
Uncovering Dimension*

An Introduction to the Concept of Dimension in Topology

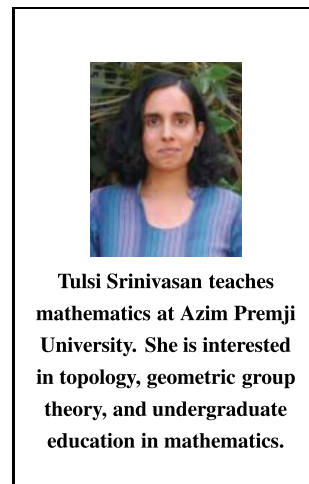
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For common objects, we have a shared intuition about what it means to be one, two or three dimensional. Translating that intuition into a precise definition requires us to closely probe what mathematical structures we are considering and examine more exotic objects, for which dimension is less intuitive. This article explores the idea of dimension and its historical development through the lens of topology.

Introduction

What do we mean when we say the real line is one-dimensional or that the Cartesian plane is two-dimensional? For most of us, the answer would probably involve coordinates. The Cartesian plane, \mathbb{R}^2 , is two-dimensional because every point can be uniquely specified by a pair of coordinates. The coordinates are also independent, in the sense that once we choose the x -coordinate, the y -coordinate can be chosen as any real number, and it still gives us a point on the plane.

Similarly, any point in Euclidean space \mathbb{R}^3 , can be uniquely given by three independent coordinates, and any point on the real line \mathbb{R} can be uniquely specified by a single coordinate. More formally, these spaces are vector spaces (over \mathbb{R}), whose dimension can be deduced from the number of elements in a basis. For example, the set $\{(1, 0), (0, 1)\}$ is a basis for \mathbb{R}^2 , as the plane consists of exactly those points of the form $(x, y) = x(1, 0) + y(0, 1)$, and $(1, 0)$ cannot be obtained from $(0, 1)$. Similarly, \mathbb{R} and \mathbb{R}^3 have $\{1\}$ and $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ respectively as bases.



Keywords

Dimension, topology.

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Figure 1. The disk and the circle of radius 1 centered at the origin.

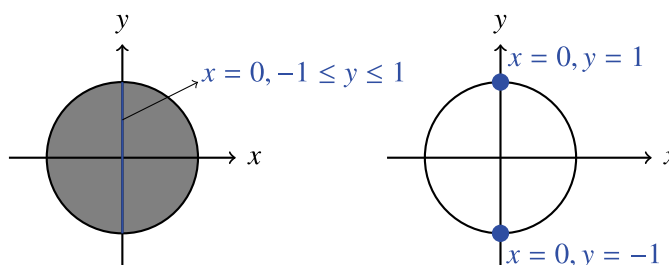
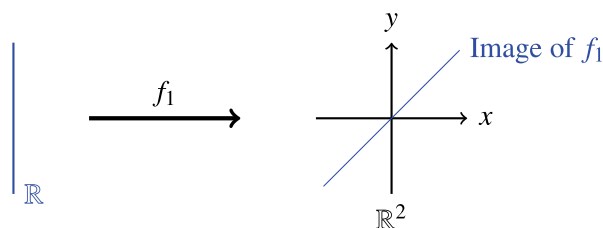


Figure 2. The image of f_1 in the plane.



The circle and the disk are not vector spaces, and it may not immediately be clear what their dimension should be.

Contrast these spaces with the disk or the circle as in *Figure 1*; again, any point is given by two coordinates. In the disk, once we choose the x -coordinate to be 0, while there are infinitely many choices for the y -coordinate, they all have to lie between -1 and 1 . In the circle, once we choose the x -coordinate to be 0, there are exactly two choices for the y -coordinate. These are not vector spaces, and it may not immediately be clear what their dimension should be. However, as we will soon see, they are among the best-behaved objects whose dimensions we can study.

