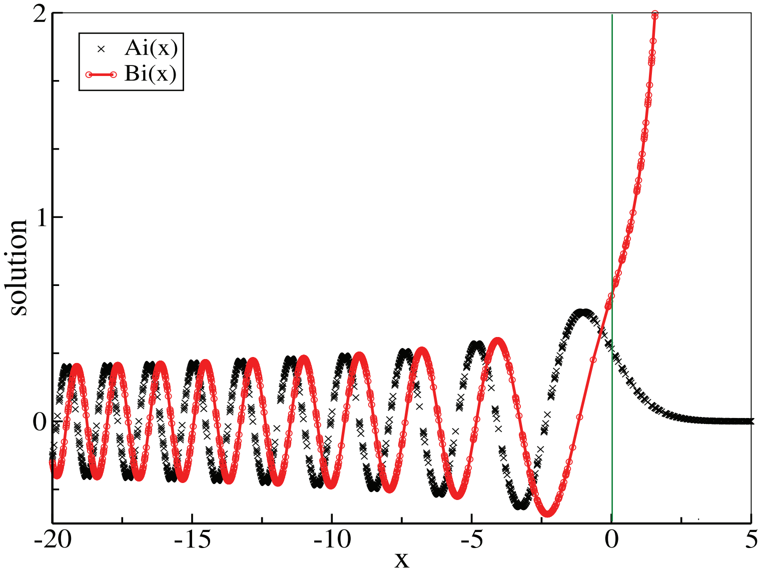

Figure 1. The behavior of the Airy function - $Ai(x)$ and the Bairy function - $Bi(x)$ for $x \in \mathbb{R}$.

$$Bi(z) = \sqrt{3}\frac{3^{-\frac{2}{3}}}{\pi}\sin\left(\frac{2\pi}{3}\right)\left(-\frac{2}{3}\right)!\left(1+\frac{z^3}{2.3}+\frac{z^6}{2.3.5.6}+\ldots\right)$$
$$-\sqrt{3}\frac{3^{-\frac{1}{3}}}{\pi}\sin\left(\frac{4\pi}{3}\right)\left(-\frac{1}{3}\right)!\left(z+\frac{z^4}{3.4}+\frac{z^7}{3.4.6.7}+\ldots\right). \quad (2)$$

The series described by Eq.[1] and Eq.[2] are identical except for a factor of $\sqrt{3}$ and a negative sign in the second term in Eq.[2]. For $x \in \mathbb{R}^+$, another form of the Airy functions are the asymptotic series [5]. These series are useful approximations at $x >> 0$. These general asymptotic series for any order for the entire real axis are given as,

Eq.[3] and Eq.[4] are valid for $|\mathrm{ph}(z)| \leq \pi - \delta$ and $|\mathrm{ph}(z)| \leq \frac{1}{3}\pi - \delta$ respectively in the complex plane when $x = z \in \mathbb{C}$ [11]. But here we consider only real arguments i.e. $x \in \mathbb{R}$.

$$Ai(x) = \frac{x^{-\frac{1}{4}}}{2\sqrt{\pi}}e^{-\frac{2}{3}x^{\frac{3}{2}}}\left(1-\frac{1.5}{1!.\,48}\frac{1}{x^{\frac{3}{2}}}+\frac{1.5.7.11}{2!.\,(48)^2}\frac{1}{x^3}+\ldots\right), \quad (3)$$

$$Bi(x) = \frac{x^{-\frac{1}{4}}}{\sqrt{\pi}}e^{\frac{2}{3}x^{\frac{3}{2}}}\left(1+\frac{1.5}{1!.\,48}\frac{1}{x^{\frac{3}{2}}}+\frac{1.5.7.11}{2!.\,(48)^2}\frac{1}{x^3}+\ldots\right). \quad (4)$$

and we write $-x = xe^{i\pi}$ along the negative real axis,

$$Ai(-x) = \frac{x^{-\frac{1}{4}}}{\sqrt{\pi}}\left(\sin\left(\frac{2}{3}x^{\frac{3}{2}}+\frac{\pi}{4}\right)L_1 - \cos\left(\frac{2}{3}x^{\frac{3}{2}}+\frac{\pi}{4}\right)L_2\right), \quad (5)$$

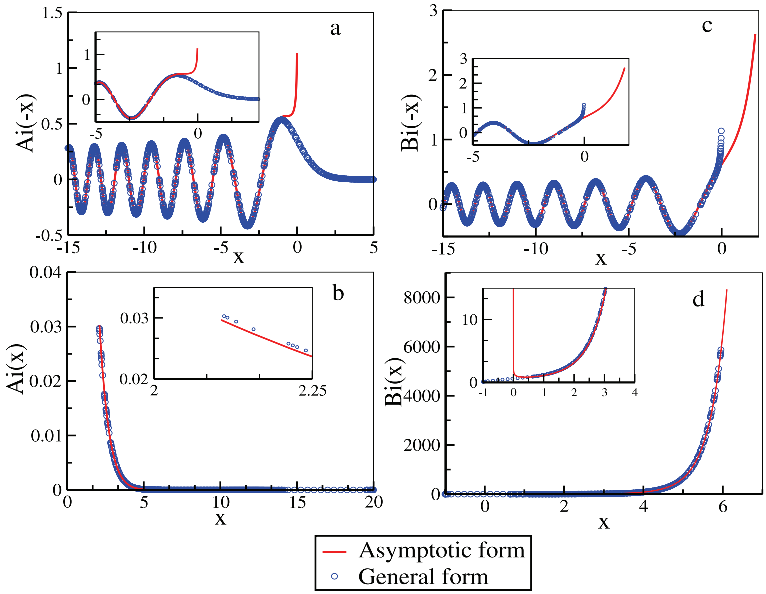

Figure 2. Graph of Airy functions and their first order asymptotic series for $x \in \mathbb{R}^+$.

$$Bi(-x) = \frac{x^{-\frac{1}{4}}}{\sqrt{\pi}} \left(\cos\left(\frac{2}{3}x^{\frac{3}{2}} + \frac{\pi}{4}\right) L_1 + \sin\left(\frac{2}{3}x^{\frac{3}{2}} + \frac{\pi}{4}\right) L_2 \right). \quad (6)$$

Eq.[5] and Eq.[6] are valid for $|\mathrm{ph}(z)| \leq \frac{2}{3}\pi - \delta$ and $|\mathrm{ph}(z)| \leq \frac{2}{3}\pi - \delta$ respectively in the complex plane when $x = z \in \mathbb{C}$ [11]. But here we consider only real arguments i.e. $x \in \mathbb{R}$.

where L_1 is the series containing even terms and L_2 is the series containing odd terms given as

$$L_1 = \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^{2k} \Gamma\left(2k + \frac{1}{6}\right) \Gamma\left(2k + \frac{5}{6}\right) \frac{1}{2k!} \frac{1}{2\pi} (x)^{-3k} (-1)^k, \quad (7)$$

and

$$L_2 = \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^{2k+1} \Gamma\left(2k + 1 + \frac{1}{6}\right) \times \Gamma\left(2k + 1 + \frac{5}{6}\right) \frac{1}{2\pi(2k+1)!} \left((x)^{-3k-\frac{3}{2}} (-1)^k\right). \quad (8)$$

Figure [2] shows the graphs of Airy and Bairy functions and their first order asymptotic expansions. The asymptotic expansions have an appreciably less error for $x \to +\infty$. The insets show the plots when x is very near to 0 and we can see that these asymptotic forms show divergent behavior from the regular Airy functions. This is due to dominance of $x^{-\frac{1}{4}}$ when $x \in (0, 1)$.

3. A Brief Introduction to Bessel Functions

In the forthcoming sections, we aim at establishing the relation between Airy functions ($Ai(z)$ and $Bi(z)$, $z \in \mathbb{R}$ [10], [11]) with the well known Bessel functions [11] by simple methods. It is well known that Bessel functions form the backbone of mathematical physics and engineering. To name a few : Vibrations of a two dimensional circular membrane [12], heat transfer [12] and Airy's disc resulting form the diffraction of light from circular aperture [13]. $J_\nu(z)$ is known as the Bessel function of the first kind and of order ν ($\nu \in \mathbb{R}$) [4], [14]. We define the series form of $J_\nu(z)$ and $Y_\nu(z)$ in terms of the gamma function as,

$$J_\nu(z) = \sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(s+1)\Gamma(s+\nu+1)} \left(\frac{z}{2}\right)^{\nu+2s}, \tag{9}$$

$$Y_\nu(z) = \frac{\cos(\nu\pi)J_\nu(z) - J_{-\nu}(z)}{\sin(\nu\pi)}, \tag{10}$$

where $Y_\nu(z)$ is known as the Neumann function or the Bessel function of the second kind of ν^{th} order. Eq.[10] is valid for $\nu \notin \mathbb{Z}$ and is a linear combination of the two linearly independent solutions $J_\nu(z)$ and $J_{-\nu}(z)$ of Bessel's differential equation. $Y_\nu(z)$ is the second linearly independent solution of Bessel's differential equation for $\nu \in \mathbb{Z}$. Another way of expressing Bessel functions is to write them as Hankel functions. These functions are denoted as $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ and are known as Hankel functions of first and second kind respectively. More explicitly

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_\nu(z), \tag{11}$$

$$H_\nu^{(2)}(z) = J_\nu(z) - iY_\nu(z). \tag{12}$$

If $z = x \in \mathbb{R}^+$, then from Eq.[11] and Eq.[12] we have

$$H_\nu^{(2)}(x) = \left(H_\nu^{(1)}(x)\right)^*. \tag{13}$$

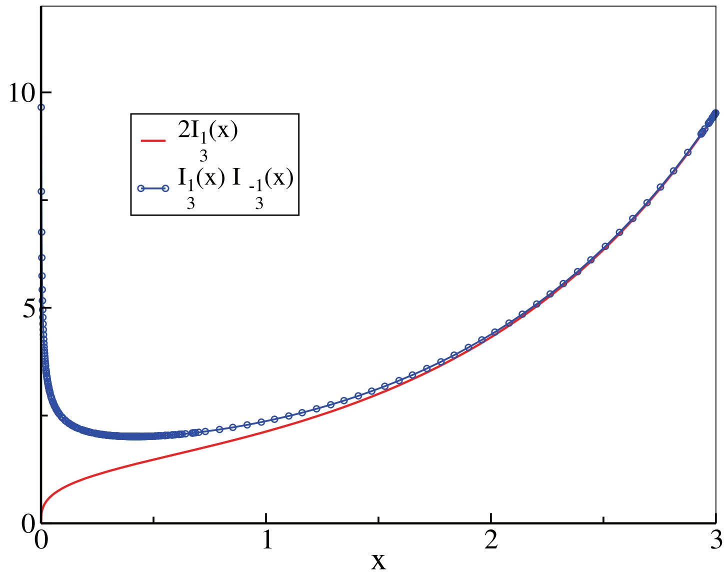

Figure 3. A plot showing the comparison between the expression $I_\nu(x) + I_{-\nu}(x)$ and $2I_\nu(x)$ for $x \in \mathbb{R}^+$ and $\nu = \frac{1}{3}$.

