An Elementary Introduction to Driven Damped Oscillators*

Carl E. Mungan

The steady-state motion of a sinusoidally driven, linearly damped oscillator is deduced and graphed without using phasors or differential equation methods. Only basic trigonometry and derivatives are employed to make the analysis accessible to introductory students. The frequency dependence of the phase shift is motivated physically. Application to simple ac filter circuits is made.

Carl Mungan has been a professor of physics at the United States Naval Academy in Annapolis, Maryland since 2000. He teaches across the undergraduate curriculum in physics for both majors and nonmajors. All midshipmen are required to take two semesters of calculus-based introductory physics.

1. Introduction

Driven damped oscillations arise in introductory physics, both for a mass on a spring in a viscous fluid and for an inductor, resistor, and capacitor, wired in series [1]. Some textbooks state the equations for the frequency dependence of the resulting amplitude and phase shift without deriving them. Books that do present a derivation typically adopt one of two methods to do so. A mathematical approach is to solve Newton's second law in the case of a mechanical oscillator or Kirchhoff's voltage loop rule in the case of an ac circuit written as a second-order, linear, inhomogeneous, ordinary differential equation. Alternatively, an engineering approach is the method of phasors. However, neither of these methods is particularly accessible to introductory physics students. Instead, the present article only uses elementary derivatives and trigonometry to obtain the key results.

Keywords
Oscillations, resonance, damping, ac circuits, impedance, quality factor, filters, trigonometry.

The following trigonometric relations are needed in the analysis. The top panel in *Figure* 1 plots $\sin\phi$ over one cycle of oscillation from 0 to 2π. Since there are equal areas above as below the

*Vol.27, No.6, DOI: https://doi.org/10.1007/s12045-022-1395-1

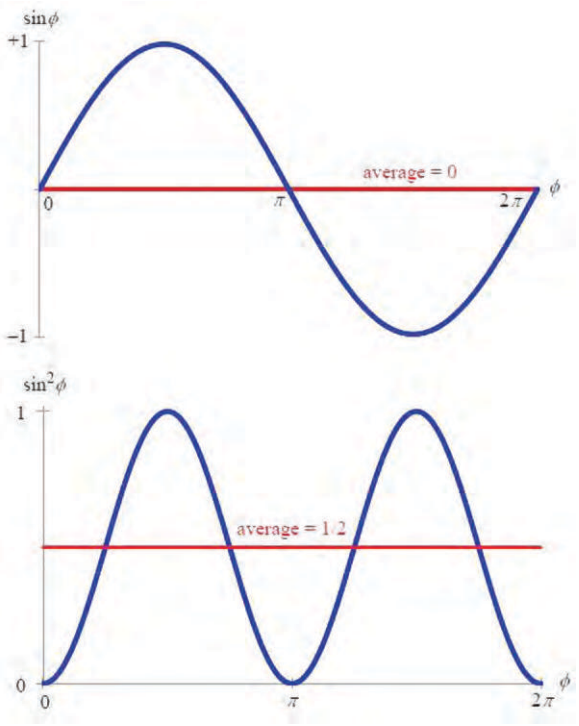

Figure 1. Graphs of one cycle of a sine function and its square, used to deduce their mean values without mathematical calculation.

horizontal axis of this graph, the mean value (denoted by angular brackets) is

$$\langle \sin\phi \rangle = 0 \tag{1}$$

The average value of a sinusoidal function can be obtained graphically.

over any integer number of cycles. The bottom panel in *Figure* 1 plots $\sin^2\phi$ over the same angular range from 0 to 2π. Again, there are equal areas above as below a horizontal line drawn midway between the peaks and troughs of oscillation, so that

$$\left\langle \sin^2\phi \right\rangle = \tfrac{1}{2}. \tag{2}$$

These mean values are unaffected if one moves the vertical axes of these two graphs horizontally by any arbitrary angle. For example, shifting the vertical axis rightward by $\pi/2$ implies that

$$\langle \cos\phi \rangle = 0 \tag{3}$$

and

$$\left\langle \cos^2 \phi \right\rangle = \tfrac{1}{2}. \tag{4}$$

Alternatively, one can use this idea that $\sin^2 \phi$ and $\cos^2 \phi$ must have equal mean values and combine it with the Pythagorean identity $\sin^2 \phi + \cos^2 \phi = 1$ to deduce (2) and (4).

Key trig identities needed in the derivation are the double-angle formulas.