1. Distorting Vector Spaces

What happens when we apply functions to \mathbb{R} , \mathbb{R}^2 or \mathbb{R}^3 ? For example, the function $f_1 : \mathbb{R} \rightarrow \mathbb{R}^2$ given by $f_1(t) = (t, t)$, has its image (see *Figure 2*) as the line $y = x$ in the plane. The original space, or domain, is a one-dimensional vector space with basis $\{1\}$, and its image is also one-dimensional with basis $\{(1, 1)\}$. This is an example of a linear transformation, a map that preserves vector space structure.



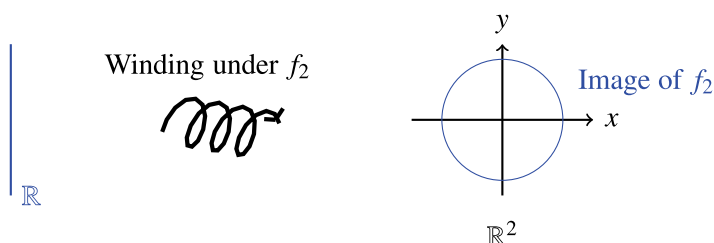


Figure 3. The image of f_2 in the plane.

Linear transformations are not too interesting as far as dimension is concerned. They take vector spaces to vector spaces and can either preserve or lower dimension. As one studies in linear algebra, the dimension of the image is given by the rank of the map. On the other hand, if we look at the function $f_2 : \mathbb{R} \rightarrow \mathbb{R}^2$ given by $f_2(t) = (\cos(t), \sin(t))$, its image is a circle in the plane (*Figure 3*). This, as we have seen, is not a vector space.

We could argue that the circle is two-dimensional, as it cannot be accommodated in one-dimensional space or because each point has two coordinates. Or we could switch to polar coordinates (which have some but not all properties of the Cartesian coordinates), under which the circle can be described as points where one coordinate (radius) is fixed, and the other (angle) varies, which could lead us to say the circle is one-dimensional. Or, since f_2 is differentiable, we could apply our earlier arguments to look at the rank of the derivative of f_2 at each point. Or, if we decide to look around single points rather than at the whole circle, we might observe that when we zoom in to a ‘neighbourhood’ of any point, it looks like a line, which is one-dimensional. In contrast, if one zooms in around a point on the disk, any neighbourhood looks like a plane, which is two-dimensional. This line of reasoning leads to defining the dimension of objects called manifolds, which, informally, are spaces that ‘locally’ resemble \mathbb{R}^n .

What happens if the function is not differentiable? If one looks at just continuous functions, things get even worse (or better, depending on how much chaos one finds desirable). For example, the steps in *Figure 4* lead to a continuous map from the interval $[0, 1]$ to the plane, whose image is called the Sierpiński triangle.



Figure 4. The first few steps in constructing the Sierpiński triangle.

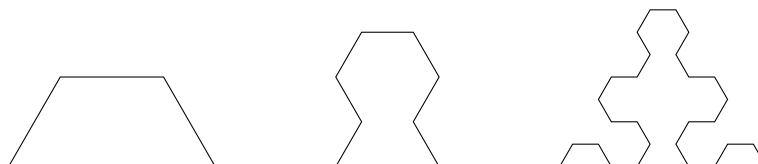
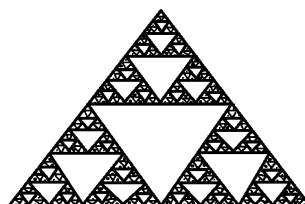


Figure 5. The Sierpiński triangle.



The image of the limit of these maps looks something like the lacy fractal (*Figure 5*), and it is hard to say whether the twists and turns have produced a one-dimensional or a two-dimensional figure or whether the notion of dimension is even a meaningful one for it.

In this article, we explore the concept of dimension that applies to vector subspaces of \mathbb{R}^n when considered along with distortions under continuous maps. So we need to think of these spaces, not as vector spaces, or even manifolds but as something less rigid. Then, we have to find a new definition of dimension that makes sense for the circle, the disk, and even pathological objects like the Sierpiński triangle.