Using Eq.[11] and Eq.[12] and considering $z = x \in \mathbb{R}$ we can see that

$$J_\nu(x) = \frac{1}{2}\left(H_\nu^{(1)}(x) + H_\nu^{(2)}(x)\right). \tag{14}$$

$I_\nu(z)$ is known as the modified Bessel function and of order ν ($\nu \in \mathbb{R}$) which is defined as,

$$I_\nu(z) = \sum_{s=0}^{\infty} \frac{1}{\Gamma(s+1)\Gamma(s+\nu+1)}\left(\frac{z}{2}\right)^{\nu+2s}. \tag{15}$$

For $x \in \mathbb{R}^+$, $I_\nu(x)$ is an exponentially increasing function which is real and thus from the convergent series form given by Eq.[15] we can say that when x and s are very large as compared to $|\nu|$ the expressions $I_\nu(x) + I_{-\nu}(x)$ and $2I_\nu(x)$ are equivalent with an appreciably low error for $\nu \in \mathbb{R}$ (refer *Figure* [3] where $\nu = \frac{1}{3}$), while $I_\nu(x) = I_{-\nu}(x)$ for $\nu \in \mathbb{Z}$. However when x is not real and positive $I_\nu(x) + I_{-\nu}(x)$ and $2I_\nu(x)$ are not equivalent as the imaginary terms of $I_\nu(x)$ and $I_{-\nu}(x)$ do not match. Now we define the modified Bessel function of the second kind $K_\nu(z)$ for $z \in \mathbb{C}$. These functions are also known as Macdonald functions. The Macdonald function of ν^{th} order is defined as [14]

$$K_\nu(z) = \frac{\pi}{2}\left(\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}\right). \tag{16}$$

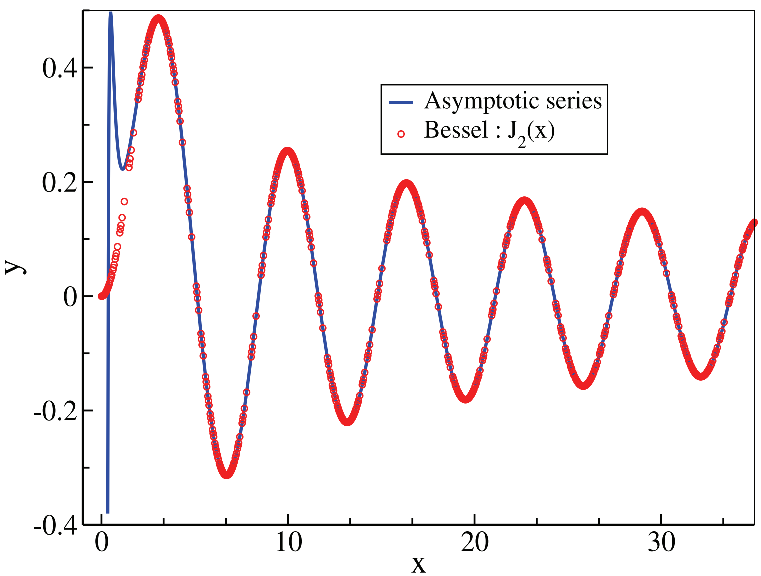

Figure 4. Plot of the Bessel function $J_2(x)$ of second order and its asymptotic form for $x > 0$ showing a decreasing behavior which is oscillatory in nature.

Eq.[16] is valid for $\nu \notin \mathbb{Z}$ and is a linear combination of the two linearly independent solutions $I_\nu(z)$ and $I_{-\nu}(z)$ of modified Bessel's differential equation. $K_\nu(z)$ is the second linearly independent solution of modified Bessel's differential equation for $\nu \in \mathbb{Z}$. For $x \in \mathbb{R}^+$, we have the relation between modified Bessel functions and the Hankel functions [14] as,

$$I_\nu(x) = \frac{e^{-\frac{i\nu\pi}{2}}}{2}\left(H_\nu^{(1)}\left(e^{i\frac{\pi}{2}}x\right) + H_\nu^{(2)}\left(e^{i\frac{\pi}{2}}x\right)\right), \tag{17}$$

and the relation between Macdonald functions and Bessel functions is

$$K_\nu(x) = \frac{i\pi}{2}e^{\frac{i\nu\pi}{2}}H_\nu^{(1)}(xe^{i\frac{\pi}{2}}). \tag{18}$$

An important ingredient in deriving the relationship between Airy functions and the family of Bessel functions are the asymptotic forms of the family of Bessel functions [4], [14]. An elegant derivation of these asymptotic series is given in the Appendix of this paper. The asymptotic series of $J_\nu(x), I_\nu(x)$, and $K_\nu(x)$ are [7], [15], [16].

Note that for $x \in \mathbb{R}^+$ and $\nu \in \mathbb{R}$, the functions $J_\nu(x)$, $Y_\nu(x)$, $I_\nu(x)$ and $K_\nu(x)$ are all real.

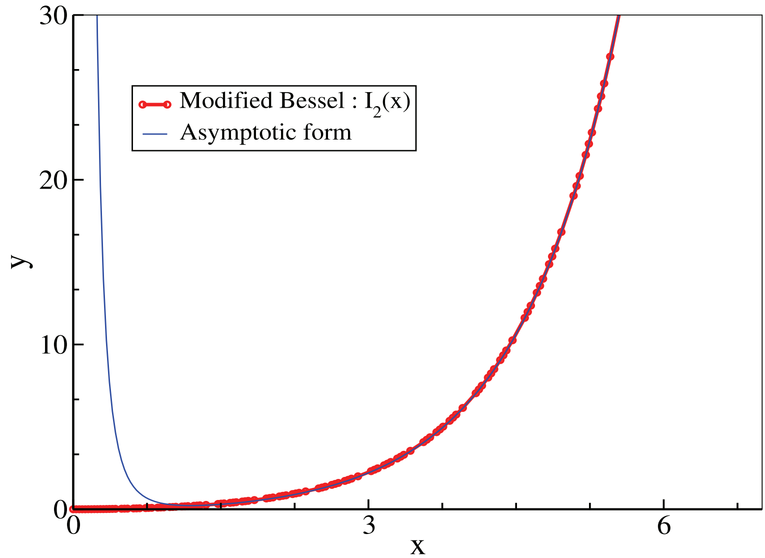

Figure 5. The plot of modified Bessel function of second order $I_2(x)$ and its asymptotic form for $x > 0$ showing an exponentially increasing behavior.

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\Bigg(\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \times \left(1 - \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)}{8^2.\ 2!\ x^2} + \ldots\right) - \sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\left(\frac{(4\nu^2 - 1^2)}{8.\ 1!\ x} - \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)(4\nu^2 - 5^2)}{8^3.\ 3!\ x^3} + \ldots\right)\Bigg), \tag{19}$$

$$I_\nu(x) = \frac{e^x}{\sqrt{2\pi x}}\left(1 - \frac{(4\nu^2 - 1^2)}{8.\ 1!\ x} + \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)}{8^2.\ 2!\ x^2} - \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)(4\nu^2 - 5^2)}{8^3.\ 3!\ x^3} + \ldots\right), \tag{20}$$

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x}\left(1 + \frac{(4\nu^2 - 1^2)}{8.\ 1!\ x} + \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)}{8^2.\ 2!\ x^2} + \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)(4\nu^2 - 5^2)}{8^3.\ 3!\ x^3} + \ldots\right). \tag{21}$$

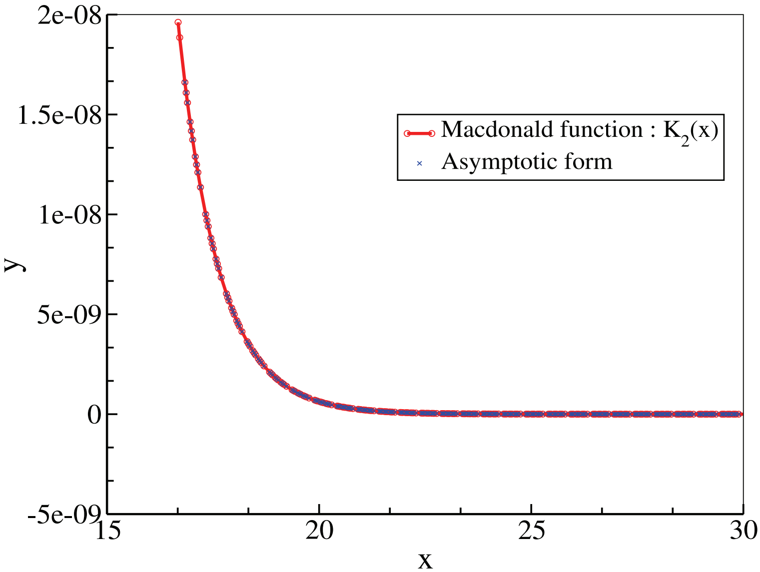

Figure 6. The plot of Macdonald function of second order $K_2(x)$ and its asymptotic form for $x > 0$ showing an exponentially decreasing behavior.

