Six Exponentials Theorem – Irrationality*

Michel Waldschmidt

Let p, q, r be three multiplicatively independent positive rational numbers and u a positive real number such that the three numbers p^u, q^u, r^u are rational. Then u is also rational. We prove this result by introducing a parameter L and a square $L \times L$ matrix, the entries of which are functions $(p^{s_1} q^{s_2} r^{s_3})^{(t_0+t_1 u)x}$. The determinant $\Delta(x)$ of this matrix vanishes at a real point $x \neq 0$ if and only if u is rational. From the hypotheses, it follows that $\Delta(1)$ is a rational number; one easily estimates a denominator of it. An upper bound for $|\Delta(1)|$ follows from the fact that the first $L(L-1)/2$ Taylor coefficients of $\Delta(x)$ at the origin vanish.

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Our goal is to give a complete elementary proof of the following result:

Theorem. *Let p, q, r be three positive rational numbers which are multiplicatively independent, namely, the only relation* $p^a q^b r^c = 1$ *with integers a, b, c is for* $a = b = c = 0$. *Let u be a real number such that* p^u, q^u *and* r^u *are rational numbers. Then u is a rational number.*

Recall that for $x > 0$ and $u \in \mathbb{R}$, $x^u = \exp(u \log x)$.

This statement is a special case of the six exponentials theorem, where the assumption that p, q, r, and x^u are rational is replaced with the assumption that they are algebraic (and u may be a complex number). More information on this result from transcendental number theory is available in [1–6] for instance.

Keywords
Six exponential theorem, rational numbers, real numbers, transcendental number theory.

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From the fundamental theorem of arithmetic, it follows that any three distinct prime numbers are multiplicatively independent.

From the fundamental theorem of arithmetic, it follows that any three distinct prime numbers are multiplicatively independent. One easily checks that if u is a rational number and p a prime number such that p^u is rational, then u is an integer. Examples with an irrational u and a prime number p with p^u an integer n are obtained with $u = (\log n)/(\log p)$. We do not know whether there exists an irrational u and two multiplicatively independent rational numbers p and q with p^u and q^u rational numbers: proving that there is no such example in the four exponentials conjecture for irrationality, so far it is an open problem. Writing $p^u = r$ and $q^u = s$, we would get

$$u = \frac{\log r}{\log p} = \frac{\log s}{\log q}\cdot$$

The problem is to prove that a 2×2 matrix

$$\begin{pmatrix} \log p & \log q \\ \log r & \log s \end{pmatrix}$$

has a rank 2 when p, q, r, s are positive rational numbers with multiplicatively independent p, q and multiplicatively independent p, r.

If u is a positive real number such that x^u is an integer for each positive integer x, then u is an integer.

A consequence of the theorem is the following statement:

Corollary. *If u is a positive real number such that x^u is a rational number for each positive rational number x, then u is an integer.*

In his paper 'Transcendental Numbers' [7], Heini Halberstam quotes the following special case of the above corollary:

> *If u is a positive real number such that x^u is an integer for each positive integer x, then u is an integer.*

According to Halberstam: *This result appeared as a problem in the 1972 Putnam*[1]*. Prize competition, and not one of more than 2000 university student competitors gave a solution; the solution, though not hard, could well elude even a professional mathematician for several hours (or days).*

[1] The reference to Putnam is: 32nd Putnam 1971 Question A6. https://prase.cz/kalva/putnam/putn71.html

A proof of this special case, using the calculus of finite differences, is given in [7]. See also [5, Chapter I, Exercise 6, p.I-12–I-13] and [1]. It might be interesting to find a similar proof of the above corollary.