The double-angle formula for cosine is

$$\cos(A + B) = \cos A \cos B - \sin A \sin B. \tag{5}$$

Students who are familiar with complex exponentials can verify this formula by equating the real part of each side of the identity

$$e^{i(A+B)} = e^{iA} e^{iB}. \tag{6}$$

Similarly the double-angle formula for sine,

$$\sin(A + B) = \cos A \sin B + \sin A \cos B, \tag{7}$$

can be obtained by equating the imaginary part of each side of (6). Now change the sign of B in (5) to get

$$\cos(A - B) = \cos A \cos B + \sin A \sin B, \tag{8}$$

because $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$. Adding together (5) and (8) gives

$$\cos(A + B) + \cos(A - B) = 2 \cos A \cos B. \tag{9}$$

The rest of the article is organized as follows. Section 2 justifies an analogy between mechanical and electromagnetic oscillators so that instructors can map the results for one topic onto the other, depending on which they are teaching. Section 3 analyzes a driven *RLC* series circuit. Section 4 graphically explores the key results. Section 5 applies these ideas to series circuits treated as voltage dividers.

2. Analogous Quantities for Mechanical and Electromagnetic Oscillators

Suppose a mass m is connected to a spring obeying Hooke's law of stiffness constant k and is subject to a linear damping force

with drag coefficient b while being driven by a sinusoidal force F of amplitude $F_{\max}$ and angular frequency ω. Start a timer at the instant the driving force is at a maximum value so that Newton's second law becomes

$$-kx - b\upsilon + F_{\max}\cos\omega t = ma, \tag{10}$$

where x, υ, and a are the displacement, velocity, and acceleration of the mass and t is the elapsed time. (10) can be rewritten as

$$F_{\max}\cos\omega t = m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx. \tag{11}$$

Assuming the driver has been on long enough that the drag force has damped out any initial transients in the response of the system, the displacement x of the mass (from its equilibrium position) will be forced to oscillate sinusoidally at the same angular frequency ω as F but not necessarily in phase with it (as discussed in the *Appendix*) so that

$$x = x_{\max}\sin(\omega t - \delta), \tag{12}$$

where $x_{\max}$ is the displacement amplitude. For example, if the phase shift δ happens to equal $-\pi/2$, then (12) becomes $x = x_{\max}\cos\omega t$ according to (7), in which case the displacement is in phase with the driving force. Having chosen the trigonometric function in (12) to be a sine function, its time derivative is

$$\upsilon = \upsilon_{\max}\cos(\omega t - \delta) \text{ where } \upsilon_{\max} = \omega x_{\max}, \tag{13}$$

For a mechanical oscillator, the velocity lags *behind* the driving force by the phase shift δ.

so that δ is seen to represent the *phase lag* of the velocity with respect to the driving force. For example, $\delta = \pi/2$ would imply the velocity of the mass reaches its peak value one-quarter of a period after the driver attains a maximum value.

Now, consider the *RLC* circuit in *Figure* 2. Assume the ac power supply has been switched on for a long enough time that the resistance has damped out any initial transients in the current. A timer is started at the instant that this driving emf ε is peaking so that it can be described by a cosine function of time,

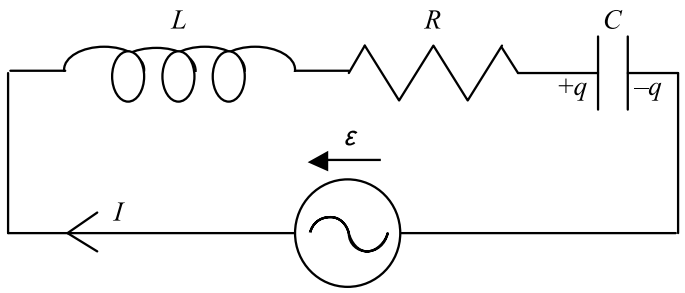

Figure 2. An *RLC* series circuit driven by a sinusoidally oscillating emf ε. The charges on the two plates of the capacitor are equal in magnitude and opposite in sign; q itself can be either positive or negative according to (18) and is consistent with $I = dq/dt$ for the indicated current direction. The arrow next to the power supply indicates that ε equals the difference in potential between its left and right terminals (rather than the other way around); ε itself can be either positive or negative, according to (14).

$$\varepsilon = \varepsilon_{\max} \cos \omega t, \tag{14}$$

with amplitude $\varepsilon_{\max}$ and angular frequency ω. According to Kirchhoff's voltage loop rule, the rise in potential across the power supply equals the sum of the potential drops across the inductance L, resistance R, and capacitance C so that

$$\varepsilon_{\max} \cos \omega t = L\frac{dI}{dt} + IR + \frac{q}{C}, \tag{15}$$

which can be rewritten as

$$\varepsilon_{\max} \cos \omega t = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q. \tag{16}$$