We can write Eq.[21] compactly as

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}}e^{-x}\left(\sum_{n=0}^{\infty} a_n \left(\frac{1}{x}\right)^n\right), \tag{22}$$

where $a_0 = 1$ and $a_n = \dfrac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)\ldots(4\nu^2 - (2n-1)^2)}{2^{3n}.n!}$ where n starts from 1. *Figure* [4] shows the graph between the Bessel function of second order ($J_2(x)$) plotted along with its asymptotic series. The error between the plots decreases as $x \to +\infty$, but very close to 0, the asymptotic expansion shows a divergent behavior since the function $\frac{1}{\sqrt{x}}$ is dominant as $x \to 0$, we have $\frac{1}{\sqrt{x}} \to +\infty$. *Figure* [5] shows the graphs of the modified Bessel function of second order ($I_2(x)$) and its asymptotic series. The error between the plots decreases as $x \to +\infty$, but very close to 0, the asymptotic expansion shows a divergent behavior for a similar reason as mentioned in the case of $J_2(x)$. *Figure* [6] shows the graph of Macdonald function of second order ($K_2(x)$) and its asymptotic series which shows an exponential decrease as $x \to +\infty$.

In studying nuclear physics, with constant potential profiles, the Schrödinger equation in spherical and cylindrical coordinates reduce to Bessel's differential equations and these asymptotic forms of Bessel functions are the solutions.

4. Relation Between Airy Functions and the Family of Bessel Functions

The relation between Airy functions and the family of Bessel functions exist only for $x \in \mathbb{R}$ and not for any other argument of x.

We will be proving the following results using two completely different methods. For $x \in \mathbb{R}^+$, we have the relations

$$Ai(x) = \frac{1}{\pi}\sqrt{\frac{x}{3}}K_{\pm\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right), \tag{23}$$

$$Bi(x) = \sqrt{\frac{x}{3}}\left(I_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + I_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)\right), \tag{24}$$

$$Ai(-x) = \frac{\sqrt{x}}{3}\left(J_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)\right), \tag{25}$$

$$Bi(-x) = \sqrt{\frac{x}{3}}\left(-J_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)\right). \tag{26}$$

METHOD 1. ***Derivation using the asymptotic series of the functions***

Our main aim is to find the relation between Airy functions and the family of Bessel functions in the domain of the entire real axis by using their asymptotic series. On observing the plots of $Ai(x)$ and $K_\nu(x)$ for $x \in \mathbb{R}^+$ from *Figure* [1] and *Figure* [6] respectively and in the domain considered, both these functions are exponentially decreasing, this indicates a possible relation between them. A similar conclusion can be made in the case of $Bi(x)$ and $I_\nu(x)$, both being exponentially increasing functions in the domain considered. Consider the general asymptotic series of $Ai(x)$ from Eq.[3], a replacement of x by $\frac{2}{3}x^{\frac{3}{2}}$ in the asymptotic series of Macdonald function, as given in Eq.[22] will depict the behavior of $Ai(x)$ to a good approximation, effecting this in Eq.[22]

$$K_\nu\left(\frac{2}{3}x^{\frac{3}{2}}\right) = \sqrt{\frac{\pi}{2\left(\frac{2}{3}x^{\frac{3}{2}}\right)}}e^{-\frac{2}{3}x^{\frac{3}{2}}}\left(\sum_{n=0}^{\infty} a_n\left(\frac{1}{\frac{2}{3}x^{\frac{3}{2}}}\right)^n\right), \tag{27}$$

$$K_\nu\left(\frac{2}{3}x^{\frac{3}{2}}\right) = \frac{\sqrt{3}}{2}\frac{\sqrt{\pi}}{x^{\frac{3}{4}}}e^{-\frac{2}{3}x^{\frac{3}{2}}}\sum_{n=0}^{\infty} a_n\left(\frac{3}{2}\right)^n\left(x^{-\frac{3}{2}}\right)^n. \tag{28}$$

To obtain at least the first term of the asymptotic series of $Ai(x)$ in $K_\nu(x)$, we need to multiply Eq.[28] on both sides by $\left(\frac{x^{\frac{1}{2}}}{\pi}\frac{1}{\sqrt{3}}\right)$, this gives

$$\frac{1}{\pi}\sqrt{\frac{x}{3}}K_\nu\left(\frac{2}{3}x^{\frac{3}{2}}\right)=\left(\frac{x^{-\frac{1}{4}}}{2\sqrt{\pi}}e^{-\frac{2}{3}x^{\frac{3}{2}}}\right)\sum_{n=0}^{\infty}a_n\left(\frac{3}{2}\right)^n\left(x^{-\frac{3}{2}}\right)^n, \tag{29}$$

expanding the R.H.S. to get

$$\frac{1}{\pi}\sqrt{\frac{x}{3}}K_\nu\left(\frac{2}{3}x^{\frac{3}{2}}\right)=\left(\frac{x^{-\frac{1}{4}}}{2\sqrt{\pi}}e^{-\frac{2}{3}x^{\frac{3}{2}}}\right)\times\left(1+\frac{(4\nu^2-1)}{1!\,8}\left(\frac{3}{2}\right)x^{-\frac{3}{2}}+\frac{(4\nu^2-1^2)(4\nu^2-3^2)}{2!\,8^2}\left(\frac{3}{2}\right)^2x^{-3}+\ldots\right). \tag{30}$$

Now if we select $\nu=\frac{1}{3}$ in Eq.[30], we have

$$\frac{1}{\pi}\sqrt{\frac{x}{3}}K_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)=\left(\frac{x^{-\frac{1}{4}}}{2\sqrt{\pi}}e^{-\frac{2}{3}x^{\frac{3}{2}}}\right)\times\left(1+\frac{(\frac{4}{9}-1)}{1!\,8}\left(\frac{3}{2}\right)x^{-\frac{3}{2}}+\frac{(\frac{4}{9}-1)(\frac{4}{9}-9)}{2!\,8^2}\left(\frac{3}{2}\right)^2x^{-3}+\ldots\right). \tag{31}$$

$$=\left(\frac{x^{-\frac{1}{4}}}{2\sqrt{\pi}}e^{-\frac{2}{3}x^{\frac{3}{2}}}\right)\left(1+\frac{(4-9)}{9.\,1!\,8}\left(\frac{3}{2}\right)x^{-\frac{3}{2}}+\frac{(4-9)(4-81)}{9.\,9.\,2!\,8^2}\left(\frac{3}{2}\right)^2x^{-3}+\ldots\right). \tag{32}$$

$$=\left(\frac{x^{-\frac{1}{4}}}{2\sqrt{\pi}}e^{-\frac{2}{3}x^{\frac{3}{2}}}\right)\left(1-\frac{1.5}{1!.\,48}x^{-\frac{3}{2}}+\frac{1.5.7.11}{2!.\,48^2}x^{-3}+\ldots\right). \tag{33}$$

Thus from Eq.[3] and Eq.[33], we get the relation

$$Ai(x)=\frac{1}{\pi}\sqrt{\frac{x}{3}}K_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right). \tag{34}$$

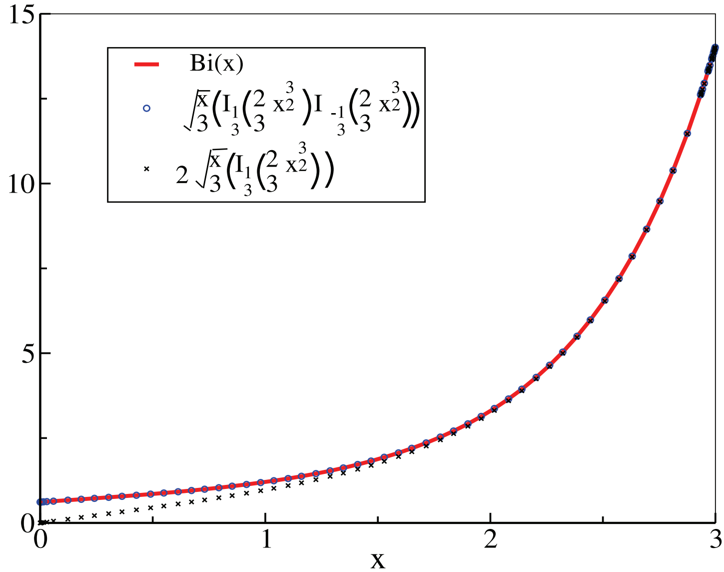

Figure 7. The plot clearly showing the comparison between $Bi(x)$, $\sqrt{\frac{x}{3}}\left(I_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)+I_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)\right)$ and $2\sqrt{\frac{x}{3}}I_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)$.