Here is the idea of the proof of the theorem. Given that the six numbers p, q, r, p^u, q^u and r^u are rational. Then the three functions of a real variable p^x, q^x and r^x take rational values at all points of the form $\xi_{\underline{t}} = t_0 + t_1 u$ with $\underline{t} = (t_0, t_1) \in \mathbb{Z}^2$. For $\underline{s} = (s_1, s_2, s_3) \in \mathbb{Z}^3$, the same is true for the function $f_{\underline{s}}(x) = (p^{s_1} q^{s_2} r^{s_3})^x$. Select a sufficiently large integer N (we will make this assumption explicit at the end of the proof). Set $S = N^2$, $T = N^3$, $L = N^6$, so that $L = S^3 = T^2$. The determinant[2]

$$\Delta = \det\Big(f_{\underline{s}}(\xi_{\underline{t}})\Big)_{\substack{0 \le s_j < S \\ 0 \le t_i < T}},$$

is a rational number.

[2]This determinant is well-defined up to its sign, depending on the ordering of the $\underline{s}$ and of the $\underline{t}$.

Let D be a common denominator of p, q, r, p^u, q^u and r^u. Since $s_j \ge 0$ and $t_i \ge 0$ are integers, the numbers

$$D^{6ST} f_{\underline{s}}(\xi_{\underline{t}}) = (Dp)^{s_1 t_0} (Dq)^{s_2 t_0} (Dr)^{s_3 t_0} (Dp^u)^{s_1 t_1} (Dq^u)^{s_2 t_1} (Dr^u)^{s_3 t_1},$$

are integers, hence $D^{6LST}\Delta$ is a rational integer. We will produce an upper bound for $|\Delta|$, in particular, for sufficiently large N, we will check $|\Delta| < D^{-6LST}$, hence $\Delta = 0$. And we will show that the condition $\Delta = 0$ implies that u is rational.

Let us start by proving this last claim. The condition $\Delta = 0$ means that there are rational numbers $a_{\underline{s}}$, not all of which are zero, such that the function

$$F(x) = \sum_{s_1=0}^{S-1} \sum_{s_2=0}^{S-1} \sum_{s_3=0}^{S-1} a_{\underline{s}} f_{\underline{s}}(x).$$

satisfies

$$F(\xi_{\underline{t}}) = 0 \quad \text{for} \quad 0 \le t_0, t_1 < T. \tag{1}$$

Since p, q, r are multiplicatively independent, the three numbers $\log p, \log q, \log r$ are $\mathbb{Q}$–linearly independent. Using the next lemma for

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$$\{w_1, \ldots, w_n\} = \{s_1 \log p + s_2 \log q + s_3 \log r \mid 0 \le s_1, s_2, s_3 < S\}$$

with $n = L$, we deduce that the conditions (1) imply that the numbers $\xi_{\underline{t}}$ are not all distinct, hence u is a rational number.

Lemma 1. *Let $w_1, \ldots, w_n$ be pairwise distinct real numbers and $a_1, \ldots, a_n$ real numbers, not all of which are zero. Then the number of real zeroes of the function*

$$F(x) = a_1 \mathrm{e}^{w_1 x} + \cdots + a_n \mathrm{e}^{w_n x}$$

is $\leq n - 1$.

Proof. We use the following result, known as Rolle Theorem (1691)[3]: *If a real function of a real variable of class C^1 (continuously derivable) has at least m real zeroes, then its derivative has at least $m - 1$ zeroes.*

[3]Already stated by Bhāskarācārya (Bhāskara II, 1114–1185).

We prove Lemma 1 by induction on n. The statement is true for $n = 1$: the function $a_1 \mathrm{e}^{w_1 x}$ has no zero. Assume that the result holds for $n - 1$ for some $n \geq 2$. Assume also, without loss of generality, that $a_1, \ldots, a_{n-1}$ are not all zero. The derivative $G(x)$ of the function $\mathrm{e}^{-w_n x} F(x)$ can be written

If a real function of a real variable of class C^1 (continuously derivable) has at least m real zeroes, then its derivative has at least $m - 1$ zeroes.