Comparing (11) and (16), the following analogs are evident. The driving force F is analogous to the driving voltage ε; the displacement x is analogous to the charge q; the velocity $v = dx/dt$ is analogous to the current $I = dq/dt$; the mass m, which represents the inertia opposing changes in velocity is analogous to the inductance L opposing changes in current; the drag coefficient b, which gives rise to mechanical energy losses (transformed into thermal energy in the viscous fluid) is analogous to the resistance R, which gives rise to electrical energy losses (transformed into thermal energy in the resistor); and the stiffness constant k, which is the ratio of the spring force to the displacement is analogous to the reciprocal of the capacitance because $1/C$ is the ratio of the capacitor voltage to the charge. So in accordance with these analogs, (13) and (12) respectively become

$$I = I_{\max} \cos(\omega t - \delta) \tag{17}$$

For an electromagnetic oscillator, the current lags *behind* the driving emf by the phase shift δ.

and

$$q = q_{\max} \sin(\omega t - \delta) \quad \text{where} \quad q_{\max} = \frac{I_{\max}}{\omega}. \tag{18}$$

3. Circuit Analysis

With reference to *Figure* 2 and (14) and (17), the problem is the following. Assuming that the values of L, R, C, $\varepsilon_{\max}$, and ω are known, find the values of $I_{\max}$ and δ. From $I_{\max}$ and δ one can deduce anything else one might want to know about the circuit, such as the charge on the capacitor plates, or the voltages across and power delivered to or from each of the circuit elements. Thus the complete *response* of the circuit to a given *driver* will have been determined.

The potential drop across the resistor is in phase with the current,

$$V_R = IR = I_{\max} R \cos(\omega t - \delta) \tag{19}$$

using (17). The potential drop across the capacitor lags the current by a quarter cycle,

$$V_C = \frac{q}{C} = I_{\max} X_C \sin(\omega t - \delta) \quad \text{where} \quad X_C = \frac{1}{\omega C} \tag{20}$$

using (18). The potential drop across the inductor leads the current by a quarter cycle,

$$V_L = L\frac{dI}{dt} = -I_{\max} X_L \sin(\omega t - \delta) \quad \text{where} \quad X_L = \omega L \tag{21}$$

using (17). According to Kirchhoff's voltage loop rule, the rise in potential ε across the power supply equals the sum of the potential drops across the inductor, resistor, and capacitor so that

$$\frac{\varepsilon}{I_{\max}} = \frac{V_R}{I_{\max}} + \frac{V_L}{I_{\max}} + \frac{V_C}{I_{\max}} \tag{22}$$

after dividing every term by the current amplitude. Define the impedance of the series *RLC* combination to be the amplitude of the voltage applied across it (by the power supply) divided by the amplitude of the current to it,

$$Z = \frac{\varepsilon_{\max}}{I_{\max}}. \tag{23}$$

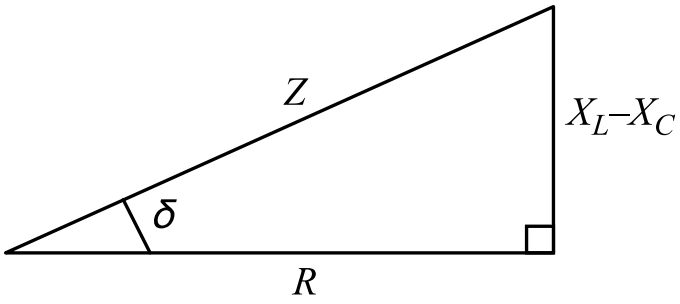

Figure 3. Triangle diagram relating key quantities relevant to ac circuits. The side adjacent the acute angle δ (representing the phase shift) is the resistance R, the side opposite δ is the difference in the inductive and capacitive reactances $X_L - X_C$, and the hypoteneuse is the total impedance Z.

If one can find Z, this definition gives $I_{\max}$. Thus the problem is now to find Z and δ.