Since for any value of ν we have $K_\nu(x) = K_{-\nu}(x)$, we can also write Eq.[34] as

$$Ai(x) = \frac{1}{\pi}\sqrt{\frac{x}{3}}K_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right). \tag{35}$$

For $Bi(x)$, we can consider the relation between $Bi(x)$ and $I_\nu(x)$ as already mentioned. To this end consider the general series of $Bi(x)$ and $I_\nu(x)$, from Eq.[4] and Eq.[20] a similar approach as done for $Ai(x)$ suggests that we replace x as $\frac{2}{3}x^{\frac{3}{2}}$ in Eq.[20] and simplifying

$$\begin{aligned} I_\nu\left(\frac{2}{3}x^{\frac{3}{2}}\right) = \frac{\sqrt{3}}{2\sqrt{\pi}}\frac{e^{\frac{2}{3}x^{\frac{3}{2}}}}{x^{\frac{3}{4}}}\Bigg(1 - \frac{(4\nu^2-1^2)}{8.\ 1!\ x^{\frac{3}{2}}}\left(\frac{3}{2}\right) \\ + \frac{(4\nu^2-1^2)(4\nu^2-3^2)}{8^2.\ 2!\ x^3}\left(\frac{3}{2}\right)^2 \\ - \frac{(4\nu^2-1^2)(4\nu^2-3^2)(4\nu^2-5^2)}{8^3.\ 3!\ x^{\frac{9}{2}}}\left(\frac{3}{2}\right)^3 + \dots\Bigg). \end{aligned} \tag{36}$$

Multiplying Eq.[36] by $\sqrt{\frac{x}{3}}$ and choosing ν as $\frac{1}{3}$, we get

$$\sqrt{\frac{x}{3}}I_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) = \frac{1}{2\sqrt{\pi}}\frac{e^{\frac{2}{3}x^{\frac{3}{2}}}}{x^{\frac{1}{4}}}\Bigg(1 - \frac{(\frac{4}{9}-1^2)}{8.\ 1!\ x^{\frac{3}{2}}}\left(\frac{3}{2}\right) + \frac{(\frac{4}{9}-1^2)(\frac{4}{9}-3^2)}{8^2.\ 2!\ x^3}\left(\frac{3}{2}\right)^2 - \frac{(\frac{4}{9}-1^2)(\frac{4}{9}-3^2)(\frac{4}{9}-5^2)}{8^3.\ 3!\ x^{\frac{9}{2}}}\left(\frac{3}{2}\right)^3 + \dots\Bigg). \quad (37)$$

$$\sqrt{\frac{x}{3}}I_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) = \frac{1}{2\sqrt{\pi}}\frac{e^{\frac{2}{3}x^{\frac{3}{2}}}}{x^{\frac{1}{4}}}\Bigg(1 + \frac{1.5}{48.\ 1!\ x^{\frac{3}{2}}} + \frac{1.5.7.11}{(48)^2.\ 2!\ x^3} + \dots\Bigg). \quad (38)$$

In order to obtain first term of the asymptotic series of $Bi(x)$ we can either multiply Eq.[38] by 2 or consider $\sqrt{\frac{x}{3}}\Big(I_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + I_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)\Big)$. These two ideas give the same answer for $x >> 0$ but when x is very close to 0 the expression $\sqrt{\frac{x}{3}}\Big(I_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + I_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)\Big)$ gives a better match with $Bi(x)$ as verified numerically, refer *Figure* [7]. This will become more clear in the next method as seen from Eq.[82]. Thus we choose the suitable combination of $I_\nu(x)$ and $I_{-\nu}(x)$ with ν to be $\frac{1}{3}$,

$$\sqrt{\frac{x}{3}}\left(I_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + I_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)\right) = \frac{1}{\sqrt{\pi}}\frac{e^{\frac{2}{3}x^{\frac{3}{2}}}}{x^{\frac{1}{4}}}\Bigg(1 + \frac{1.5}{48.\ 1!\ x^{\frac{3}{2}}} + \frac{1.5.7.11}{(48)^2.\ 2!\ x^3} + \dots\Bigg). \quad (39)$$

Thus from Eq.[4] and Eq.[39], we have the relation

$$Bi(x) = \sqrt{\frac{x}{3}}\left(I_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + I_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)\right). \quad (40)$$

In order to find the possible relation between the Airy functions and the family of Bessel functions for the case when argument is along the negative real axis of Airy functions, we consider $x \in \mathbb{R}^+$, on observing the behavior of the functions $Ai(-x)$,

$Bi(-x)$ and $J_\nu(x)$, it is suggested that these functions are related closely as all of these functions decrease in an oscillatory fashion. We will consider the same approach that we had used to obtain Eq.[34] and Eq.[40] for $x \in \mathbb{R}^+$. Consider the general asymptotic series of $J_\nu(x)$ and $Ai(x)$ as given by Eq.[19] and Eq.[5].

$$Ai(xe^{i\pi}) = Ai(-x) = \frac{x^{-\frac{1}{4}}}{\sqrt{\pi}}\left(\sin\left(\frac{2}{3}x^{\frac{3}{2}} + \frac{\pi}{4}\right)L_1 - \cos\left(\frac{2}{3}x^{\frac{3}{2}} + \frac{\pi}{4}\right)L_2\right), \quad (41)$$

where

$$L_1 = 1 - \frac{1.5.7.11}{2!(48)^2x^3} + \ldots +, \quad (42)$$

and

$$L_2 = \frac{1.5}{1!(48)x^{\frac{3}{2}}} - \frac{1.5.7.11.13.17}{3!(48)^3x^{\frac{9}{2}}} + \ldots + . \quad (43)$$

In order for the functions $Ai(-x)$ and $J_\nu(x)$ to be related, from Eq.[19] and Eq.[41] we can predict that argument of J_ν should be $\frac{2}{3}x^{\frac{3}{2}}$, thus effecting this in Eq.[19] and simplifying by choosing $\nu = \frac{1}{3}$, we have

$$J_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) = x^{-\frac{3}{4}}\sqrt{\frac{3}{\pi}}\left(\cos\left(\frac{2}{3}x^{\frac{3}{2}} - \frac{5\pi}{12}\right)\times \left(1 - \frac{1.5.7.11}{2!(48)^2x^3} + \ldots\right) + \sin\left(\frac{2}{3}x^{\frac{3}{2}} - \frac{5\pi}{12}\right)\times \left(\frac{1.5}{1!(48)x^{\frac{3}{2}}} - \frac{1.5.7.11.13.17}{3!(48)^3x^{\frac{9}{2}}} + \ldots\right)\right). \quad (44)$$

An analogous procedure by choosing ν as $-\frac{1}{3}$ in Eq.[19] and simplifying, we get

$$J_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) = x^{-\frac{3}{4}}\sqrt{\frac{3}{\pi}}\left(\cos\left(\frac{2}{3}x^{\frac{3}{2}} - \frac{\pi}{12}\right)\times \left(1 - \frac{1.5.7.11}{2!(48)^2x^3} + \ldots\right) + \sin\left(\frac{2}{3}x^{\frac{3}{2}} - \frac{\pi}{12}\right)\times \left(\frac{1.5}{1!(48)x^{\frac{3}{2}}} - \frac{1.5.7.11.13.17}{3!(48)^3x^{\frac{9}{2}}} + \ldots\right)\right). \quad (45)$$

An important point to note from Eq.[44] and Eq.[45] is that the R.H.S. of both these equations are related to the the expressions of L_1 and L_2 as given by Eq.[42] and Eq.[43]. In order that the first term of the asymptotic series of $Ai(-x)$ to match with that of $J_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)$ and $J_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)$ we multiply Eq.[44] and Eq.[45] with $\sqrt{x}$ on both sides. Since the coefficients of the two series in $\sqrt{x}J_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)$ and $\sqrt{x}J_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)$ do not match with that of $Ai(-x)$, we consider a linear combination of the form $a\sqrt{x}J_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + b\sqrt{x}J_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)$, where a and b are the constants to be found. We start by writing

$$Ai(-x) = a\sqrt{x}J_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + b\sqrt{x}J_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right), \qquad (46)$$

substituting Eq.[41], Eq.[44], and Eq.[45] in Eq.[46], we get

$$\begin{aligned}
&\frac{x^{-\frac{1}{4}}}{\sqrt{\pi}}\left(\sin\left(\frac{2}{3}x^{\frac{3}{2}}+\frac{\pi}{4}\right)L_1 - \cos\left(\frac{2}{3}x^{\frac{3}{2}}+\frac{\pi}{4}\right)L_2\right) \\
&\quad = \frac{x^{-\frac{1}{4}}}{\sqrt{\pi}}\left(\left(a\sqrt{3}\cos\left(\frac{2}{3}x^{\frac{3}{2}}-\frac{5\pi}{12}\right) + b\sqrt{3}\cos\left(\frac{2}{3}x^{\frac{3}{2}}-\frac{\pi}{12}\right)\right)L_1\right. \\
&\quad\quad \left. + \left(a\sqrt{3}\sin\left(\frac{2}{3}x^{\frac{3}{2}}-\frac{5\pi}{12}\right) + b\sqrt{3}\sin\left(\frac{2}{3}x^{\frac{3}{2}}-\frac{\pi}{12}\right)\right)L_2\right). \qquad (47)
\end{aligned}$$

By comparing the coefficients of L_1 and L_2 in Eq.[47], we get two linear simultaneous equations in two unknowns a and b as follows

$$\sin\left(\frac{2}{3}x^{\frac{3}{2}}+\frac{\pi}{4}\right) = a\sqrt{3}\cos\left(\frac{2}{3}x^{\frac{3}{2}}-\frac{5\pi}{12}\right) + b\sqrt{3}\cos\left(\frac{2}{3}x^{\frac{3}{2}}-\frac{\pi}{12}\right), \qquad (48)$$

$$-\cos\left(\frac{2}{3}x^{\frac{3}{2}}+\frac{\pi}{4}\right) = a\sqrt{3}\sin\left(\frac{2}{3}x^{\frac{3}{2}}-\frac{5\pi}{12}\right) + b\sqrt{3}\sin\left(\frac{2}{3}x^{\frac{3}{2}}-\frac{\pi}{12}\right), \qquad (49)$$

Eq.[48] and Eq.[49] can be solved for a and b by using Cramer's rule and a little bit of straight forward algebra to get $a = \frac{1}{3}$ and $b = \frac{1}{3}$. Using these values of a and b so obtained in Eq.[46], we get the relation between Airy function and Bessel function as,

$$Ai(-x) = \frac{\sqrt{x}}{3}\left(J_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)\right). \qquad (50)$$

We use a similar approach to find the relation between $Bi(-x)$ and the Bessel function by considering the behavior of these functions

