The Mathematical Legacy of Jacques Tits*

B. Sury

The 'Erlangen Program' of Felix Klein worked by 'reducing' problems in geometry to the study of their symmetry groups—thereby algebraizing geometry. Jacques Tits's work goes in the opposite direction—he made fundamental contributions to the abstract theory of groups via geometric methods. His geometric techniques apply to not only finite groups, but also to rather diverse situations such as groups defined over the p-adic numbers, and to the so-called arithmetic groups etc. Tits's ideas have enriched many of the important advances in group theory and geometry in the last six decades. He designed the theory of so-called 'buildings' which incorporates geometrically the algebraic structure of linear groups. Amazingly, these ideas have also led to applications in subjects like the study of Riemannian manifolds of higher rank that are seemingly remote from the original developments.

Sury is with the Indian Statistical Institute in Bangalore since 1999. He learnt some of the rich mathematics talked about in this article at the Tata Institute in Bombay during 1981–1999. He was the National co-ordinator for the Mathematical Olympiad Programme in the country and likes to communicate with young students at the school as well as college levels. He introduces this article with the limerick:
There once was a Belgian
who talked of pairs BN.
He was the king
of many a building.
We can only say 'Très Bien!'

Keywords
Groups, buildings, geometry, immeubles, Tits group, apartments.

Introduction

One is hard put to attempt to describe the decisive impact that Jacques Tits's ideas have had in the mathematical landscape over the last six decades. Nevertheless, we are emboldened to try and communicate some of the legacy of Tits's everlasting work.

During one of his lectures, Tits said: "It has to be expected that, in my lectures, geometry will often take its revenge from the Erlangen program, the theory of groups serving as a pretext this time."

Remarkably, Tits' work also yielded some key applications of group theory to geometry. Some of the principal subjects that were highly influenced by his ideas include the theory of Lie

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groups and linear algebraic groups, their infinite-dimensional versions like Kac–Moody groups (which are of interest to physicists as well), and combinatorial aspects like Coxeter groups (also of interest to theoretical computer scientists).

Before we go on to give a brief description of the manifold contributions of Tits, we point out that notions such as 'buildings' invented for convenience by mathematicians are superficial analogues of these objects in daily life. They cannot be taken literally. For instance, in the mathematical theory of buildings, there are apartments, walls, chambers etc. but a chamber may belong to two different apartments!

We start with a brief peek into at his early life and background.

Early Life, French Citizenship and Honors

Jacques Tits was born in Uccle, a place in the outskirts of Brussels, Belgium on August 12, 1930 to Léon Tits and Louisa André. His mother Louisa was a piano teacher and his father Léon was a mathematician. Jacques was something of a child prodigy and was allowed to skip many years in school. As his father died when Jacques was 13, he took to tutoring students several years senior to him, teaching them differential and integral calculus. At the age of 14, he gained admission to the Free University of Brussels when he passed the university's entrance examination. He earned his doctorate there at the age of 20, under the direction of Paul Libois—his thesis was titled 'Généralisation des groupes projectifs basés sur la notion de transitivité.' He produced a complete classification (this is the only place where we state a result using group theoretic language without explanation) of the finite sharply n-transitive groups for all $n \geq 2$. His PhD thesis includes a novel geometry of projective and affine spaces (over a field) in terms of a transformation group from which the geometry is extracted and treated infinite groups as well. Indeed, two of the 'gems' from his thesis (obtained already at the age of 17) are the following assertions (see Reference **1**):

Sharply 4*-transitive permutation groups are necessarily finite.*

Let G be a permutation group that is triply transitive on a set S. Let $p \neq q$ in S, and assume that the G-stabilizer of p and q is abelian. Then G must be the projective linear group of 2×2 matrices over some field, acting on the projective line.

In 1956, Jacques married a historian, Marie-Jeanne Dieuaide when he was an assistant at the university. Since 1956, they had been inseparable until the passing of Jacques Tits on the 5th of December, 2021; they had no children. Until 1964, Tits was a Professor at the Free University in Brussels. Tits's list of doctoral students includes several well-known names such as Francis Buekenhout, Jens Jantzen, Guy Rousseau, Jean-Pierre Tignol and Olivier Mathieu.

Buekenhout has made an interesting remark about Tits's way of working (see Reference **1**). He says that unlike most mathematicians, Tits constructed all of his mathematics in his head, without using paper and pen, including the proofs of difficult results!