To do so, substitute (14), (19), (20), and (21) into (22) to obtain

$$Z\cos\omega t = R\cos(\omega t - \delta) - (X_L - X_C)\sin(\omega t - \delta) \tag{24}$$

using (23). (24) must hold for all values of t. Two particular values of t can be used to solve for the two sought quantities. First, choose the instant in time when $\omega t = \delta$ so that (24) becomes

$$Z\cos\delta = R. \tag{25}$$

Second, substitute $\omega t = \delta - \pi/2$ into (24) to get

$$Z\sin\delta = X_L - X_C \tag{26}$$

using (8). (25) and (26) uniquely specify the right triangle sketched in *Figure* 3. From this *triangle diagram* one can read off three key equations. The power factor (so named because of (38) below) is

$$\cos\delta = \frac{R}{Z}, \tag{27}$$

the impedance is

$$Z = \sqrt{R^2 + (X_L - X_C)^2}, \tag{28}$$

and the phase shift between the driving emf ε and the resulting current I is

$$\delta = \tan^{-1}\frac{X_L - X_C}{R}. \tag{29}$$

The impedance Z and phase shift δ completely determine the *responding* current to the *driving* emf.

The triangle diagram is also the key result of a phasor analysis.

The power factor, impedance, and phase shift are directly read off the triangle diagram.

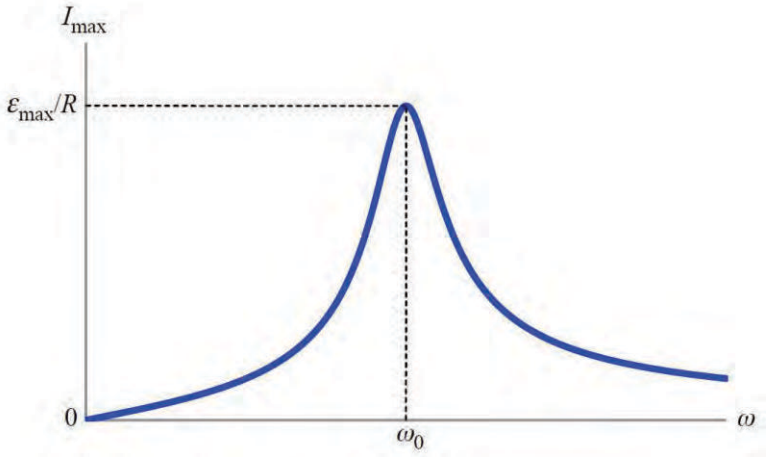

Figure 4. Current amplitude I_{max} for the circuit in *Figure* 2 as a function of the angular frequency ω of the power supply. The frequency and current at resonance are labeled. The circuit elements were chosen to satisfy $L/R = 25RC$. The horizontal axis ranges from 0 to $2\omega_0$.

For this *series* circuit, the current is the same in every element, and the voltages across the components sum up to the voltage of the power supply.

(28) and (29) complete the solution of the problem . It is worth emphasizing to students that for single-loop series circuits, the *same* current I flows out of the power supply and to every circuit element R, L, and C. In contrast, the voltages across each of the four of them are in general *different*.

4. Graphical Presentations of the Solution

Substituting (28) into (23) results in

$$I_{max} = \frac{\mathcal{E}_{max}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \tag{30}$$

expressed in terms of the five givens. To plot it as a function of the driving frequency, consider two limiting values. When $\omega \to 0$ then $X_C \to \infty$ so that $I_{max} = 0$ because the capacitor prevents a dc current from flowing across the gap between its plates. On the other hand, as $\omega \to \infty$ then $X_L \to \infty$ so that $I_{max} = 0$ again, this time because the inductor blocks sudden changes in current. However, I_{max} is positive at all other values of ω, and thus one anticipates it must peak at what is called the resonant angular frequency ω_0. Specifically, I_{max} peaks when the denominator of (30) is a minimum. In turn, that occurs when the quantity inside

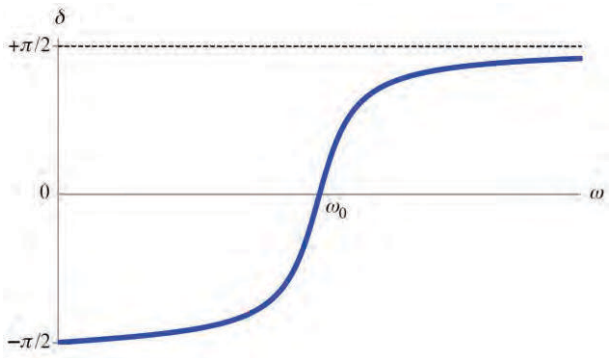

Figure 5. Phase shift δ of the current relative to the driving emf for the circuit in *Figure* 2 as a function of the angular frequency ω of the power supply. The values at zero, the resonant, and infinite frequencies are labeled. The circuit elements were chosen to satisfy $L/R = 25RC$. The horizontal axis ranges from 0 to $2\omega_0$.

the parentheses is zero, implying

$$\omega_0 L = \frac{1}{\omega_0 C} \quad \Rightarrow \quad \omega_0 = \frac{1}{\sqrt{LC}}. \tag{31}$$

The current amplitude is maximum at the resonant frequency.