2. An Informal Introduction to Topology

The first glimpse into the puzzling nature of dimension was obtained by Cantor.

What if we drop all structure and focus just on sets? Perhaps the first glimpse into the puzzling nature of dimension was obtained by Cantor. Georg Cantor and Richard Dedekind were German mathematicians who are considered pioneers of set theory. Cantor asked if we can have a one-to-one correspondence or bijection between the line and the plane (that is, if the two sets have the same cardinality). Against his own intuition, Cantor was able to



show exactly this during his correspondence with Dedekind in the 1870s. Cantor then remarked

...Rather the distinction which exists between figures of different dimension numbers must be sought in entirely different aspects than in the number of independent coordinates, which is normally held to be characteristic. [1]

For example, our earlier observations about the possible dimension of the disk or the circle don't make sense when we consider them as sets with no further structure. Cantor's function started as a relatively simple function from \mathbb{R}^n to \mathbb{R} that interwove the decimal expansions of n -tuples to form a single decimal expansion. Through feedback from his correspondence with Dedekind, he was able to correct the errors to produce a more complicated function. As Dedekind pointed out, this function was not continuous [2].

In real and multivariable calculus, one encounters the infamous ϵ - δ definition of continuity. For maps from \mathbb{R} to \mathbb{R} , an equivalent definition is that a map is continuous precisely when the preimage of an open interval is an open interval or a union of open intervals. For maps from \mathbb{R}^n to \mathbb{R}^m , similarly, a map is continuous precisely when the preimage of an "open m -ball" is a union of "open n -balls". Here, an open 1-ball is an open interval, an open 2-ball is a disk without its boundary, an open 3-ball is a solid ball without its boundary, and so on.

Since we have chosen to study continuous distortions of \mathbb{R}^n , what we need is a structure that is compatible with continuity, known as the topology of that space. This is what Henri Poincaré described as a "third geometry" (along with metric and projective geometry), where "quantity is completely banned" [3]. Since open balls seem to determine continuity, the topology on a space is a collection of subsets, called open sets, satisfying certain conditions with respect to unions and intersections. The collection of open n -balls in \mathbb{R}^n determines its standard topology. As all our examples so far are subsets of \mathbb{R}^n , we consider them with the topology they

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—Georg Cantor



inherit, i.e., a topology determined by intersecting open n -balls with the subset in question.

3. Invariance of Dimension

Cantor's map, being discontinuous, is not compatible with the standard topology on \mathbb{R} and \mathbb{R}^2 . One of the natural questions to arise after Cantor's finding was that whether his bijection could be made continuous. A slightly stronger form of the question asked if the inverse of this bijection could be made continuous too.

A continuous bijection with a continuous inverse is called a homeomorphism and intuitively allows stretching, bending and shrinking, but not tearing or glueing.

A continuous bijection with a continuous inverse is called a homeomorphism and intuitively allows stretching, bending and shrinking, but not tearing or glueing. Thus, an object distorted from one form to another under a homeomorphism can be changed back to its original form under another homeomorphism. Topologists consider two objects to be the same if there is a homeomorphism between them. For example, the letter 'U' is considered the same as 'V', but different from the letter 'O'.

The question of invariance of dimension asked if there can ever be a homeomorphism between \mathbb{R}^n and \mathbb{R}^m for n different from m . In other words, if we stretch, bend and shrink \mathbb{R}^n without glueing or tearing, does the resulting object still have dimension n ? It was Cantor's work that made the question even worth considering, as mathematicians in the nineteenth century took the answer to be an obvious yes. In his microscopically detailed history of the invariance of dimension, Dale Johnson describes how, after Cantor constructed his bijection, his sceptical correspondent Dedekind "warns Cantor not to polemize openly against the original 'article of faith' of manifold theory unless he (Cantor) examines the conjectured question of invariance". [1]

Further doubt was cast on the issue in 1890 when Guiseppe Peano defined a 'dimension-raising' continuous map from the interval $[0, 1]$ whose image is a square. We omit the description of the map here, but like the function whose image is the Sierpiński triangle above, it is defined as the limit of a sequence of functions, each of whose images seems to occupy more and more



space of the square. One might then well imagine there could be a map combining the properties of those given by Cantor and Peano, shattering natural intuitions about dimension being (topologically) an intrinsic feature of \mathbb{R}^n . When we looked at the Sierpiński triangle, we asked whether it is meaningful to explore the dimension of the image of \mathbb{R}^n under continuous maps. This is very much tied to whether or not there is a definition of dimension of \mathbb{R}^n (equalling n) that is actually a feature of its topology and, therefore, to the question of invariance of dimension.