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Tits accepted a position at the University of Bonn in 1963 where he remained for a decade. In 1973, he moved to Collège de France in Paris as the Chair of Group Theory. At that time, one had to have French citizenship to teach there and Tits became a French citizen in 1974. He had to renounce Belgian nationality then as the Belgian rules did not allow dual nationality. He remained in Collège de France until he retired in the year 2000 as an Emeritus professor.

Throughout the last six decades, Tits had been an essential part of the mathematical landscape in myriad ways. One of the most important contributions to the Bourbaki volumes is due to Tits, when he shared the contents of his work 'Groupes et Géométries de Coxeter'—a work in which he generously acknowledged H.S.M. Coxeter's pioneering ideas. This appeared in Chapter 4 of Bourbaki's *Groupes et algèbres de Lie* in the 1960's and Tits's paper was not published until 2001! In fact, the nomenclature 'Coxeter group', 'Coxeter Graph' which are ubiquitous in the theory of Lie and algebraic groups were introduced by Tits. Some of the awards and honors that were accorded to Tits include the Wettrems Prize

of the Royal Belgium Academy of Sciences in 1958, the Grand Prix of the French Academy of Sciences in 1976, the Wolf Prize in 1993, the Cantor medal in 1996, and the Abel Prize (along with John Thompson) in 2008. Tits remained the Editor in Chief of the mathematical publications of the IHES from 1980 to 1999. He also founded the premier journal *Inventiones Mathematicae* and served as an editor during 1966–1975. He was an invited speaker in the ICMs (International Congress of Mathematicians) in 1962, 1970 and 1974. He was also a member of the Fields Medal committees in 1978 and 1994. When he read the Fields Medal laudatio for Margulis in 1978, Margulis had not been allowed by the Soviet authorities to attend the Congress. Tits had boldly remarked then:

"*This is probably neither the time nor the place to start a polemic. However, I cannot but express my deep disappointment – no doubt shared by many people here – in the absence of Margulis from this ceremony. In view of the symbolic meaning of this city of Helsinki, I had indeed grounds to hope that I would have a chance at last to meet a mathematician whom I know only through his work and for whom I have the greatest respect and admiration.*"

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– Jacques Tits

It is remarkable that Tits's address was delivered in Finlandia Hall, where the 1975 Helsinki Agreements dealing with security and cooperation in Europe were concluded. Margulis was finally able to visit and stay at the University of Bonn for three months in 1979. In a small ceremony there, Tits handed over the Fields medal to Margulis.

The Birth of 'Immeubles'

Galois (1811–1832) is considered the father of modern group theory. By the 1870's, his work was much better understood through the works of Jordan. This was followed by Lie's developments of the theory of continuous transformation groups (now called Lie groups). Elie Cartan and Killing had classified the simple Lie groups by late 19th century. At this point, it was realized by some

researchers (especially Dickson) that these groups defined over real and complex numbers have analogues over other fields such as the rational numbers or even over finite fields. There was a period of stangancy—especially, in the understanding of the so-called exceptional groups of type E, F, G—until after the second world war. The year 1955 saw the appearance of the seminal work of Claude Chevalley which not only gave a complete classification of these groups but proved, at one stroke, the 'simplicity' of these groups. During this period, Tits had been developing his theory of buildings on his own, especially motivated by his wish to understand the exceptional groups to which he was introduced by Hopf during a visit to Zürich. The simple Lie groups of classical type were known to have geometric interpretations in terms of projective geometry. The exceptional groups proved elusive for a long time. In 1954, Tits made a profound geometric construction in terms of the group and its so-called parabolic subgroups. This world of incidence geometries is referred to by Jeremy Gray as "Tits created a world out of nothing." These constructions fit in perfectly with the projective spaces and were pre-cursors to the theory of buildings. Tits constructed what is now known as the "Tits group" which is an addition to the 26 so-called sporadic groups and, is often called the 27th sporadic group. The first computer-free proof of the existence of the simple group called the Hall–Janko group was given in an elegant geometric manner by Tits using an extension of the generalized hexagon of order 2. Tits came out with a full-fledged theory in 1974 that not only gave a complete understanding of the structure of these groups but went much beyond its originally envisioned aims. Tits's theory of (what is now known as) Tits buildings became a unifying principle that is central to many developments in group theory. In that sense, Tits was a *'theory builder'*—pun intended!