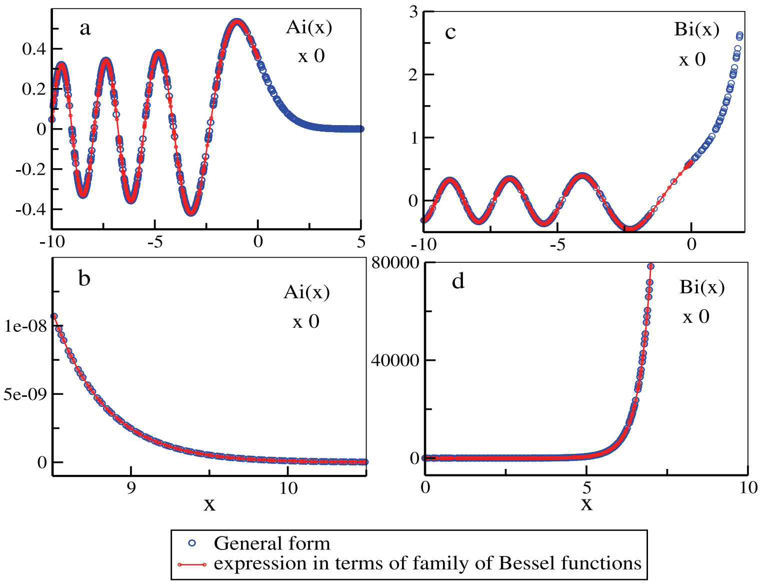

Figure 8. The plot of Airy functions and their equivalent expressions in terms of Bessel functions.

from *Figure* [1] and *Figure* [4]. Thus we assume $Bi(-x)$ equal to the linear combination

$$Bi(-x) = c\sqrt{x}J_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + d\sqrt{x}J_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right), \tag{51}$$

the values of constants c and d need to be found. Substituting Eq.[6], Eq.[44], and Eq.[45] in Eq.[51], we get

$$\begin{aligned}\frac{x^{-\frac{1}{4}}}{\sqrt{\pi}}&\left(\cos\left(\frac{2}{3}x^{\frac{3}{2}}+\frac{\pi}{4}\right)L_1+\sin\left(\frac{2}{3}x^{\frac{3}{2}}+\frac{\pi}{4}\right)L_2\right)\\ &=\frac{x^{-\frac{1}{4}}}{\sqrt{\pi}}\left(\left(c\sqrt{3}\cos\left(\frac{2}{3}x^{\frac{3}{2}}-\frac{5\pi}{12}\right)+d\sqrt{3}\cos\left(\frac{2}{3}x^{\frac{3}{2}}-\frac{\pi}{12}\right)\right)L_1\right.\\ &\left.+\left(c\sqrt{3}\sin\left(\frac{2}{3}x^{\frac{3}{2}}-\frac{5\pi}{12}\right)+d\sqrt{3}\sin\left(\frac{2}{3}x^{\frac{3}{2}}-\frac{\pi}{12}\right)\right)L_2\right),\end{aligned} \tag{52}$$

giving us two equations of the form

$$\cos\left(\frac{2}{3}x^{\frac{3}{2}}+\frac{\pi}{4}\right) = c\sqrt{3}\cos\left(\frac{2}{3}x^{\frac{3}{2}}-\frac{5\pi}{12}\right)+d\sqrt{3}\cos\left(\frac{2}{3}x^{\frac{3}{2}}-\frac{\pi}{12}\right), \tag{53}$$

$$\sin\left(\frac{2}{3}x^{\frac{3}{2}}+\frac{\pi}{4}\right) = c\sqrt{3}\sin\left(\frac{2}{3}x^{\frac{3}{2}}-\frac{5\pi}{12}\right)+d\sqrt{3}\sin\left(\frac{2}{3}x^{\frac{3}{2}}-\frac{\pi}{12}\right). \tag{54}$$

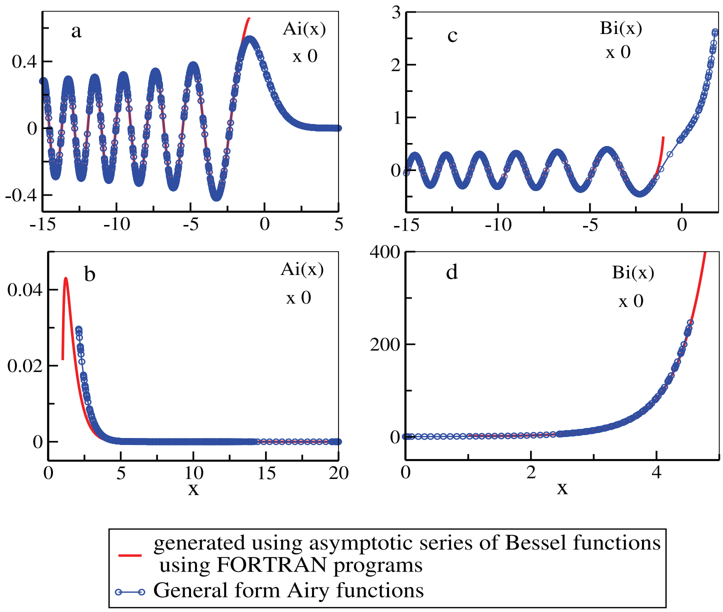

Figure 9. The plot of Airy functions and their equivalent expressions generated using the asymptotic series of Bessel functions.

Again using Cramer's rule to solve Eq.[53] and Eq.[54] with a bit of algebra gives $c = -\frac{1}{\sqrt{3}}$ and $d = \frac{1}{\sqrt{3}}$. Using these values of c and d so obtained in Eq.[51], we get the relation between Bairy function and Bessel function as,

$$Bi(-x) = \sqrt{\frac{x}{3}}\left(-J_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)\right). \tag{55}$$

Figure [8]a and *Figure* [8]b show the graphs of $Ai(x)$ (for $x \in \mathbb{R}$) (in blue, circle symbol) and the relative expressions in terms of the family of Bessel functions (in red, line symbol) for $x < 0$ and $x > 0$ respectively. *Figure* [8]c and *Figure* [8]d show the graphs of $Bi(x)$ (in blue, circle symbol) and the relative expressions in terms of the family of Bessel functions (in red, line symbol) for $x < 0$ and $x > 0$ respectively. Similarly *Figure* [9]a and *Figure* [9]b show the graphs of $Ai(x)$ (in blue, circle symbol) and that generated using the asymptotic series of the family of Bessel functions (in red, line symbol), by using fortran programs for $x < 0$ and $x > 0$ respectively. *Figure* [9]c and *Figure* [9]d show the graphs of $Bi(x)$ (in blue, circle symbol) and that generated using the asymptotic series of the family of Bessel functions (in red, line symbol), by using fortran programs for $x < 0$ and $x > 0$ respectively. The discrepancy near $x = 0$ between the plots is due to the fact that asymptotic series show divergent nature in that region as the case should be.

METHOD 2. ***Derivation using convergent series of the functions***

Another method of finding the relation between the Airy functions and Bessel functions is by comparing their convergent series [6], in this case we initially obtain the Bessel functions in terms of $Ai(-x)$ and $Bi(-x)$ ($x \in \mathbb{R}^+$). Consider the convergent series for $Ai(z)$ for any complex argument z as given by Eq.[1]. In order that Eq.[1] can be extended to the negative real axis set $z = -x$ where $x \in \mathbb{R}^+$ in Eq.[1], to get

$$Ai(-x) = \frac{3^{-\frac{2}{3}}}{\pi}\sin\left(\frac{2\pi}{3}\right)\left(-\frac{2}{3}\right)!\left(1 - \frac{x^3}{2.3} + \frac{x^6}{2.3.5.6} + \dots\right) + \frac{3^{-\frac{1}{3}}}{\pi}\sin\left(\frac{4\pi}{3}\right)\left(-\frac{1}{3}\right)!\left(-x + \frac{x^4}{3.4} - \frac{x^7}{3.4.6.7} + \dots\right). \quad (56)$$

A similar procedure to extend $Bi(z)$ to the negative real axis can be achieved by setting $z = -x$ with $x \in \mathbb{R}^+$ in Eq.[2], to get

$$Bi(-x) = \sqrt{3}\frac{3^{-\frac{2}{3}}}{\pi}\sin\left(\frac{2\pi}{3}\right)\left(-\frac{2}{3}\right)!\left(1 - \frac{x^3}{2.3} + \frac{x^6}{2.3.5.6} + \dots\right) - \sqrt{3}\frac{3^{-\frac{1}{3}}}{\pi}\sin\left(\frac{4\pi}{3}\right)\left(-\frac{1}{3}\right)!\left(-x + \frac{x^4}{3.4} - \frac{x^7}{3.4.6.7} + \dots\right) \quad (57)$$

By looking at the convergent series of $Ai(-x)$ and $Bi(-x)$, we can easily see that it is made up of two series expansions. Subtracting the series involving $Ai(-x)$ and $Bi(-x)$ will help us to get rid of the first term of both the series, however addition of Eq.[56] and Eq.[57] will get rid of the second term. The point worth noting is that the prefactors of Eq.[56] and Eq.[57] are different. To see whether there is a possibility to extract the Bessel functions in its series form by using Eq.[56] and Eq.[57], we suitably manipulate the functions $Ai(-x)$ and $Bi(-x)$. After some trials and mathematically motivated guesses we hit upon the following linear combination by carefully observing the prefactors to the series in Eq.[56] and Eq.[57] as,

$$3Ai(-x) - \sqrt{3}Bi(-x), \tag{58}$$

this will give us the series

$$\begin{aligned} &3Ai(-x) - \sqrt{3}Bi(-x) \\ &= 2x^{\frac{1}{2}}\left(-\frac{3^{-\frac{1}{3}}}{\pi}\Gamma\left(-\frac{1}{3}\right)\sin\left(\frac{4\pi}{3}\right) \times \right. \\ &\left.\left(-x^{\frac{1}{2}} + \frac{x^{\frac{7}{2}}}{3.4} - \frac{x^{\frac{13}{2}}}{3.4.6.7} + \ldots\right)\right). \end{aligned} \tag{59}$$