$$G(x) = a_1(w_1 - w_n)\mathrm{e}^{(w_1 - w_n)x} + \cdots + a_{n-1}(w_{n-1} - w_n)\mathrm{e}^{(w_{n-1} - w_n)x}$$

with coefficients $a_1(w_1 - w_n), \ldots, a_{n-1}(w_{n-1} - w_n)$, not all of which are zero, while in the exponent $w_1 - w_n, \ldots, w_{n-1} - w_n$ are pairwise distinct. From the inductive hypothesis, we deduce that $G(x)$ has at most $n - 2$ zeroes. From Rolle Theorem it follows that $\mathrm{e}^{-w_n x} F(x)$, hence also $F(x)$, has at most $n - 1$ zeroes. □

It remains only to estimate $|\Delta|$ from above. The upper bound will not use arithmetic assumptions: it holds also when the numbers p, q, r, p^u, q^u and r^u are not assumed to be rational, only real numbers.

We introduce the function

$$\Psi(x) = \det\Big(f_{\underline{s}}(\xi_{\underline{t}} x) \Big)_{\substack{0 \leq s_j < S \\ 0 \leq t_i < T}},$$

so that $\Delta = \Psi(1)$. We expand the determinant and write

$$\Psi(x) = \sum_{\sigma \in \mathfrak{S}_L} \epsilon(\sigma) \mathrm{e}^{w_\sigma x},$$

where $\mathfrak{S}_L$ is the set with $L!$ elements which are the bijective maps $\sigma : \underline{s} \to (t_{0,\sigma(\underline{s})}, t_{1,\sigma(\underline{s})})$ from the set of $\underline{s} = (s_1, s_2, s_3)$ $(0 \le s_j < S$, $j = 1, 2, 3)$ onto the set of $\underline{t} = (t_0, t_1)$, $(0 \le t_i < T$, $i = 1, 2)$, $\epsilon(\sigma)$ is the signature of σ (depending on the order which was chosen for the $\underline{s}$ and the $\underline{t}$), and, for $\sigma \in \mathfrak{S}_L$,

$$w_\sigma = \sum_{s_1=0}^{S-1} \sum_{s_2=0}^{S-1} \sum_{s_3=0}^{S-1} (s_1 \log p + s_2 \log q + s_3 \log r)(t_{0,\sigma(\underline{s})} + t_{1,\sigma(\underline{s})} u).$$

We will use the upper bound

$$|w_\sigma| \le LST(1+u)\log(pqr). \tag{2}$$

We write the Taylor expansion at the origin of ψ:

$$\Psi(x) = \sum_{m \ge 0} \alpha_m x^m.$$

The next Lemma shows that

$$\alpha_0 = \alpha_1 = \cdots = \alpha_{M-1} = 0$$

with $M = L(L-1)/2$.

Let us recall that an analytic function at 0 is the sum in a neighbourhood of 0 of a convergent series: this series is the Taylor expansion of the function at the origin.

Lemma 2. *Let $f_1, \ldots, f_L$ be analytic functions at* 0 *and $\xi_1, \ldots, \xi_L$ be complex numbers. The Taylor expansion at the origin of the function*

$$F(x) = \det\Big(f_\lambda(\xi_\mu x)\Big)_{1 \le \lambda, \mu \le L},$$

say

$$F(x) = \sum_{m \ge 0} \alpha_m x^m,$$

satisfies

$$\alpha_0 = \alpha_1 = \cdots = \alpha_{M-1} = 0.$$

Proof. From the multilinearity of the determinant, it is sufficient to prove this lemma when each $f_\lambda(x)$ is a monomial x^{n_λ}. If the determinant

$$\det\Big((\xi_\mu x)^{n_\lambda}\Big)_{1 \le \lambda, \mu \le L} = x^{n_1 + n_2 + \cdots + n_L} \det\Big(\xi_\mu^{n_\lambda}\Big)_{1 \le \lambda, \mu \le L}$$

is not zero, then $n_1, \ldots, n_L$ are pairwise distinct, hence

$$n_1 + n_2 + \cdots + n_L \geq 0 + 1 + \cdots + (L - 1) = M.$$

□

In order to prove the expected upper bound for $|\Delta|$, we introduce an auxiliary parameter $R > 1$; we will choose $R = \mathrm{e}$, the basis of the Napierian logarithms, but any constant > 1 would do.

