

The Other Way Round*

It is very common in mathematics to come across a problem which begets another problem when one looks at it *the other way round*. In this note, we discuss a few such problems and their solutions. Every effort has been made to make the exposition lucid and instructive.

The following problem appeared in the Pre-Regional Mathematical Olympiad (PRMO) 2019, the first step of the Indian Mathematical Olympiad Programme toward the International Mathematical Olympiad (IMO).

Let $f(x) = x^2 + ax + b$. If for all nonzero real x , $f(x + 1/x) = f(x) + f(1/x)$, and the roots of $f(x) = 0$ are integers, what is the value of $a^2 + b^2$?

This problem is very straightforward, and many students gave the correct answer. But an interesting problem emerges with minor modifications of the statement of the original problem. Here it is.

Find all non-constant polynomials f with real coefficients which satisfy $f(x + 1/x) = f(x) + f(1/x)$ for all nonzero real x .

In a way, this is the converse problem of the PRMO problem. By asking this question, we are going *the other way round*. But how do we go about solving this new problem? Let us try some simple cases first. Since f is a non-constant polynomial, the simplest candidate for f is a linear polynomial of the form $f(x) = ax + b$, where a, b are real numbers. Does this satisfy the given functional equation? Here we have

$$a(x + 1/x) + b = ax + b + a/x + b,$$

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Prithwijit De
Homi Bhabha Centre for Science
Education
Tata Institute of Fundamental
Research
V.N.Purav Marg, Mankhurd
Mumbai 400 088
Email:
prithwijit@hbcse.tifr.res.in

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whence $b = 0$. Thus $f(x) = ax$ is indeed a solution. Moving ahead, if we consider $f(x) = ax^2 + bx + c$ with $a \neq 0$ then we obtain

$$a(x + 1/x)^2 + b(x + 1/x) + c = (ax^2 + bx + c) + (a/x^2 + b/x + c).$$

Observe that f is a solution if, and only if $c = 2a$. But for $f(x) = ax^3 + bx^2 + cx + d$ with $a \neq 0$ to be a solution we must have

$$a(x+1/x)^3+b(x+1/x)^2+c(x+1/x)+d = (ax^3+bx^2+cx+d)+(a/x^3+b/x^2+c/x+d),$$

which yields

$$3a(x + 1/x) = d - 2b.$$

This is impossible unless $a = 0$. So a cubic polynomial cannot be a solution of the given functional equation. At this juncture, we might conjecture that no non-constant polynomial of degree at least 3 is a solution. But how do we settle it? Assume

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0,$$

with $n \geq 3$ and $a_n \neq 0$ is a solution. Then the coefficient of x^{n-2} in $f(x) + f(1/x)$ is a_{n-2} and that of $f(x + 1/x)$, which is observed to originate from

$$a_n(x + 1/x)^n + a_{n-1}(x + 1/x)^{n-1} + a_{n-2}(x + 1/x)^{n-2},$$

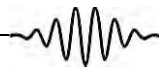
is $na_n + a_{n-2}$. Therefore,

$$na_n + a_{n-2} = a_{n-2},$$

and hence $a_n = 0$, a contradiction. So $n \leq 2$.

Thus, as we have seen earlier, any f satisfying the given equation must be of the form $f(x) = ax^2 + bx + 2a$ where a, b are real numbers. Conversely, it is patently true that any such f indeed satisfies the given equation.

The next two examples stem from two simple observations.



CLASSROOM

1. Consider three consecutive nonnegative integers $n - 1, n, n + 1$.
Observe that

$$n^2 - (n - 1)(n + 1) = 1.$$

(Incidentally, this identity holds for any real or complex number but the notion of *succession* is lost in these cases.)

2. Consider three consecutive terms F_{n-1}, F_n, F_{n+1} of the Fibonacci sequence. Observe that

$$F_n^2 - F_{n-1}F_{n+1} = (-1)^n.$$

Consider three consecutive nonnegative integers $n - 1, n, n + 1$. Observe that $n^2 - (n - 1)(n + 1) = 1$.

The first observation naturally lends itself to the following problem:

Let $\{a_n\}_{n \geq 1}$ be a sequence of nonnegative real numbers with $a_1 = 1, a_2 = 2$ and $a_n^2 - a_{n-1}a_{n+1} = 1$ for $n \geq 2$. Determine all such sequences a_n .