From the well known relation of gamma functions $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$, when $z = \frac{4}{3}$ we have $\Gamma\left(\frac{4}{3}\right)\Gamma\left(1-\frac{4}{3}\right) = \frac{\pi}{\sin\left(\frac{4\pi}{3}\right)}$, thus we can write $\frac{1}{\pi}\sin\left(\frac{4\pi}{3}\right)\Gamma\left(-\frac{1}{3}\right)$ as $\frac{1}{\Gamma\left(\frac{4}{3}\right)}$ in Eq.[59], to get

$$\begin{aligned} &3Ai(-x) - \sqrt{3}Bi(-x) = \\ &2x^{\frac{1}{2}}\left(3^{-\frac{1}{3}}\frac{1}{\Gamma\left(\frac{4}{3}\right)}\left(x^{\frac{1}{2}} - \frac{x^{\frac{7}{2}}}{3.4} + \frac{x^{\frac{13}{2}}}{3.4.6.7} + \ldots\right)\right), \end{aligned} \tag{60}$$

$$\begin{aligned} = 2x^{\frac{1}{2}}\Bigg(&\frac{x^{\frac{1}{2}}}{0!\, 3^{\frac{1}{3}}\, \Gamma\left(\frac{4}{3}\right)} - \frac{x^{\frac{1}{2}+3}}{1!\, 3^{\frac{1}{3}+2}\, \Gamma\left(\frac{4}{3}+1\right)} \\ &+ \frac{x^{\frac{1}{2}+6}}{2!\, 3^{\frac{1}{3}+4}\, \Gamma\left(\frac{4}{3}+2\right)} + \ldots\Bigg), \end{aligned} \tag{61}$$

$$3Ai(-x) - \sqrt{3}Bi(-x) = 2x^{\frac{1}{2}}\sum_{s=0}^{\infty}\frac{(-1)^s}{\Gamma(s+1)\Gamma\left(s+\frac{1}{3}+1\right)}\left(\frac{\frac{2}{3}x^{\frac{3}{2}}}{2}\right)^{\frac{1}{3}+2s}. \tag{62}$$

Comparing the R.H.S. of Eq.[62] and Eq.[9], it immediately follows that

$$3Ai(-x) - \sqrt{3}Bi(-x) = 2x^{\frac{1}{2}}J_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right). \tag{63}$$

Another natural linear combination involving $Ai(-x)$ and $Bi(-x)$ is by considering the positive sign as,

$$3Ai(-x) + \sqrt{3}Bi(-x), \tag{64}$$

which is equal to

$$= 2x^{\frac{1}{2}}\left(\frac{3^{+\frac{1}{3}}}{\pi}\Gamma\left(1-\frac{2}{3}\right)\sin\left(\frac{2\pi}{3}\right)\left(x^{-\frac{1}{2}} - \frac{x^{\frac{5}{2}}}{2.3} + \frac{x^{\frac{11}{2}}}{2.3.5.6} + \ldots\right)\right), \tag{65}$$

From the well known relation of gamma functions $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$, when $z = \frac{2}{3}$ we have $\Gamma\left(\frac{2}{3}\right)\Gamma\left(1-\frac{2}{3}\right) = \frac{\pi}{\sin(\frac{2\pi}{3})}$, thus we can write $\frac{1}{\pi}\sin\left(\frac{2\pi}{3}\right)\Gamma\left(1-\frac{2}{3}\right)$ as $\frac{1}{\Gamma(\frac{2}{3})}$ in Eq.[65]

$$\begin{aligned} &3Ai(-x) + \sqrt{3}Bi(-x) \\ &\quad = 2x^{\frac{1}{2}}\left(3^{+\frac{1}{3}}\frac{1}{\Gamma\left(\frac{2}{3}\right)}\left(x^{-\frac{1}{2}} - \frac{x^{\frac{5}{2}}}{2.3} + \frac{x^{\frac{11}{2}}}{2.3.5.6} + \ldots\right)\right), \end{aligned} \tag{66}$$

$$\begin{aligned} = 2x^{\frac{1}{2}}\Bigg(&\frac{x^{-\frac{1}{2}}}{0!\,3^{-\frac{1}{3}}\,\Gamma\left(\frac{2}{3}\right)} - \frac{x^{-\frac{1}{2}+3}}{1!\,3^{-\frac{1}{3}+2}\,\Gamma\left(\frac{2}{3}+1\right)} \\ &+\frac{x^{-\frac{1}{2}+6}}{2!\,3^{-\frac{1}{3}+4}\,\Gamma\left(\frac{2}{3}+2\right)} + \ldots\Bigg), \end{aligned} \tag{67}$$

$$3Ai(-x)+\sqrt{3}Bi(-x) = 2x^{\frac{1}{2}}\sum_{s=0}^{\infty}\frac{(-1)^s}{\Gamma(s+1)\Gamma\left(s-\frac{1}{3}+1\right)}\left(\frac{\frac{2}{3}x^{\frac{3}{2}}}{2}\right)^{-\frac{1}{3}+2s}. \tag{68}$$

Comparing Eq.[68] with Eq.[9], we get the relation

$$3Ai(-x) + \sqrt{3}Bi(-x) = 2x^{\frac{1}{2}}J_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right). \tag{69}$$

Solving Eq.[63] and Eq.[69] simultaneously for $Ai(-x)$ and $Bi(-x)$ we get

$$Ai(-x) = \frac{\sqrt{x}}{3}\left(J_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)\right), \tag{70}$$

$$Bi(-x) = \sqrt{\frac{x}{3}}\left(-J_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + J_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)\right), \tag{71}$$

Now for $x > 0$, as we had discussed earlier that $Ai(x)$ and $K_\nu(x)$ show an exponentially decreasing behavior as in *Figure* [1] and *Figure* [6] while $Bi(x)$ and $I_\nu(x)$ show exponentially increasing behavior as in *Figure* [1] and *Figure* [5]. So it appears that there must exist a relation between $Ai(x)$ and $K_\nu(x)$, $Bi(x)$ and $I_\nu(x)$. Since by Eq.[16] we know that $K_\nu(x)$ is a function of $I_\nu(x)$ and $I_{-\nu}(x)$, it means $Ai(x)$ is in turn related to $I_\nu(x)$ and $I_{-\nu}(x)$. Thus $Ai(x)$ and $Bi(x)$ will naturally be linear combinations of $I_\nu(x)$ and $I_{-\nu}(x)$ with different coefficients. For $z = x \in \mathbb{R}^+$ Eq.[1] and Eq.[2] becomes

$$Ai(x) = \frac{3^{-\frac{2}{3}}}{\pi}\sin\left(\frac{2\pi}{3}\right)\left(-\frac{2}{3}\right)!\left(1 + \frac{x^3}{2.3} + \frac{x^6}{2.3.5.6} + \ldots\right) + \frac{3^{-\frac{1}{3}}}{\pi}\sin\left(\frac{4\pi}{3}\right)\left(-\frac{1}{3}\right)!\left(x + \frac{x^4}{3.4} + \frac{x^7}{3.4.6.7} + \ldots\right), \quad (72)$$

$$Bi(x) = \sqrt{3}\frac{3^{-\frac{2}{3}}}{\pi}\sin\left(\frac{2\pi}{3}\right)\left(-\frac{2}{3}\right)!\left(1 + \frac{x^3}{2.3} + \frac{x^6}{2.3.5.6} + \ldots\right) - \sqrt{3}\frac{3^{-\frac{1}{3}}}{\pi}\sin\left(\frac{4\pi}{3}\right)\left(-\frac{1}{3}\right)!\left(x + \frac{x^4}{3.4} + \frac{x^7}{3.4.6.7} + \ldots\right). \quad (73)$$

Consider Eq.[72] which can be written as

$$Ai(x) = \frac{1}{\pi}x^{\frac{1}{2}}\left(3^{-\frac{2}{3}}\sin\left(\frac{2\pi}{3}\right)\left(-\frac{2}{3}\right)! \times \left(x^{-\frac{1}{2}} + \frac{x^{\frac{5}{2}}}{2.3} + \frac{x^{\frac{11}{2}}}{2.3.5.6} + \ldots\right) + 3^{-\frac{1}{3}}\sin\left(\frac{4\pi}{3}\right)\left(-\frac{1}{3}\right)!\left(x^{\frac{1}{2}} + \frac{x^{\frac{7}{2}}}{3.4} + \frac{x^{\frac{13}{2}}}{3.4.6.7} + \ldots\right)\right). \quad (74)$$

Using the relation for gamma functions, $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$, replace $\sin\left(\frac{4\pi}{3}\right)$ by $\frac{\pi}{\Gamma(\frac{4}{3})\Gamma(-\frac{1}{3})} = \frac{\pi}{\Gamma(\frac{4}{3})(-\frac{1}{3}-1)!}$ and $\sin\left(\frac{2\pi}{3}\right)$ by $\frac{\pi}{\Gamma(\frac{2}{3})\Gamma(\frac{1}{3})} =$

$\frac{\pi}{\Gamma(\frac{2}{3})(-\frac{2}{3})!}$ in Eq.[74] and after some straightforward algebra

$$\begin{aligned} Ai(x) &= \frac{1}{\pi}x^{\frac{1}{2}}\Bigg(\pi\left(\frac{1}{3}\right)\Bigg(\frac{x^{-\frac{1}{2}}}{0!\;3^{-\frac{1}{3}}\;\Gamma\left(\frac{2}{3}\right)}+\frac{x^{-\frac{1}{2}+3}}{1!\;3^{-\frac{1}{3}+2}\;\Gamma\left(\frac{2}{3}+1\right)} \\ &+\frac{x^{-\frac{1}{2}+6}}{2!\;3^{-\frac{1}{3}+4}\;\Gamma\left(\frac{2}{3}+2\right)}+\ldots\Bigg) \\ &-\pi\left(\frac{1}{3}\right)\Bigg(\frac{x^{\frac{1}{2}}}{0!\;3^{\frac{1}{3}}\Gamma\left(\frac{4}{3}\right)}+\frac{x^{\frac{1}{2}+3}}{1!\;3^{\frac{1}{3}+2}\;\Gamma\left(\frac{4}{3}+1\right)} \\ &+\frac{x^{\frac{1}{2}+6}}{2!\;3^{\frac{1}{3}+4}\;\Gamma\left(\frac{4}{3}+2\right)}+\ldots\Bigg)\Bigg). \end{aligned} \quad (75)$$