After efforts by several mathematicians, the question was settled in 1912, with Dutch mathematician LEJ Brouwer's proof that \mathbb{R}^n and \mathbb{R}^m are not homeomorphic for $n \neq m$. Brouwer is probably best-known for his fixed point theorem, which says that a continuous map from an n -ball to itself always leaves some point fixed, and he laid the foundations for many ideas in general and algebraic topology. He also had a deep interest in philosophy and a somewhat quarrelsome nature where other mathematicians were concerned. His proof of invariance of dimension introduced some ideas about the degree of a map and simplicial approximations that play an important role in algebraic topology and also led to a professional rivalry with Lebesgue. However, it did not indicate any satisfactory topological definition of dimension, and this continued to occupy Brouwer [1].

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4. Inductive Dimension: Poincaré, Brouwer, Menger and Urysohn

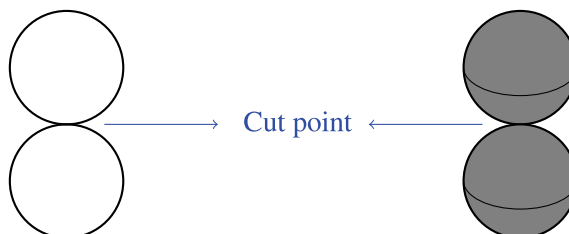
The first step in defining dimension topologically was probably made by Poincaré, who observed that dimension can be defined inductively, saying

...To divide space, cuts that are called surfaces are necessary; to divide surfaces, cuts that are called lines are necessary; to divide lines, cuts that are called points are necessary; we can go no further and a point can not be divided, not being a continuum. [4]

While there is deep insight in this paragraph, it does not imme-



Figure 6. The figure-eight and two 3-balls joined at a point.



diately give a mathematical definition. For example, most people would agree that the figure-eight and the object consisting of two 3-balls meeting at a point shown in *Figure 6* have different dimensions (whatever the exact values are). But both can be divided by a cut at a single point.

The quotation above appears in a book published in 1913, and similar ideas appear in a paper he wrote in a philosophy journal in 1912 [3]. 1912 was also the year of Poincaré’s death, so it was left to others to mathematically refine his idea.

In 1913, a year after he proved the invariance of dimension, Brouwer introduced a topological invariant¹ called *Dimensionsgrad* that made use of the inductive nature of dimension pointed out by Poincaré. He did this by refining the notion of a cut so that the kind of problem we saw above does not occur and was able to show that the dimension of \mathbb{R}^n is indeed n with his definition [5]. After this, Brouwer did not publish in dimension theory for over a decade.

In the meantime, in the early 1920s, two mathematicians—Pavel Urysohn and Karl Menger—also began working on a definition of topological dimension. Urysohn was at Moscow University, and Menger was at the University of Vienna, and their initial work appears to be independent of Brouwer as well as one another [2].

The essence of their ideas became what is today known as the small inductive dimension. In this idea, informally, a zero-dimensional space is one in which the boundary of a small ‘neighbourhood’ around any point is empty; a one-dimensional space is one in which the boundary of a small neighbourhood around any point

¹A property that remains unchanged under homeomorphisms.