Lemma 3. *Let $w_1, \ldots, w_J$, $a_1, \ldots, a_J$ be real numbers. If the Taylor expansion at the origin of the function*

$$F(x) = \sum_{j=1}^{J} a_j \mathrm{e}^{w_j x},$$

say

$$F(x) = \sum_{m \geq 0} \alpha_m x^m,$$

has

$$\alpha_0 = \alpha_1 = \cdots = \alpha_{M-1} = 0,$$

then

$$|F(1)| \leq R^{-M} \sum_{j=1}^{J} |a_j| \mathrm{e}^{|w_j| R}.$$

Proof. We have

$$F(x) = \sum_{j=1}^{J} a_j \sum_{m \geq 0} \frac{w_j^m}{m!} x^m = \sum_{m \geq 0} \sum_{j=1}^{J} a_j \frac{w_j^m}{m!} x^m,$$

hence

$$\alpha_m = \sum_{j=1}^{J} a_j \frac{w_j^m}{m!}$$

and

$$|\alpha_m| \leq \sum_{j=1}^{J} |a_j| \frac{|w_j|^m}{m!}.$$

Therefore

$$|F(1)| = \left|\sum_{m\geq M}\alpha_m\right| \leq \sum_{m\geq M}|\alpha_m| \leq R^{-M}\sum_{m\geq M}|\alpha_m|R^m$$
$$\leq R^{-M}\sum_{m\geq M}\sum_{j=1}^{J}|a_j|\frac{|w_j|^m}{m!}R^m \leq R^{-M}\sum_{j=1}^{J}|a_j|\mathrm{e}^{|w_j|R}.$$

□

Thanks to Lemma 2, we can use the upper bound given by Lemma 3 for the function Ψ with $J = L!$ and $a_j \in \{-1, 1\}$; since $\Psi(1) = \Delta$, we deduce from (2):

$$|\Delta| \leq R^{-M}L!(pqr)^{LST(1+u)R}.$$

It remains to check

$$L!(pqr)^{LST(1+u)R}D^{6LST} < R^M, \tag{3}$$

for sufficiently large N. Recall the choice of parameters

$$L = N^6, \quad S = N^2, \quad T = N^3, \quad M = \frac{1}{2}L(L-1).$$

One checks that the condition (3) is satisfied with $R = \mathrm{e}$ as soon as

$$N > 12\log D + 2\mathrm{e}(1+u)\log(pqr) + 1.$$

Comments. Where does this determinant Δ come from? There is a long history behind it. The transcendence proofs originate in the proof by Hermite of the transcendence of the number e; they have been developed since 1873 by many a mathematician, including Siegel, Lang and Ramachandra, who are at the origin of the six exponentials theorem. The first occurrence of this theorem is in a paper by Alaoglu and Erdős [8] on Ramanujan highly composite numbers, where they also study superabundant and colossally abundant numbers. They asked Siegel whether it was true that the conditions that p^u and q^u are integers with p and q distinct primes imply that u is an integer. Siegel replied that he did not know how to prove such a result (which is still an open problem), but that he knew how to get the conclusion if one added r^u, like in the theorem.

[8, p. 449] *This question leads to the following problem in Diophantine analysis. If p and q are different primes, is it true that p^x and q^x are both rational only if x is an integer?*

[8, p. 455] *It is very likely that q^x and p^x cannot be rational at the same time except if x is an integer. At present, we cannot show this. Professor Siegel has communicated to us the result that q^x, r^x and s^x cannot be simultaneously rational except if x is an integer*

The proofs by Lang and Ramachandra are given in [2] and [4]. These proofs involve auxiliary functions. To replace these functions with the so-called interpolation determinant Δ is an idea of M. Laurent [3]. There is already a similar determinant introduced by Cantor and Straus in their paper [9] on a theorem of Dobrowolski dealing with a question of Lehmer. Further references are given in [5, 6].

The interested reader may compare this proof with the proof in [1].

Suggested Reading

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