The resonant frequency equals the natural frequency of an undamped undriven LC circuit. At this frequency, the reactance of the inductor and capacitor cancel each other. So the current amplitude simplifies to $\varepsilon_{\max}/R$ according to (30), just as it would if the power supply were connected across the resistor alone. Thus a graph of $I_{\max}$ has the shape shown in *Figure* 4.

The other component of the solution is the phase lag of I relative to ε as given by (29),

$$\delta = \tan^{-1} \frac{\omega L - \frac{1}{\omega C}}{R}. \tag{32}$$

An overall mnemonic is "ELI the ICE man."

It was remarked in connection with (20) and (21) that the current leads the potential drop across a capacitor (which can be remembered as frozen water 'ICE', indicating that the current 'I' for a capacitor 'C' comes before the emf 'E' across it) but lags the potential drop across an inductor (remembered as a man's name 'ELI'). Accordingly, X_L has a positive sign in front of it, and X_C a negative sign in the numerator of the ratio in (29). To plot this phase shift versus ω, again consider the two limiting values. As $\omega \to 0$ one gets $\delta = \tan^{-1}(-\infty) = -\pi/2$, whereas $\delta = \tan^{-1}(+\infty) = +\pi/2$ as $\omega \to \infty$. Furthermore, at resonance

one has $\delta = \tan^{-1}(0) = 0$ because the inductor and capacitor cancel each other's effects so that the emf is effectively applied only across the resistor, in which case the current is in phase with it. The graph is, therefore, as sketched in *Figure* 5.

Next, consider the power delivered by the generator to the *RLC* series combination. As for any two-lead device, the instantaneous power is given by the product of the current supplied to the combination from (17) and the voltage across it from (14),

$$P = I\varepsilon = I_{\max}\varepsilon_{\max}\cos(\omega t - \delta)\cos\omega t$$
$$= \tfrac{1}{2}I_{\max}\varepsilon_{\max}\left[\cos(2\omega t - \delta) + \cos\delta\right] \quad (33)$$

using (9) in the last step with $A = \omega t - \delta$ and $B = \omega t$. Now time average this result to obtain

$$P_{\text{avg}} = \tfrac{1}{2}I_{\max}\varepsilon_{\max}\cos\delta \quad (34)$$

using (3) with $\phi = 2\omega t - \delta$. This expression can be written more compactly using rms ('root-of-the-mean-of-the-square') values. For the current from (17) one gets

$$I_{\text{rms}} = \sqrt{\left\langle I_{\max}^2\cos^2\phi\right\rangle} = \frac{I_{\max}}{\sqrt{2}} \quad (35)$$

using (4) with $\phi = \omega t - \delta$. Likewise for the supply emf from (14) one finds

$$\varepsilon_{\text{rms}} = \sqrt{\left\langle \varepsilon_{\max}^2\cos^2\phi\right\rangle} = \frac{\varepsilon_{\max}}{\sqrt{2}} \quad (36)$$

using (4) with $\phi = \omega t$. These two results imply that (23) can be rewritten as

$$Z = \frac{\varepsilon_{\text{rms}}}{I_{\text{rms}}}. \quad (37)$$

The average power equals the product of the rms current, rms voltage, and power factor.

Substituting (35) and (36) into (34) leads to

$$P_{\text{avg}} = I_{\text{rms}}\varepsilon_{\text{rms}}\cos\delta. \quad (38)$$

Using the expression for the power factor $\cos\delta$ found from the triangle diagram in (27), this result can be rewritten in two other ways. Eliminating ε_{rms} using (37) gives

$$P_{\text{avg}} = I_{\text{rms}}^2 R, \quad (39)$$

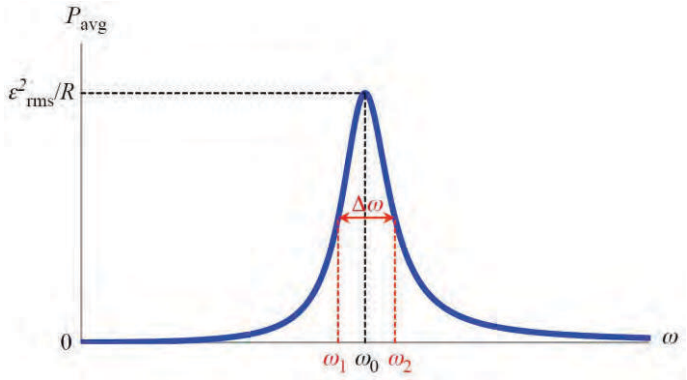

Figure 6. Average power P_{avg} dissipated in the resistor for the circuit in *Figure* 2 as a function of the angular frequency ω of the power supply. The values at resonance and the frequencies at half the maximum height are labeled for ease in identifying the width $\Delta\omega$ of the peak. The circuit elements were chosen to satisfy $L/R = 25RC$. The horizontal axis ranges from 0 to $2\omega_0$.