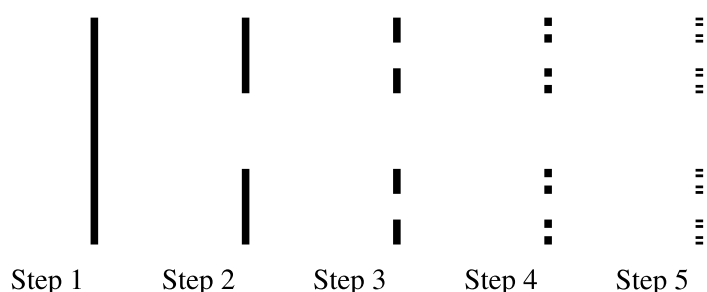


Figure 7. The first few steps in constructing the Cantor set.

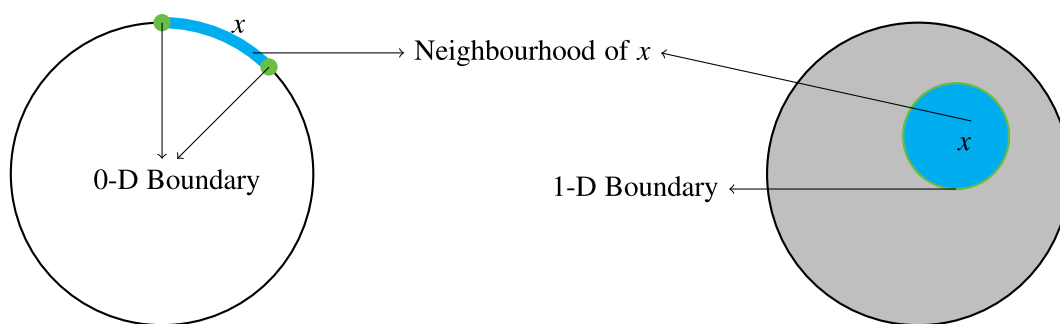


Figure 8. Inductive dimension of the circle and disk.

is at most zero-dimensional, a two-dimensional space is one in which the boundary of a neighbourhood of any point is at most one-dimensional, etc. Here, the boundary of a set can be thought of as the place where the set and its complement meet.

For example, singletons, finite sets, the natural numbers, and the rationals all have dimension zero, as small neighbourhoods of points, being points, have no boundary. Even the Cantor set, which is obtained from an infinite process of removing successive middle-thirds from an interval, the first few steps of which are shown in *Figure 7*, is zero-dimensional.

On the other hand, the circle has dimension one, as the boundary of an open arc is zero-dimensional, as can be seen in *Figure 8*. Similarly, the disk has dimension two, as the boundary of an open 2-ball, being a circle, is one-dimensional.

Brouwer's definition, after a small technical error was corrected, coincides with this in some common spaces. However, the new

Singletons, finite sets, the natural numbers, and the rationals all have dimension zero, as small neighbourhoods of points, being points, have no boundary.



Figure 9. Tiling of I by intervals that overlap at most pairwise.



definition proved more useful as the theory developed. There was some animosity between Brouwer and Menger over how serious the error was and, therefore, on whether Brouwer’s ideas were precursors to dimension theory or the main discovery that only needed to be refined and formalised [6]. On the other hand, van Dalen says Brouwer viewed Urysohn, who pointed the error in Brouwer’s definition, as the “rightful inheritor of his own topology”, and took an interest in Urysohn’s mathematical legacy after his tragically early death [7].

5. Lebesgue’s Covering Dimension

During his attempts to prove the invariance of dimension, Lebesgue made a profound observation about n -dimensional cubes, which is sometimes referred to as the tiling or paving principle.

A very different path was taken by Lebesgue. During his attempts to prove the invariance of dimension, Lebesgue made a profound observation about n -dimensional cubes, which is sometimes referred to as the tiling or paving principle. The 1-cube I is just the closed interval $[0, 1]$. The 2-cube I^2 is the product of the interval with itself, which is the square with vertices $(0, 0), (1, 0), (0, 1)$ and $(1, 1)$. The 3-cube is I^3 , which is a cube with vertices $(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)$, etc. Higher-dimensional I^n cannot be sketched but can be described mathematically.