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Before we give a brief, informal sketch of the theory of buildings, we recall what Tits himself had to say (on the occasion of his winning the Abel Prize along with John Thompson) about his motivations in developing these notions (see References **1**, **3**):

"I studied these objects because I wanted to understand these

exceptional Lie groups geometrically. In fact, I came to mathematics through projective geometry; what I knew about was projective geometry. In projective geometry you have points, lines, and so on. When I started studying exceptional groups I sort of looked for objects of the same sort. For instance, I discovered – or somebody else discovered, actually – that the group E_6 is the collineation group of the octonion projective plane. And a little bit later, I found some automatic way of proving such results, starting from the group to reconstruct the projective plane. I could use this procedure to give geometric interpretations of the other exceptional groups, e.g., E_7 and E_8. That was really my starting point. Then I tried to make an abstract construction of these geometries. In this construction I used terms like skeletons, for instance, and what became apartments were called skeletons at the time." . . .

"I should say that the terminology like buildings, apartments, and so on is not mine. I discovered these things, but it was Bourbaki who gave them these names. They wrote about my work and found that my terminology was a shambles. They put it in some order, and this is how the notions like apartments and so on arose."

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In what follows, we use a few definitions that come from basic algebraic topology. For those who are not familiar with these notions, it suffices to say that points, line segments, triangles, tetrahedra and their higher dimensional analogues called 'regular polytopes' are the 'simplices' being spoken of below.

A "building" of dimension n (called 'immeuble' in French) is a simplicial complex Δ that is obtained by gluing together certain subcomplexes called apartments which satisfy certain properties (see References **4**, **5**, **6**).

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The idea is to encode all the information in constructing the building in terms only of adjacency properties of simplices of maximal

dimension. More precisely, the properties are:
(a) for any $r < n$, every r-simplex is contained in at least three chambers (another name for n-simplices);
(b) every pair of simplices lies in a common apartment;
(c) every $(n-1)$-simplex in an apartment is contained in exactly two adjacent chambers;
(d) any two apartments containing two simplices are isomorphic, with an isomorphism that fixes the simplices.

Every apartment has associated to it, a 'Coexeter group'; in this manner, apartments are also 'Coxeter complexes' (see **3.**). In fact, for any two chambers intersecting in a 'panel' (an $(n-1)$-dimensional simplex), there is a simplicial automorphism of order 2 (also called a reflection) that carries one chamber to the other while fixing all the common points. The finite set S of reflections generate a Coxeter group (W, S). For any two of the generators $s_i \neq s_j$ in S, there is a relation $(s_i s_j)^{m_{ij}} = 1$ where $2 \leq m_{ij} = m_{ji} \leq \infty$. The group of all permutations of a finite set, the group of isometries of the plane generated by the affine reflections with respect to the sides of an equilateral triangle and—more generally—the Weyl groups of Lie groups—can be viewed as Coxeter groups. Given a Coxeter group (W, S), subsets J of S give 'standard' subgroups W_J of W generated by the elements in J. The partially ordered set of standard cosets, where E is said to be a face of F (and write $E \leq F$) if $E \supseteq F$ is the Coxeter complex associated to (W, S) and is called its geometric realization.
The apartment whose Coxeter group is (W, S) is then the geometric realization of (W, S).

When W is a finite group, the corresponding apartment is topologically a sphere; so, the corresponding building is said to be spherical. If we consider the Coxeter group consisting of the group of isometries of the plane generated by the affine reflections with respect to the sides of an equilateral triangle, then the corresponding Coxeter complex is simply the whole plane tiled by equilateral triangles.

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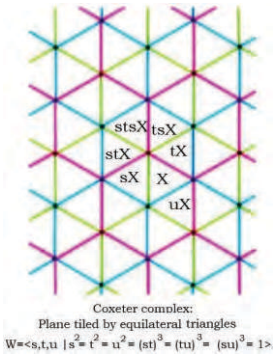
The 'flag' complex corresponding to (the projective space of) a finite-dimensional vector space V over any field is an example of a

building. The simplices are chains of proper, non-zero subspaces

$$0 \subsetneq V_1 \subsetneq V_2 \cdots \subsetneq V_r \subsetneq V.$$

The chambers (maximal simplices) are those chains as above with $r = dim(V) - 1$; so, necessarily, $dim(V_i) = i$.