which emphasizes that power is only dissipated on average in the resistor; the inductor and capacitor alternately store and release electromagnetic energy as the charge and current increase and decrease during the oscillations. For plotting purposes, instead, eliminate I_{rms} using (37) to find

$$P_{avg} = \frac{\varepsilon^2_{rms} R}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}. \tag{40}$$

The graph of this expression in *Figure* 6 looks nominally similar to that in *Figure* 4. But keep in mind that I_{max} is inversely proportional to Z, whereas P_{avg} is inversely proportional to Z^2 and thus their shapes are not actually the same. However, both curves peak at exactly ω_0 given by (31). Again at that resonance frequency the reactances of the inductor and capacitor cancel, so that the peak value of P_{avg} is ε^2_{rms}/R according to (40). The full width at half maximum (FWHM) is $\Delta\omega$ given by the difference in the angular frequencies ω_2 and ω_1 labeled in *Figure* 6 at which $P_{avg} = \varepsilon^2_{rms}/2R$. Substituting that average power into the left-hand side of (40) and solving for the positive roots of the two resulting quadratic equations, one finds [2],

The average power is maximum at the resonant frequency.

$$\Delta\omega = \frac{R}{L}, \tag{41}$$

which is exact regardless of how large R is. Consequently, if the quality factor is defined as the reciprocal fractional width of the

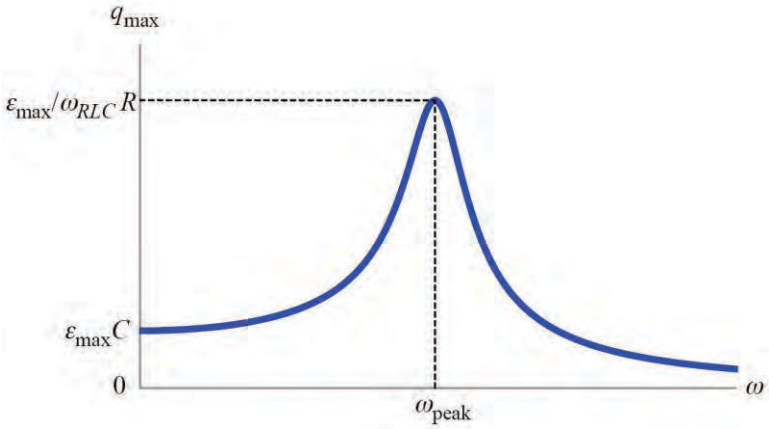

Figure 7. Charge amplitude q_{max} on the capacitor for the circuit in *Figure* 2 as a function of the angular frequency ω of the power supply. The values at the peak and at zero frequency are labeled. The circuit elements were chosen to satisfy $L/R = 25RC$. The horizontal axis ranges from 0 to $2\omega_0$.

resonance peak,

$$Q = \frac{\omega_0}{\Delta\omega}, \tag{42}$$

A large quality factor results in oscillations for many cycles before decaying away when there is no driver.

it must also be exactly equal to

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}}, \tag{43}$$

which expresses it in terms of the three circuit elements.

In contrast to I_{max} and P_{avg}, which peak at exactly ω_0, the charge amplitude does not. According to (18) and (30),

$$q_{max} = \frac{\varepsilon_{max}}{\sqrt{R^2\omega^2 + L^2(\omega^2 - \omega_0^2)^2}}, \tag{44}$$

The charge amplitude does not peak at the resonant frequency.

which peaks when the argument of the square root is a minimum. By setting its derivative to zero, that is found to occur at an angular frequency of

$$\omega_{peak} = \omega_0\sqrt{1 - \frac{1}{2Q^2}} \quad \text{provided} \quad Q \geq \frac{1}{\sqrt{2}}. \tag{45}$$

If $Q >> 1$ then $\omega_{peak} \approx \omega_0$, but q_{max} peaks at zero frequency when $R \geq (2L/C)^{1/2}$. Substituting $\omega = \omega_{peak}$ into (44), the peak value of q_{max} is found to be $\varepsilon_{max}(\omega_{RLC}R)^{-1}$ where ω_{RLC} is the angular frequency of oscillation of an undriven underdamped *RLC* circuit