Compactly writing, Eq.[75] boils down to

$$\begin{aligned} Ai(x) &= \frac{1}{\pi}x^{\frac{1}{2}}\pi\frac{1}{3}\Bigg(\sum_{s=0}^{\infty}\frac{1}{\Gamma(s+1)\Gamma\left(s+\frac{2}{3}\right)}\left(\frac{x^{-\frac{1}{2}+3s}}{3^{-\frac{1}{3}+2s}}\right) \\ &-\sum_{s=0}^{\infty}\frac{1}{\Gamma(s+1)\Gamma\left(s+\frac{4}{3}\right)}\left(\frac{x^{\frac{1}{2}+3s}}{3^{\frac{1}{3}+2s}}\right)\Bigg), \end{aligned} \quad (76)$$

doing a little bit of simple manipulations, Eq.[76] can be recasted as

$$\begin{aligned} Ai(x) &= \frac{1}{\pi}x^{\frac{1}{2}}\;\pi\;\frac{1}{3}\Bigg(\sum_{s=0}^{\infty}\frac{1}{\Gamma(s+1)\Gamma\left(s-\frac{1}{3}+1\right)}\left(\frac{\frac{2}{3}x^{\frac{3}{2}}}{2}\right)^{-\frac{1}{3}+2s} \\ &-\sum_{s=0}^{\infty}\frac{1}{\Gamma(s+1)\Gamma\left(s+\frac{1}{3}+1\right)}\left(\frac{\frac{2}{3}x^{\frac{3}{2}}}{2}\right)^{\frac{1}{3}+2s}\Bigg). \end{aligned} \quad (77)$$

On comparing Eq.[77] with Eq.[15], we can easily see that

$$Ai(x)=\frac{1}{\pi}\sqrt{\frac{x}{3}}\;\frac{\pi}{2}\left(\frac{I_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)-I_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)}{\sin\left(\frac{\pi}{3}\right)}\right). \quad (78)$$

From the definition of Macdonald function as given by Eq.[16], we finally get the required relation

$$Ai(x)=\frac{1}{\pi}\sqrt{\frac{x}{3}}K_{\pm\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right). \quad (79)$$

Now for $Bi(x)$, consider Eq.[73] and after trivial simplifications we get

$$Bi(x) = \sqrt{3}\frac{1}{\pi}x^{\frac{1}{2}}\Bigg(3^{-\frac{2}{3}}\sin\left(\frac{2\pi}{3}\right)\left(-\frac{2}{3}\right)! \times \left(x^{-\frac{1}{2}} + \frac{x^{\frac{5}{2}}}{2.3} + \frac{x^{\frac{11}{2}}}{2.3.5.6} + \ldots\right) -3^{-\frac{1}{3}}\sin\left(\frac{4\pi}{3}\right)\left(-\frac{1}{3}\right)!\left(x^{\frac{1}{2}} + \frac{x^{\frac{7}{2}}}{3.4} + \frac{x^{\frac{13}{2}}}{3.4.6.7} + \ldots\right)\Bigg). \quad (80)$$

Using the relation for gamma functions, $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$, replace $\sin\left(\frac{4\pi}{3}\right)$ by $\frac{\pi}{\Gamma(\frac{4}{3})\Gamma(-\frac{1}{3})} = \frac{\pi}{\Gamma(\frac{4}{3})(-\frac{1}{3}-1)!}$ and $\sin\left(\frac{2\pi}{3}\right)$ by $\frac{\pi}{\Gamma(\frac{2}{3})\Gamma(\frac{1}{3})} = \frac{\pi}{\Gamma(\frac{2}{3})(-\frac{2}{3})!}$ in Eq.[80] and after some straightforward algebra, we get

$$\begin{aligned} Bi(x) = \sqrt{3}\frac{1}{\pi}x^{\frac{1}{2}}\Bigg(&\pi\left(\frac{1}{3}\right)\Bigg(\frac{x^{-\frac{1}{2}}}{3^{-\frac{1}{3}}\ 0!\ \Gamma\left(\frac{2}{3}\right)} + \frac{x^{\frac{5}{2}}}{3^{-\frac{1}{3}+2}\ 1!\ \frac{2}{3}\ \Gamma\left(\frac{2}{3}\right)} \\ &+\frac{x^{\frac{11}{2}}}{3^{-\frac{1}{3}+4}\ 2!\ \left(1+\frac{2}{3}\right)\ \frac{2}{3}\Gamma\left(\frac{2}{3}\right)} + \ldots\Bigg) \\ &+\pi\left(\frac{1}{3}\right)\Bigg(\frac{x^{\frac{1}{2}}}{3^{\frac{1}{3}}\ 0!\ \Gamma\left(\frac{4}{3}\right)} + \frac{x^{\frac{7}{2}}}{3^{\frac{1}{3}+2}\ 1!\ \frac{4}{3}\ \Gamma\left(\frac{4}{3}\right)} \\ &+\frac{x^{\frac{13}{2}}}{3^{\frac{1}{3}+4}\ 2!\ \left(1+\frac{4}{3}\right)\ \frac{4}{3}\Gamma\left(\frac{4}{3}\right)} + \ldots\Bigg)\Bigg). \end{aligned} \quad (81)$$

Many physicists like Albert Hofmann [17] while working on synchrotron radiation have come across electric field profiles that can be described as Airy functions and their derivatives. There also exist relations between these derivatives of Airy functions and Bessel functions. It is at times convenient to express the electric field as Bessel functions.

Compactly writing, Eq.[81] boils down to

$$\begin{aligned} Bi(x) = \sqrt{3}\frac{1}{\pi}x^{\frac{1}{2}}\ \pi\ \frac{1}{3}\Bigg(&\sum_{s=0}^{\infty}\frac{1}{\Gamma(s+1)\Gamma\left(s-\frac{1}{3}+1\right)}\left(\frac{\frac{2}{3}x^{\frac{3}{2}}}{2}\right)^{-\frac{1}{3}+2s} \\ &+\sum_{s=0}^{\infty}\frac{1}{\Gamma(s+1)\Gamma\left(s+\frac{1}{3}+1\right)}\left(\frac{\frac{2}{3}x^{\frac{3}{2}}}{2}\right)^{\frac{1}{3}+2s}\Bigg). \end{aligned} \quad (82)$$

On comparing Eq.[82] with Eq.[15], we can see that

$$Bi(x) = \sqrt{\frac{x}{3}}\left(I_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + I_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)\right). \quad (83)$$

Thus we have established the relationship between Airy functions and the family of Bessel functions using the convergent series method.

5. Appendix : Derivation of Higher Order Asymptotic Series of Bessel Functions

For the sake of completeness, for igniting the interests in the serious reader, we give below a clear derivation of the asymptotic forms of $H_\nu^{(1)}(x)$, $H_\nu^{(2)}(x)$ and $J_\nu(x)$ for $x \in \mathbb{R}^+$ [4], [14]. The appendix may be quite useful for postgraduste students and researchers who take courses on nuclear and particle physics or in any other area where these functions occur. Here we derive the most general asymptotic forms of higher order of the Bessel function $J_\nu(x)$. To this end we consider the well known integral forms of Hankel functions $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$ as given below [7].

$$H_\nu^{(1)}(x) = \sqrt{\frac{2}{\pi x}} \frac{e^{i\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)}}{\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^{+\infty} e^{-u} u^{\nu-\frac{1}{2}} \left(1+\frac{ui}{2x}\right)^{\nu-\frac{1}{2}} du. \quad (84)$$

From Eq.[13], we have

$$H_\nu^{(2)}(x) = \sqrt{\frac{2}{\pi x}} \frac{e^{-i\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)}}{\Gamma\left(\nu+\frac{1}{2}\right)} \int_0^{+\infty} e^{-u} u^{\nu-\frac{1}{2}} \left(1-\frac{ui}{2x}\right)^{\nu-\frac{1}{2}} du. \quad (85)$$

Define a function $f(x)$ such that Eq.[84] becomes

$$H_\nu^{(1)}(x) = \sqrt{\frac{2}{\pi x}} \frac{e^{i\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)}}{\Gamma\left(\nu+\frac{1}{2}\right)} f\left(\frac{1}{x}\right). \quad (86)$$

Thus

$$f(x) = \int_0^{+\infty} e^{-u} u^{\nu-\frac{1}{2}} \left(1+\frac{uxi}{2}\right)^{\nu-\frac{1}{2}} du. \quad (87)$$

Differentiating Eq.[87] with respect to x, we get

$$\frac{d}{dx} f(x) = \frac{d}{dx} \int_0^{+\infty} e^{-u} u^{\nu-\frac{1}{2}} \left(1+\frac{uxi}{2}\right)^{\nu-\frac{1}{2}} du. \quad (88)$$

Since the integrand $e^{-u}u^{\nu-\frac{1}{2}}\left(1+\frac{uxi}{2}\right)^{\nu-\frac{1}{2}}$ is continuous in the range $u \in [0,+\infty)$, we can differentiate with respect to x under the integral sign of Eq.[87]. We denote the n^{th} derivative of $f(x)$ as $f^n(x)$, where n starts from 1 and is given as

$$f^n(x) = \frac{(2\nu-1)(2\nu-3)\ldots(2\nu-(2n-1))}{2^n}\left(\frac{i}{2}\right)^n \times \int_0^{+\infty} e^{-u}\left(u^{\nu-\frac{1}{2}+n}\right)\left(1+\frac{uxi}{2}\right)^{\nu-\frac{2n+1}{2}} du. \quad (89)$$