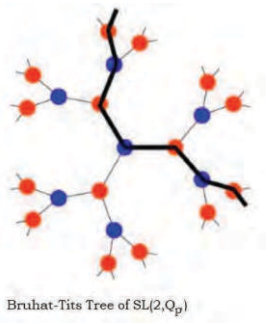


Coxeter complex:
Plane tiled by equilateral triangles
W=<s,t,u | $s^2 = t^2 = u^2 = (st)^3 = (tu)^3 = (su)^3 = 1$>

If we have a linear group like the group $G := SL_n(k)$ of $n \times n$ matrices of determinant 1 over a field k, there is a natural building associated to it. This association depends on the subgroup structure of the group. It depends on the subgroup B of upper triangular matrices in G, the subgroup N of monomial matrices in G (those with exactly one non-zero entry in each row and each column), and the subgroup T of diagonal matrices in G. The quotient group $W := N/T$ is called the Weyl group of G, and comes with a set S of reflections which generate it; (W, S) is naturally a Coxeter group. The building corresponding to G is called its Tits building and the group G acts on its building by simplicial automorphisms. It is possible to obtain group theoretic consequences using the action of G and its subgroups. This 4-tuple (G, B, N, S) is formally now known as a Tits system. For instance, the formalism allows one to obtain for free the simplicity (that non-existence of non-trivial normal subgroups) theorem:
For any field k (with a few exceptions of small finite cardinality), and any simple linear algebraic group G defined, and isotropic over k (that is, it has a subgroup isomorphic over k to a group of diagonal matrices), the abstract group (the uninitiated may think of groups like $SL_n(k), SO_n(k), Sp_n(k)$ etc.) is, modulo its finite center, a simple group.

Although Tits's motivation to introduce buildings was to study the structure of linear algebraic groups—especially, the elusive "exceptional groups" (see References **5**, **6**), there are buildings which are not associated with groups. However, in 1974, Tits proved the remarkable theorem that (thick, irreducible, spherical) buildings of rank > 2 must come from groups.
After the initial introduction of these Tits systems, it was realized that there are more general Tits systems arising when we study groups of matrices defined over other fields like p-adic

fields. These "Bruhat–Tits buildings" have led to several far-reaching consequences. For instance, the Bruhat–Tits building for the group $SL_2(\mathbb{Q}_p)$ of 2×2 matrices of determinant 1 with entries in the field $\mathbb{Q}_p$ of p-adic numbers is 1-dimensional; that is, it is a tree—see the figure.



Using the Bruhat–Tits tree, one can prove:
Every discrete subgroup of $SL_2(\mathbb{Q}_p)$ which is torsion-free (that is, has no nontrivial elements of finite order) must be a free group.

The Bruhat–Tits building for p-adic groups is the analogue of the Riemannian symmetric spaces over the reals. One can use the Bruhat–Tits building to prove conjugacy theorems or results on open compact subgroups as fixed point theorems. For example, a theorem of Bruhat, Tits and Rousseau shows that the reductive groups G over p-adic fields k for which $G(k)$ is compact are precisely those which do not possess any k-split tori (subgroups isomorphic over k to a group of diagonal matrices).

More remarkably, Lubotzky, Philips and Sarnak made crucial use of the Bruhat–Tits tree of $SL_2(\mathbb{Q}_p)$ to construct expander graphs called Ramanujan graphs; higher dimensional Bruhat–Tits buildings have been later used to construct higher dimensional expanders.

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It is natural for mathematicians to look at the p-adic numbers (a nonarchimedean construct) even to understand the real or the complex picture. In fact, one studies all these together (sometimes known as the world of adelic numbers). If this is counter-intuitive, it may be appropriate to recall the following metaphysical quote by Yuri Manin in this context about physics (see page 297 of Reference **7**):

"*On the fundamental level our world is neither real nor p-adic; it is adelic. For some reasons, reflecting the physical nature of our kind of living matter (e.g. the fact that we are built of massive particles), we tend to project the adelic picture onto its real side. We can equally well spiritually project it upon its non-Archimedean side and calculate most important things arithmetically. The relations between "real" and "arithmetical" pictures of the world*

is that of complementarity, like the relation between conjugate observables in quantum mechanics."

Other applications of the theory of buildings include those to rigidity and super-rigidity theorems in discrete subgroups of Lie groups, representation theory, construction of infinite-dimensional Kac–Moody groups where 'twin buildings' play a role, study of hyperbolic groups and manifolds of negative curvature.