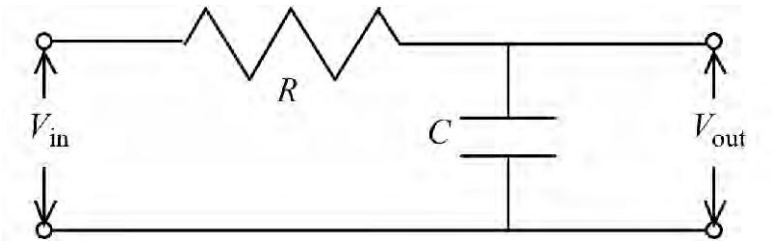

Figure 8. An *RC* series circuit used as a low-pass filter where the input signal V_{in} is applied across both components but the output V_{out} is across the capacitor, which shorts out at high frequencies.

[3]. Specifically $\omega_{RLC} = \omega_0[1 - (4Q^2)^{-1}]^{1/2}$ where Q is defined by (43), but the FWHM of the charge peak is not given by (41) and so (42) does not apply to its graph.

Also, in contrast to I_{max} and P_{avg}, which are zero at $\omega = 0$, at that frequency $q_{max} = \varepsilon_{max}C$ according to (44). A graph of q_{max} is presented in *Figure* 7.

5. Application to Series Filter Circuits

Suppose voltage output leads (which draw minimal current) are connected across the resistor in *Figure* 2. From (19) and (23), it follows that

$$\frac{V_{out\ max}}{V_{in\ max}} = \frac{I_{max}R}{\varepsilon_{max}} = \frac{R}{Z}, \tag{46}$$

where $V_{out} = V_R$ and $V_{in} = \varepsilon$. (46) is a *voltage divider* equation: the ratio of the output to the input voltage amplitudes is the ratio of the output to the input impedances. According to the graph in *Figure* 4, this ratio equals 100% at resonance but 0% at zero and infinite frequencies. Thus it acts like a *bandpass filter*, cutting off low and high frequencies but outputting the range from approximately ω_1 to ω_2 in *Figure* 6.

Series filter circuits can be quickly analyzed using the voltage divider equation.

Likewise, for the *low-pass filter* sketched in *Figure* 8, one finds

$$\frac{V_{out\ max}}{V_{in\ max}} = \frac{X_C}{Z} = \frac{1}{\sqrt{1 + (\omega RC)^2}}, \tag{47}$$

after putting $X_L = 0$ in (28) because there is no inductor in the circuit. This response is graphed in *Figure* 9. One can instead get a *high-pass filter* by replacing the capacitor in *Figure* 8 with an

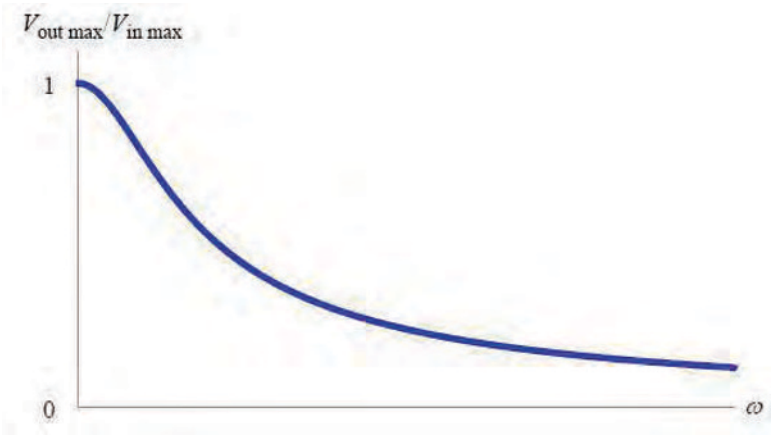

Figure 9. Ratio of the output to the input amplitudes as a function of frequency for the low-pass filter sketched in *Figure* 8. The horizontal axis ranges from 0 to $8(RC)^{-1}$.

inductor so that

$$\frac{V_{\text{out max}}}{V_{\text{in max}}} = \frac{X_L}{Z} = \frac{1}{\sqrt{1+\left(\frac{R}{\omega L}\right)^2}}, \tag{48}$$

after putting $X_C = 0$ in (28). (48) is plotted in *Figure* 10.

Appendix: Motivating the Phase Shift

Perhaps the two hardest aspects of driven oscillators students have to accept are—there is a phase shift between the driver and response, and it is frequency-dependent. The purpose of this *Appendix* is to illustrate the necessity of introducing $\delta(\omega)$ in the analysis.