We begin with $I = [0, 1]$ and some small positive ε . We can easily ‘cover’ I with open intervals of length ε such that no point lies in more than two of these open intervals. *Figure 9* shows small open intervals, considered to be tiles, coloured blue or red, that cover I . The purple sections indicate an overlap of two tiles.

It would be easy to add more tiles and get three, four or more tiles overlapping at various points. However, since the intervals are open, if we try and remove the overlaps, the intervals will no longer fully cover I . If we take any collection of open sets that cover I , we can always find a sufficiently small ε so that each ε -length tile in the pattern above lies completely in some set of the



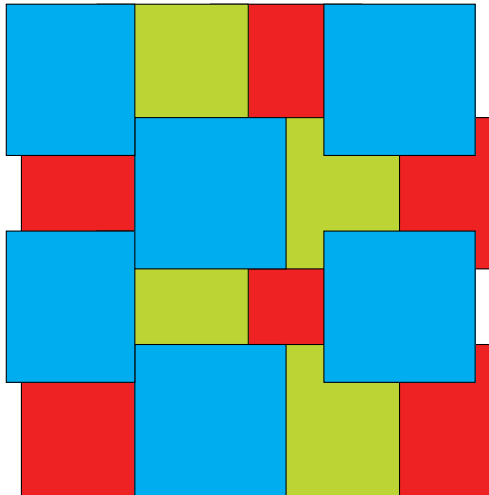


Figure 10. Tiling of I^2 with at most triple-wise overlaps.

cover. We say that we have refined the cover by tiles that overlap no more than pairwise (i.e., no more than two at a time).

Next, we consider the unit square I^2 . *Figure 10* shows a tiling with blue, yellow and red open squares. The dark blue-green patches are where three tiles overlap; at all other points, we have at most two tiles overlapping. The maximum number of tiles at each overlap can easily be increased by adding more squares or moving them around but cannot be reduced without creating gaps in the cover. Imitating the same tiling, we can cover I^2 by tiles of side any small positive ε , such that every point lies in at most three tiles. Similar to earlier, any open cover can be refined by square tiles that overlap no more than triple-wise (i.e., no more than three at a time).

We can cover I^2 by tiles of side any small positive ε , such that every point lies in at most three tiles.

Going further, one can find a tiling by open cubes so that any open cover of I^3 can be refined by cubic tiles that overlap no more than quadruple-wise (i.e., no more than four at a time). Tiles that overlap at most triple-wise will not cover I^3 . A zero-dimensional space, on the other hand, is one in which any open cover can be refined by non-overlapping tiles.

Unfortunately, this insight was not proved rigorously in Lebesgue's 1911 paper, and was corrected by Brouwer. Lebesgue himself of-



ferred another proof of it more than a decade later. However, he did not extract a definition from it, and his observations were formally given as a topological invariant called covering dimension by Eduard Čech in 1933 [6]. This definition does not agree with inductive dimension everywhere, but it does in a large class of spaces (separable metric spaces).

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One can use a tiling by small open sets that overlap no more than two at a time to explain why the dimension of the circle or figure-eight is one. With a little more work, one can also show that the dimension of the Cantor set is zero, while that of the Sierpiński triangle is, perhaps surprisingly, one.

6. Conclusion

With the initial difficulties settled, dimension theory was able to grow and flourish through the mid-twentieth century. There are further notions of dimension that we have not talked about, including the much more common notion of dimension of a manifold (which does not apply to fractals like the Sierpiński triangle), Hausdorff dimension (which takes on non-integer values for fractals, but is not a topological invariant), extension dimension, homological and cohomological dimension, and several others.

However, even a patchy telling of the story of dimension brings together several themes that frequently occur in mathematics: a seemingly commonplace notion that turned baffling after a little probing, a question about ordinary maps on ordinary spaces that forced the construction of deeply counter-intuitive objects, answers that drew on physical intuition, philosophical ideas, and mathematical arguments, and the people who supported, challenged, riled, and illuminated the path for one another through the process of discovery.

Acknowledgement

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Suggested Reading

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