The Tits Alternative

An amazing general result on groups of matrices shows a dichotomy that holds; viz.
A finitely generated group of matrices is either virtually solvable (has a solvable subgroup of finite index) or contains a free, nonabelian group.
A group of matrices is solvable means roughly that the matrices commute at a higher level. A free nonabelian group is in a sense the other extreme from a solvable group.

A free subgroup of G is roughly one which consists of all 'words' in an alphabet made up of some of the elements of G and their inverses, and where the only simplifications of words allowed are consequences of $xx^{-1} = 1$ and $x^{-1}x = 1$.

Tits's result was a logical culmination of the ideas of such a dichotomy holding good. This has led to other analogous situations where such a dichotomy has been observed, and gives a pathway to study geometric and other questions involving linear groups. Interestingly, instead of considering groups of matrices with entries in a field, if we consider the entries from a skew field, Tits's alternative can fail.

For a group with a finite set of generators, it is an important problem to determine the growth behaviour in terms of n for the number of words of length n. The motivation is that the rate of volume growth of the universal cover of a compact Riemannian manifold coincides with the rate of growth of the fundamental group. This also has applications to the curvature of the manifold. An imme-

diate consequence of the above theorem of Tits is that a finitely generated linear group either has exponential growth or has polynomial growth. A striking result in the converse direction is Gromov's theorem on word growth which shows that if the number of words of length n grows like a polynomial in n, then the group is virtually nilpotent.

An immediate consequence of the above theorem of Tits is that a finitely generated linear group either has exponential growth or has polynomial growth.

Here is a word about how one goes about proving a subgroup is free on an alphabet set. This is referred to usually by the expressive phrase 'Ping-Pong lemma'—the lemma is originally due to Klein.

If two subgroups A, B act on the union of two disjoint sets S, T and a point p outside them in such a way that nontrivial elements of A carry $S \cup \{p\}$ to T and those of B carry $T \cup \{p\}$ to S. Then, no 'word' $a_1 b_1 a_2 b_2 \cdots a_r b_r$ involving elements of both A, B can be the identity element. The reason is that the point p is carried to either S or T by the last syllable of the word after which the other syllables play 'ping-pong' between the sets S and T and the point never reaches back to p. The subgroups A, B then generate their 'free product'. A free group is just the free product of infinite cyclic groups. This method can be used for instance to prove that the two matrices

Klein's ping-pong lemma is often used to prove certain groups are free.

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

generate a free group of rank 2—by defining appropriate subsets of the plane where they play ping-pong. The same proof works when we have matrices where 2 is replaced by a complex number w such that $|w| \geq 2$ while this is not necessarily true when $|w| < 2$.

Borel–Tits, Bruhat–Tits and the Tits Classification

Some of Tits's important joint work with Armand Borel deals with the structure of the abstract groups $G(k)$ and abstract homomorphisms between such groups when G is a linear algebraic group. Tits mentions that this happened in the following curious fashion. He says that both Borel and Tits independently reached the same conclusions often but used somewhat different methods. So, it made sense to combine and write joint papers. The celebrated papers on Borel–Tits structure theory for reductive groups over arbitrary fields are among the most quoted by mathematicians of all hues. What Borel arrived at using the internal structure was reached at by Tits by his geometric intuition.

The Kneser–Tits conjecture asserts that the abstract group $G(k)$ (as soon as G is k-isotropic; that is, contains a diagonal subgroup over k) equals the normal subgroup generated by the unipotent radicals of minimal k-parabolic subgroups. This reduces in many cases to the simplicity statement for the abstract groups $G(k)$. The Kneser–Tits conjecture was instrumental in developments in another direction—rationality questions, Whitehead group and the K-theory of central simple algebras.

As already mentioned above, Tits combined with Bruhat to construct and study the Bruhat–Tits buildings associated to a linear group over a p-adic local field. These are combinatorial in nature and are analogues of Cartan's symmetric spaces which appear in the study over real numbers. For studying groups over arithmetic fields like $\mathbb{Q}$, one must combine both and hence, it was important to frame applications of Bruhat–Tits theory in a language akin to the language of symmetric spaces.

One of the crowning achievements of Tits's work is the classification theory of semisimple groups over arbitrary fields, in analogy with the classification of quadratic forms.