Both the displacement x and the velocity υ (or both the charge q and the current I) cannot be in phase with the driver for the simple reason that they are 90° out of phase with each other. Specifically, if x is a sine function, then its derivative υ is a cosine function. Thus at least one of these two quantities must be phase shifted relative to the driver.

However, in the case of an RC circuit connected to a battery by closing a switch, the buildup of charge q on the initially uncharged capacitor plates is gradual. Although the switch causes a sudden rise in the driving emf from zero up to the battery voltage,

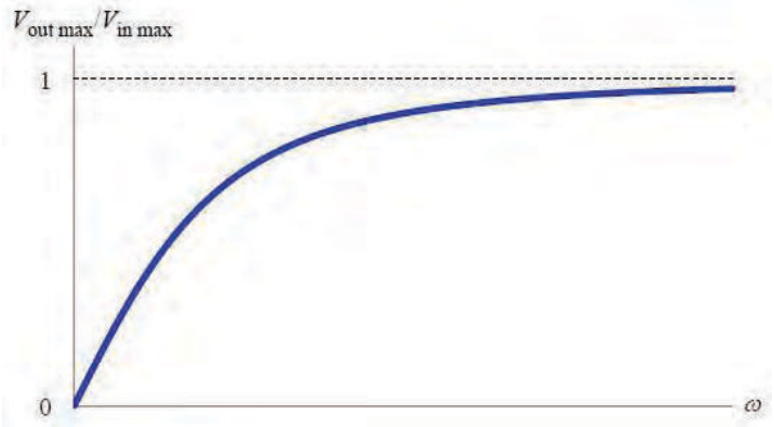

Figure 10. Voltage transmittance for a high-pass filter obtained by replacing the capacitor in *Figure* 8 with an inductor. The horizontal axis ranges from 0 to $4R/L$.

there is a delay before the capacitor acquires a substantial fraction of its maximum charge. Hence for an ac emf, there must be a phase delay δ in the function describing the charge q. But exactly the same argument holds for the function describing the current I for an LR circuit suddenly connected to a battery. Combining the ideas in this and the previous paragraph, one now expects that if say q is described by a sine function whose argument includes a phase shift δ, then I must be described by a cosine function that includes δ, justifying the form of (17) and (18), or (12) and (13).

Demonstrate the phase shift using a spring and weight.

Next, a simple demonstration is an eye-opening way to convince students that the phase shift between the driver and its response cannot merely be a constant but must vary with the driving frequency. Get a large spring. Hold the top end in your hand and attach a large weight to its bottom end such that the resonant frequency is a few hertz. Start slowly oscillating your hand vertically up and down. The weight will move approximately in phase with your hand. Then gradually increase the frequency of oscillation of your hand. The responding oscillations of the weight will begin to shift out of phase relative to your hand, approaching 180° difference at high frequencies, such that the weight is moving upward at the instant your hand is moving downward, and vice versa.

You can take advantage of the opportunity presented by this same

demonstration to show students that for a fixed amplitude of oscillation of your hand, the amplitude of oscillation of the hanging weight is greatest at some intermediate frequency (near what is called the resonant frequency) and is small at low and high frequencies. You can remind students of those observations when you later introduce the curves in *Figures* 4 and 7.

There is a competition between the angular frequency $(k/m)^{1/2}$ at which the hanging mass m wants to oscillate (when undriven and undamped) on the spring of stiffness constant k and the angular frequency at which you are shaking your hand up and down. The largest velocity response of the hanging mass occurs when these two frequencies match. At that driving frequency, the velocity of the mass must be in phase with the driving force of your hand because then maximum power—given by the dot product of the velocity and force—is delivered to the mass. That means the displacement of the mass must be 90° out of phase with the driver at resonance, agreeing with the preceding observations that they were in phase at low frequencies and 180° out of phase at high frequencies.

The average power delivered to the oscillator is maximized when the driving frequency matches the resonant frequency.

Suggested Reading

[1] F. B. Kneubil, Driven series *RLC* circuit and resonance: A graphic approach to energy, *Phys. Teach.*, Vol.58, pp.256–259, April 2020.

[2] R. L. Reese, *University Physics* (Brooks Cole, Pacific Grove CA, 2000), p. 1030.

[3] M. C. Faleski, Transient behavior of the driven *RLC* circuit, *Am. J. Phys.*, Vol.74, pp.429–437, May 2006.

Address for Correspondence
Carl E. Mungan
Physics Department
United States Naval Academy
Annapolis, MD 21402-1363
USA
Email: mungan@usna.edu