From Eq.[89] $f^n(0)$ can be computed as

$$f^n(0) = \frac{(2\nu-1)(2\nu-3)\ldots(2\nu-(2n-1))}{2^n}\times \left(\frac{i}{2}\right)^n \int_0^{+\infty} e^{-u}\left(u^{\nu+n+\frac{1}{2}-1}\right)du, \quad (90)$$

the integral in Eq.[90] is nothing but the gamma function $\Gamma\left(\nu+n+\frac{1}{2}\right)$, so

$$f^n(0) = \frac{(2\nu-1)(2\nu-3)\ldots(2\nu-(2n-1))}{2^n}\times \left(\frac{i}{2}\right)^n \Gamma\left(\nu+n+\frac{1}{2}\right), \quad (91)$$

we know that $\Gamma(r+1) = r\,\Gamma(r)$ for some r, then

$$\Gamma\left(\nu+n+\frac{1}{2}\right) = \left(\nu+n+\frac{1}{2}-1\right)\Gamma\left(\nu+n+\frac{1}{2}-1\right), \quad (92)$$

$$= \left(\nu+n+\frac{1}{2}-1\right)\left(\nu+n+\frac{1}{2}-2\right)\Gamma\left(\nu+n+\frac{1}{2}-1-1\right), \quad (93)$$

$$= \left(\nu+n+\frac{1}{2}-1\right)\ldots\left(\nu+n+\frac{1}{2}-(n-1)\right) \times\left(\nu+n+\frac{1}{2}-n\right)\Gamma\left(\nu+n+\frac{1}{2}-n\right) \quad (94)$$

$$= \left(\frac{2\nu+2n+1-2}{2}\right)\ldots\left(\frac{2\nu+2n+1-2n+2}{2}\right) \times\left(\frac{2\nu+2n+1-2n}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right) \quad (95)$$

$$= \left(\frac{2\nu+2n-1}{2}\right)\dots\left(\frac{2\nu+3}{2}\right)\left(\frac{2\nu+1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right) \quad (96)$$

$$\Gamma\left(\nu+n+\frac{1}{2}\right) = \frac{(2\nu+1)(2\nu+3)\dots(2\nu+2n-1)}{2^n}\Gamma\left(\nu+\frac{1}{2}\right). \quad (97)$$

For $\nu = 0$, Eq.[97] gets reduced to the famous Legendre duplication formula.

Now substituting Eq.[97] in Eq.[91], we get

$$f^n(0) = \frac{(2\nu-1)(2\nu-3)\dots(2\nu-(2n-1))}{2^n}\left(\frac{i}{2}\right)^n \times \frac{(2\nu+1)(2\nu+3)\dots(2\nu+2n-1)}{2^n}\Gamma\left(\nu+\frac{1}{2}\right), \quad (98)$$

$$= (i)^n \frac{(4\nu^2-1^2)(4\nu^2-3^2)\dots(4\nu^2-(2n-1)^2)}{8^n}\Gamma\left(\nu+\frac{1}{2}\right). \quad (99)$$

We can write $f\left(\frac{1}{x}\right)$ in terms of Maclaurin series as,

$$f\left(\frac{1}{x}\right) = f(0) + \frac{f^1(0)}{1!x} + \frac{f^2(0)}{2!x^2} + \dots + \frac{f^n(0)}{n!x^n} + \dots +, \quad (100)$$

thus using Eq.[87] (to find $f(0)$) and Eq.[99] in the above we get

$$f\left(\frac{1}{x}\right) = \Gamma\left(\nu+\frac{1}{2}\right)\left(1 + i\frac{(4\nu^2-1^2)}{8.\ 1!\ x} - \frac{(4\nu^2-1^2)(4\nu^2-3^2)}{8^2.\ 2!\ x^2} - i\frac{(4\nu^2-1^2)(4\nu^2-3^2)(4\nu^2-5^2)}{8^3.\ 3!\ x^3} + \dots\right). \quad (101)$$

Substituting Eq.[101] in Eq.[86], we get the final asymptotic forms for $H_\nu^{(1)}(x)$ as

$$H_\nu^{(1)}(x) = \sqrt{\frac{2}{\pi x}}\, e^{i\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)}\left(1 + i\frac{(4\nu^2-1^2)}{8.\ 1!\ x} - \frac{(4\nu^2-1^2)(4\nu^2-3^2)}{8^2.\ 2!\ x^2} - i\frac{(4\nu^2-1^2)(4\nu^2-3^2)(4\nu^2-5^2)}{8^3.\ 3!\ x^3} + \dots\right). \quad (102)$$

Using Eq.[13], the final asymptotic form of $H_\nu^{(2)}(x)$ as

$$H_\nu^{(2)}(x) = \sqrt{\frac{2}{\pi x}}\, e^{-i\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)}\left(1 - i\frac{(4\nu^2-1^2)}{8.\ 1!\ x} - \frac{(4\nu^2-1^2)(4\nu^2-3^2)}{8^2.\ 2!\ x^2} + i\frac{(4\nu^2-1^2)(4\nu^2-3^2)(4\nu^2-5^2)}{8^3.\ 3!\ x^3} + \dots\right). \quad (103)$$

Now from Eq.[14], Eq.[102] and Eq.[103], we obtain the general asymptotic form of Bessel function of ν^{th} order as

$$\begin{aligned} J_\nu(x) &= \sqrt{\frac{2}{\pi x}}\Bigg(\cos\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right) \\ &\times\left(1-\frac{(4\nu^2-1^2)(4\nu^2-3^2)}{8^2.\ 2!\ x^2}+\ldots\right) \\ &-\sin\left(x-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)\left(\frac{(4\nu^2-1^2)}{8.\ 1!\ x}\right. \\ &\left.-\frac{(4\nu^2-1^2)(4\nu^2-3^2)(4\nu^2-5^2)}{8^3.\ 3!\ x^3}+\ldots\right)\Bigg). \end{aligned} \quad (104)$$

Not only the aforesaid asymptotic form of the Bessel function is important but also the asymptotic forms of modified Bessel function and the Macdonald function are important components of our method. The asymptotic form of $I_\nu(x)$ is obtained by exploiting the relationship between $I_\nu(x)$ and the Hankel functions given by Eq.[17]. To see this, consider the asymptotic series of $H_\nu^{(1)}(e^{i\frac{\pi}{2}}x)$ and $H_\nu^{(2)}(e^{i\frac{\pi}{2}}x)$ from Eq.[102] and Eq.[103] gives

$$\begin{aligned} I_\nu(x) &= \frac{e^x}{\sqrt{2\pi x}}\left(1-\frac{(4\nu^2-1^2)}{8.\ 1!\ x}+\frac{(4\nu^2-1^2)(4\nu^2-3^2)}{8^2.\ 2!\ x^2}\right. \\ &\left.-\frac{(4\nu^2-1^2)(4\nu^2-3^2)(4\nu^2-5^2)}{8^3.\ 3!\ x^3}+\ldots\right) \\ &-i\frac{e^{-x}}{\sqrt{2\pi x}}e^{-i\nu\pi}\left(1+\frac{(4\nu^2-1^2)}{8.\ 1!\ x}+\frac{(4\nu^2-1^2)(4\nu^2-3^2)}{8^2.\ 2!\ x^2}\right. \\ &\left.+\frac{(4\nu^2-1^2)(4\nu^2-3^2)(4\nu^2-5^2)}{8^3.\ 3!\ x^3}+\ldots\right). \end{aligned} \quad (105)$$

Since from Eq.[15] we can see that for $x \in \mathbb{R}^+$, $I_\nu(x)$ is real and also the second term is small as compared to the first. Thus the second term in Eq.[105] needs to be discarded [15], [16] to give the asymptotic series of $I_\nu(x)$ as

$$\begin{aligned} I_\nu(x) &= \frac{e^x}{\sqrt{2\pi x}}\left(1-\frac{(4\nu^2-1^2)}{8.\ 1!\ x}+\frac{(4\nu^2-1^2)(4\nu^2-3^2)}{8^2.\ 2!\ x^2}\right. \\ &\left.-\frac{(4\nu^2-1^2)(4\nu^2-3^2)(4\nu^2-5^2)}{8^3.\ 3!\ x^3}+\ldots\right). \end{aligned} \quad (106)$$

In *Figure* [5], as $x \to +\infty$ only the first term of Eq.[105] contributes considerably and the second term vanishes as $\frac{e^{-x}}{\sqrt{2\pi x}} \to 0$. Also $I_2(x)$ is real for $x \in \mathbb{R}^+$. In order to obtain the asymptotic series of $K_\nu(x)$, from Eq.[18], we have

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x}\left(1 + \frac{(4\nu^2 - 1^2)}{8.\ 1!\ x} + \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)}{8^2.\ 2!\ x^2} + \frac{(4\nu^2 - 1^2)(4\nu^2 - 3^2)(4\nu^2 - 5^2)}{8^3.\ 3!\ x^3} + \dots\right). \tag{107}$$

6. Summary and Conclusions

In this paper we have developed the relationship between the Airy functions and the family of Bessel functions of the real argument. Earlier methods to achieve this were not that much transparent for the beginner in the subject. In this paper, we have given two simple methods, first using the general asymptotic series and second, using the convergent series. The relevant properties of Airy and the family of Bessel functions were discussed briefly in Sec.2 and Sec.3. The actual relationship between Airy and the family of Bessel functions were derived in Sec.4. For the sake of completeness, a complete appendix is devoted to the derivation of the general asymptotic series of the family of Bessel functions for the curious reader. Further, the results obtained in this paper were studied numerically which gave us deeper insights and thereby reinforced our claims.

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