Of course $a_n = n$ with $n \in \mathbb{Z}$ is a solution. But is it the only one? For $n = m, m + 1$ with $m \geq 2$ we have

$$a_m^2 - a_{m-1}a_{m+1} = 1,$$

and

$$a_{m+1}^2 - a_m a_{m+2} = 1,$$

whence

$$a_{m+1}(a_{m+1} + a_{m-1}) = a_m(a_m + a_{m+2}). \quad (1)$$

Note that $a_n \geq 1$ for all $n \geq 1$ and so $1/a_n$ is well-defined and finite for any n . Thus we get

$$\frac{a_{m+1} + a_{m-1}}{a_m} = \frac{a_m + a_{m+2}}{a_{m+1}},$$

for $m \geq 2$. In particular we have

$$\frac{a_{m+1} + a_{m-1}}{a_m} = \frac{a_3 + a_1}{a_2} = \frac{3 + 1}{2} = 2,$$

whence

$$a_{m+1} - a_m = a_m - a_{m-1} = \dots = a_2 - a_1 = 1,$$

Consider three consecutive terms F_{n-1}, F_n, F_{n+1} of the Fibonacci sequence. Observe that $F_n^2 - F_{n-1}F_{n+1} = (-1)^n$.



which readily shows that

$$a_m = a_1 + (m - 1) \cdot 1 = 1 + m - 1 = m.$$

Let a_n be a sequence of real numbers with $a_1 = 1, a_2 = 2$ and $a_n^2 - a_{n-1}a_{n+1} = 1$ for $n \geq 2$. Determine all such sequences a_n .

One may also ask the following question:

Let a_n be a sequence of real numbers with $a_1 = 1, a_2 = 2$ and $a_n^2 - a_{n-1}a_{n+1} = 1$ for $n \geq 2$. Determine all such sequences a_n .

Note that, here, a_n can possibly be negative or zero as a priori we cannot rule this out. So, even if we proceed as we did above, we have to be careful when it comes to considering $1/a_n$ for some $n \geq 2$. If k is the least value of n such that $a_n = 0$ then from (1) we get,

$$0 = a_k(a_k + a_{k-2}) = a_{k-1}(a_{k-1} + a_{k-3}) = \dots = a_1(a_1 + a_3) = 4, \quad (2)$$

a contradiction. Thus $a_n \neq 0$ for all $n \geq 1$ and we can divide by a_n . The rest of the argument is same as the one presented in the previous problem.

The problem arising from the second observation is:

Determine all real sequences $\{a_n\}_{n \geq 0}$ such that $a_0 = a_1 = 1$ and $a_n^2 - a_{n-1}a_{n+1} = (-1)^n$ for $n \geq 1$.

Proceeding as in the previous problem, for $n = m, m + 1$ with $m \geq 1$ we obtain,

$$a_m^2 - a_{m-1}a_{m+1} = (-1)^m,$$

and

$$a_{m+1}^2 - a_m a_{m+2} = (-1)^{m+1}.$$

Adding the two and rearranging the terms lead to

$$a_m(a_{m+2} - a_m) = a_{m+1}(a_{m+1} - a_{m-1}).$$

We claim that $a_n \neq 0$ for any n . Why? The reader may find an argument by emulating the method suggested in the previous problem. Hence we end up with

$$\frac{a_{m+2} - a_m}{a_{m+1}} = \frac{a_{m+1} - a_{m-1}}{a_m} = \dots = \frac{a_2 - a_0}{a_1} = 1,$$



whence

$$a_{m+1} = a_m + a_{m-1},$$

for $m \geq 1$.

This recurrence along with the initial conditions $a_0 = a_1 = 1$ shows that every term of the sequence is a positive integer and that $\{a_n\}_{n \geq 1}$ is the well-known Fibonacci sequence.

The Fibonacci sequence $\{F_n\}_{n \geq 1}$ with $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ also satisfies

$$F_m F_n + F_{m-1} F_{n-1} = F_{m+n-1},$$

for $m, n \geq 2$. How about looking at it *the other way round*? That is, we ask the following question:

If $\{a_n\}_{n \geq 1}$ is a sequence of real numbers with $a_1 = a_2 = 1$ and $a_m a_n + a_{m-1} a_{n-1} = a_{m+n-1}$ for $m, n \geq 2$, then does it follow that $\{a_n\}_{n \geq 1}$ is the Fibonacci sequence $\{F_n\}_{n \geq 1}$?

Observe that for $m, n \geq 2$,

$$a_{m+1} a_n + a_m a_{n-1} = a_{(m+1)+n-1} = a_{m+(n+1)-1} = a_m a_{n+1} + a_{m-1} a_n,$$

whence

$$a_n(a_{m+1} - a_{m-1}) = a_m(a_{n+1} - a_{n-1}).$$

Arguing as above we can show that $a_n \neq 0$ for $n \geq 1$. This leads to

$$\frac{a_{m+1} - a_{m-1}}{a_m} = \frac{a_{n+1} - a_{n-1}}{a_n}.$$

Fixing n and decreasing m in steps of 1 brings us to

$$\frac{a_{n+1} - a_{n-1}}{a_n} = \frac{a_3 - a_1}{a_2} = 1,$$

that is, $a_{n+1} = a_n + a_{n-1}$. Since $a_1 = a_2 = 1$, we conclude that for $n \geq 1$ each a_n is an integer and $\{a_n\}_{n \geq 1}$ is the Fibonacci sequence.

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If $\{a_n\}_{n \geq 1}$ is a sequence of real numbers with $a_1 = a_2 = 1$ and $a_m a_n + a_{m-1} a_{n-1} = a_{m+n-1}$ for $m, n \geq 2$, then does it follow that $\{a_n\}_{n \geq 1}$ is the Fibonacci sequence $\{F_n\}_{n \geq 1}$?

