Fundamental Concepts of Synchronization*

An Introduction: From Classical to Modern

Nitu Kumari and Shubhangi Dwivedi

Objects with rhythms naturally synchronize. Synchronization is the coordination of events in order to run the system uniformly. Yet the phenomenon went entirely undocumented until 1665. Since the pioneering description of synchronization by Huygens, the phenomenon has been studied by various researchers in an interdisciplinary manner. Many researchers have contributed to the development of synchronization theory proving that synchronization occurs in coupled non-linear dissipative oscillators. Such oscillators range from mechanical clocks and population dynamics to human heart and neural networks. This article aims to explain the basic principles of synchronization theory. The history and applications of synchronization are discussed in real-world scenarios. We address different types of synchronization with a detailed discussion on the simplest type of synchronization. The phenomenon of synchronization applies to oscillations of different forms—periodic, noisy, and chaotic in nature. Here, we specifically discuss the oscillators which can hold synchronization. In particular, we provide an overview of self-sustained periodic and chaotic oscillators with a detailed description of different forms of these oscillators in phase space. Further, a summary of further research challenges has also been given for the future development of advanced applications based on natural synchronization phenomenon.

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Keywords
Synchronization, oscillators, chaos, periodic motion, coupling, phase locking, frequency locking.

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Figure 1. Christiaan Huygens (1629–1695).
(Image Source: *Wikipedia*)

1. History of Synchronization

In 1665, Christiaan Huygens reported that two identical clocks hung on a beam synchronized with each other after about thirty minutes.

One of the first recorded perceptions of synchronization is by the Dutch researcher Christiaan Huygens (1629–1695, *Figure* 1). In a time when science depended vigorously on perception, experimentation, and reflection, Huygens made a fortunate disclosure. He made exact pendulum clocks. In 1665, it was reported by him that two identical clocks hung on a beam synchronized with each other after about thirty minutes. He observed that two identical pendulum clocks, weakly coupled through a heavy beam (a wooden bar supported by two chairs), soon synchronized with the same period and amplitude but the two pendulums swang in opposite directions. Since disturbances did not affect the synchronous motion, Huygens believed that synchronization was caused by airflow. However, after further experimentation, he concluded that the weak coupling of the two clocks through the beam was the reason for this state of anti-phase synchronization. This impact which Huygens called 'odd kind of sympathy' is nowadays known as synchronization. *Figure* 2 shows the original drawing made by Huygens.

354 years ago, in a letter to the Royal Society of London, Christiaan Huygens described "an odd kind of sympathy" between two pendulums mounted side by side on a wooden beam. This inspired the modern studies on synchronization in coupled non-

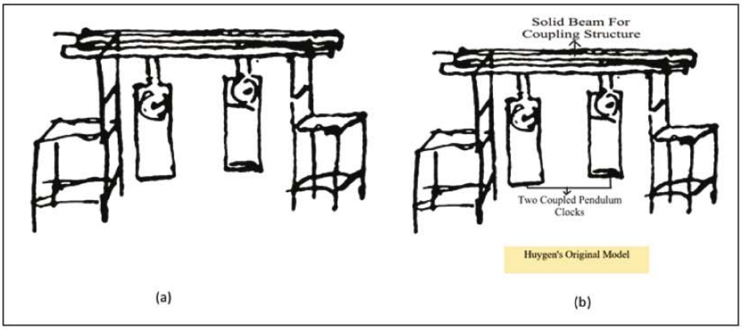

Figure 2. **(a)** Shows an original drawing of Huygens. Reprinted from [1]. **(b)** Huygens experiments with pendulum clocks.

linear oscillators [1]. He reported on his observation of synchronization of two pendulum clocks which he had invented shortly before:

"...It is quite worth noting that when we suspended two clocks so constructed from two hooks embedded in the same wooden beam, the motions of each pendulum in different swings were so much in agreement that they never receded the least bit from each other, and the sound of each was always heard simultaneously. Further, if this agreement was disturbed by some interference, it re-established itself in a short time. For a long time I was amazed at this unexpected result, but after a careful examination finally found that the cause of this is due to the motion of the beam, even though this is hardly perceptible."

Watch an experiment online here: https://youtube/Aaxw4zbULMs. The interesting phenomenon first discovered in 1665 by Christiaan Huygens has been explained in this episode. One can see that five pendulum clocks placed in a room together will mysteriously synchronize with each other. How can mechanical objects transmit an influence when they are not touching? What is the cause of this 'odd kind of sympathy'? Watch the video to find out!

One can see that five pendulum clocks placed in a room together will mysteriously synchronize with each other.

The Huygens' set-up was not precisely like the set-up as shown in the video above, because the clocks are attached to a fixed object. Still, these clocks will eventually synchronize. The fundamental idea is that although the clocks are connected to a solid object, there is still a transfer of energy taking place. To keep the pendulum clocks in time, the winding mechanism must deliver a small amount of kinetic energy to the pendulum to compensate for friction losses. This transfer of power is called a 'kick'. If there are two such clocks, then the punch of one clock transfers a small amount of energy to the other via the wall. Few examples of differential equations with regular solutions have been used in the analysis of dynamical systems. Huygens did not have the right mathematical tools at that time to explain his observations as differential calculus was not invented then. Even then, he managed to find the mechanism responsible for the sympathy in his clocks. Lacking the requisite mathematics at the time, Huygens contended that the effect was being caused by tiny vibrations in the wooden structure on which the clocks were hanging. Nevertheless, now, with the help of mathematical tools, we can show the possibility of two synchronous states by solving the resulting system of differential equations. Among these two possible synchronous states, the clocks are in exact synchronization in one, and the phase is in exact opposition in the other. The latter state is stable, confirming Huygens' observations. Similar experiments have been done on metronomes. A metronome is a tool that usually beats per minute, creates a noticeable click or other signals at a regular interval that can be controlled by the operator. Musicians use the system to perform with a steady rhythm. It is found that two coupled metronomes can exhibit typical in-phase and anti-phase locking phenomena if their frequencies almost coincide with each other or are rationally related to each other. The synchronization of two coupled metronomes can be applied to verify Huygens' famous experiment of pendulum clocks. To understand this renowned experiment via the application of mathematical tools, we recommend the article 'Anti-phase synchronization of two coupled mechanical metronomes' [2].