Finally, one of the crowning achievements of Tits's work is the classification theory of linear algebraic groups over general fields. The theory of quadratic forms (that is, homogeneous polynomials of degree 2) over fields is a rich, classical subject that is ubiquitous in mathematics. The classical theory of classification of quadratic forms proceeds by realizing what is the largest sum of copies of hyperbolic planes (forms which look like $x^2 - y^2$) that

can be cut off from the given quadratic form when one makes a linear change of variables over the given field. This number is called the Witt index (after E. Witt) of the form over that field. The remaining part is called the 'anisotropic kernel' of the quadratic form. In analogy with the classification of quadratic forms over general fields, Tits defined the 'anisotropic kernel' and the index of a semisimple algebraic group over a general field. Roughly speaking, one starts with the Cartan–Killing–Dynkin classification of the groups over algebraically closed fields (fields where all nonconstant polynomials possess roots). This is in terms of certain planar graphs called Dynkin diagrams where the number of nodes—called the rank—indicates the size of diagonal subgroups they possess (for instance, for SL_n it is $n-1$). Over a general field k, one may have several different groups that become isomorphic to the same group when considered over the algebraic closure $\overline{k}$ of k. These "k-forms" of G can be described by means of keeping track of how the Galois group of $\overline{k}$ over k acts on the Dynkin diagram. Nodes in the same orbit are placed vertically above each other (that is, one is folding the Dynkin diagram). Then, most importantly, certain orbits contribute to the diagonal subgroup over k itself, and this is indicated by circling that orbit. The 'Tits index of G over k' is thus the folded and circled Dynkin diagram where the necessary information has been incorporated. One may determine the relation between these 'simple k-roots' in terms of these data. Finally, the k-forms of G are determined in terms of the Tits index and the anisotropic kernel (made up from the part without circled orbits). This is a very general classification theorem and includes as special cases: (a) Wedderburn's theorem that each central simple k-algebra can be written uniquely as the matrix algebra $M_r(D)$ for some central division k-algebra D and $r \geq 1$, and (b) Witt's theorem alluded to above asserting that a non-degenerate quadratic form is an orthogonal sum of a uniquely determined anisotropic non-degenerate quadratic form and a sum of hyperbolic planes. If one wishes to study groups over an arbitrary field, in addition, one needs to know which Tits indices occur there. This is specific to the field and Tits also determined the admissible indices over several fields of interest. Ar-

guably, this classification paper of Tits is the one that is used by most mathematicians working on diverse aspects involving linear algebraic groups.

The exceptional groups and Tits's construction

Different forms of the exceptional group E_8 play a key role in topics like string theory and super-gravity.

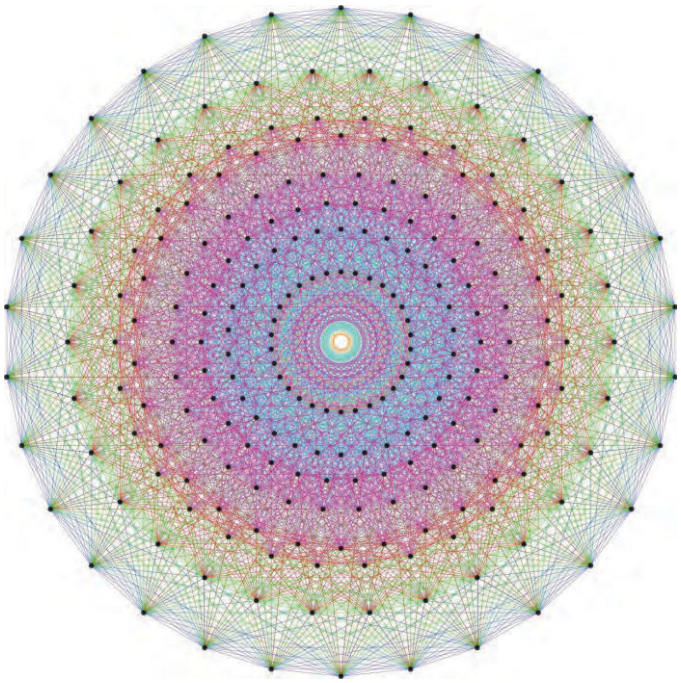
Though this short section concerns exceptional groups, we focus on the Lie group E_8 or the Lie algebra e_8 which is the largest among the exceptional groups/Lie algebras. Moreover, different forms of the algebraic group E_8 over different fields play a role in physics also. For instance, the compact real form of E_8 appears in string theory and, the split real form appears in super-gravity. A striking feature of E_8 is that the smallest dimension of a nontrivial representation as matrices is 248—equal to the dimension—and is the representation obtained by the action on its Lie algebra e_8. Thus, E_8 can be viewed as the group of automorphisms of the Lie algebra e_8. All other types of Lie groups can be nontrivially represented via matrices of sizes that are smaller than the dimension of the group. Further, it is clear that due to the diverse roles that the forms of E_8 play over different fields, it is necessary to construct E_8 over any field. Starting with the lattice $\mathbb{Z}^{248}$, one could give a Lie bracket structure making it a Lie algebra and by extending the scalars, one can obtain the 'split' form of e_8 over any field k. Its automorphism group yields an algebraic group of type E_8 over k (see Reference **2**).

Analogous to quaternion and octonion algebras, one has the notion of an Albert algebra. Any Albert algebra has dimension 27. Tits used automorphisms of octonion and Albert algebras to give an explicit construction of Lie algebras with e_8 as a possible output; this is known as Tits's construction.

Analogous to the quaternion and octonion algebras, one has the notion of an Albert algebra. Tits used automorphisms of octonion and Albert algebras to give an explicit construction of Lie algebras which produces e_8 as a possible output; this is known as Tits's construction.

A Jordan algebra over k is a commutative, not necessarily associative k-algebra A in which the so-called Jordan identity $(xy)(xx) = x(y(xx))$ holds. In particular, A is power associative. Given an associative algebra B with multiplication ., the anti-commutator $\frac{1}{2}(x.y + y.x)$ defines on B the structure of a Jordan algebra, denoted by $B+$. A Jordan algebra A is said to be special if it is

isomorphic to a Jordan subalgebra of $B+$ for an associative algebra B, and exceptional otherwise. An Albert algebra is defined to be a simple, exceptional Jordan algebra; it is known that any Albert algebra has dimension 27. The automorphism group of an Albert algebra is an exceptional algebraic group of type F_4. Tits and Weiss conjectured that the structure group of any Albert algebra is generated by certain special operators, namely, the scalar multiplications and the so called U-operators. For certain forms of type E_7 and E_8, Tits and Weiss proved that this conjecture is equivalent to the Kneser–Tits conjecture mentioned above. In very recent developments, the Tits–Weiss conjecture (and hence the Kneser–Tits conjecture) has been affirmatively answered. The fact that the root system of E_8 has such a rich structure leads it to be advertised by pretty pictures such as the one below:



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In conclusion, Tits's work is ubiquitous in modern mathematics of the last six decades. He is a theory builder and his ideas have

shaped the direction of study of linear algebraic groups, their representation theory as well as their arithmetic study which are so important to number theory. His staunch belief in geometrization of the algebra involved has led to a view point which is so fruitful in the present day that it seems inexorable and inevitable now. Needless to say, many mathematical constructs (not just the buildings) are named after him. Indeed, his work on Coxeter groups and Coxeter complexes which really put Coxeter's ideas into focus and gave a foundation, could justify adding his name too to these notions. We end by quoting a really beautiful theorem due to L. Solomon and Tits:

The spherical Tits building of a finite group of Lie type is homotopically a bouquet of spheres.

Suggested Reading

[1] ***The Abel Prize 2008–2012***, **Helge Holden and Ragni Piene (Editors), Springer-Verlag, Berlin Heidelberg 2014.**

[2] E_8, **The Most Exceptional Group, Skip Garibaldi, *Bull. Amer. Math. Soc.*, Vol.53, pp.643–671, October 2016.**

[3] **Interview with John G Thompson and Jacques Tits by Martin Raussen and Christian Skau, *Notices Amer. Math. Soc.*, Vol.54, pp.471–478, 2009.**

[4] **Buildings : Theory and Applications, Peter Abramenko & Kenneth S Brown, Graduate Texts in Mathematics, Springer Science + Business Media LLC, Vol.248, 2008.**

[5] **On buildings and their applications, J. Tits, *Proceedings of the International Congress of Mathematicians*, Vol.1, Canadian Mathematical Congress, *Montreal, Que.*, pp.209–220, 1975.**

[6] ***Buildings of Spherical Type and Finite BN-Pairs*, Jacques Tits, Springer-Verlag, Vol.386, Berlin-Heidelberg 1974, 311 pages.**

[7] ***Conformal Invariance and String Theory*, edited by P. Dita and V. Georgescu, Academic Press, 1989.**

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