It is found that two coupled metronomes can exhibit typical in-phase and anti-phase locking phenomena if their frequencies almost coincide with each other or are rationally related to each other.

2. Evolution of Synchronization

For some reason, the sympathetic motion of pendulum clocks discovered by Huygens, currently known as 'Huygens' synchronization', did not attract the attention of the scientific community at that time. After Huygens, another Dutchman, Engelbert

Kaempfer, travelled to Thailand in 1690 and witnessed the local fireflies flickering at the same time "with great regularity and precision" [3]. Two hundred years later, the English physicist John William Strutt (better known as Lord Rayleigh) found that standing two organ pipes side by side might cause the pipes to sound in complete unison, despite the inevitable significant difference [4], [5]. In the 1920s, radio engineers discovered that cabled electrical generators with different frequencies caused them to vibrate at a similar frequency (the idea behind radio communication systems) [6]. Following this, in 1967, the vibration of crickets inspired the American biologist Art Winfree to suggest a mathematical model of synchronization. Winfree's equation was very difficult to solve, but in 1974, a Japanese physicist named Yoshiki Kuramoto discussed a technique to simplify the mathematical complexity involved [7]. Yoshiki Kuramoto proposed a mathematical model to describe synchronization for the behavior of a large set of coupled oscillators such as chemical and biological oscillators. This model has found widespread applications in neuroscience, fireflies, pacemaker cells, flight starlings, chemical reactants, oscillating flame dynamics, and countless other communities of coupling oscillators. After the groundbreaking research of Pecora and Caroll in 1990, the area of chaotic synchronization has grown significantly [8]. Also in the last few years, Huygens' synchronization has become a relevant topic among scientists and researchers. Many researchers have revisited Huygens' work since then, explaining devices and demonstrations derived from his historical experiments and providing a foundation for our understanding of physical concepts [9], [10].

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Why Study Synchronization?

Before getting into the depth of this topic, it is important to know why we should be interested in this topic at all. This question can be answered with the help of the following observations:

- Everything in nature oscillates.
- Everything in nature is connected.

- Nature is fond of optimizing resources or achieving goals in the most productive and distributed way.

Advanced concepts of synchronization may help us to understand the collective behaviour of complex systems such as human body, heart cells beating together, or fish schools moving as one.

Through nature's synchrony, the principles of nature can be experienced. Advanced concepts of synchronization may help us to understand the collective behaviour of complex systems such as human body, heart cells beating together, or fish schools moving as one. The human behaviour, too, is a treasure house of synchronization phenomenon. For instance, humans perform an amazing array of tasks of varying degrees of difficulty, and they do so at a wide range of levels of activity. Even at the most leisure day of a week, we usually prepare and eat meals, participate in physical exercise, schedule events, socialize with friends and acquaintances, adapt our actions to meet the demands of casual and formal social situations, and think about our personal qualities and weakness. While on holidays, we may create music, write essays, compose poems, establish theories, try to resolve disputes, work as teams, and play games. Usually, these behaviours and modes of activity are analysed in terms of their local dynamics, and a proven approach to understanding that has created a highly compartmentalized psychology discipline. The preservation of internal stability is a crucial part of human physiological systems. Human beings may coordinate their gestures with each other— synchronous ritualistic practices belonging to different cultures, marching in line, dancing, singing, and playing music in unison are a few of the synchronized activities that can be easily noticed. The core idea is that all organizational stages of human activity, ranging from brain function to group dynamics, reflect the creation of functional groups which arises from the ability to gain coordination and to work in cooperation in order to achieve a certain task.

As illustrated by the above examples, research of synchronization phenomenon can help us to gain a better understanding of the human body and the world around us. Therefore, it is crucial that researchers— physicists and biologists— investigate the phenomenon of synchronization with great interest.

Synchronization in Nature

The second question that readers might pose is, "Where can we see synchronization?". Synchronization is ubiquitous in Nature. Following is a list of examples of synchronization phenomenon from our everyday life.

- Laser arrays.
- Electric circuits.
- Pedestrians on a bridge.
- Pacemaker cells in the heart.
- Oscillating chemical reactions.
- Synchronous firings of male fireflies.
- Metronomes placed on a moving surface.
- The spin of the moon synchronises with its orbit.
- Applause at concert halls merges to produce a harmonised sound in time. When people start clapping their hands, they tend to adjust their pace of clapping to the clapping of nearby people.
- Cellular clocks in the brain. Oscillations in the brain lead to individual electrical waves, for example, the α-waves. Some diseases like epilepsy are thought to be related to 'over-synchronization' of the brain. Hence synchronization is not always beneficial.

In 2004, Strogatz wrote a book titled as *Sync: The Emerging Science of Spontaneous Order on Synchronization* [11]. This wonderful book provides a successful account of many of the ideas related to synchronization and is easily accessible.

Synchronization cannot be seen without oscillations.

It should be mentioned here that synchronization cannot be seen without oscillations. The next query that comes in our mind is whether we can see synchronization in all oscillators? The answer is no. We discuss this in detail in the next section.

3. Study of Synchronizing Oscillators

As we mentioned earlier, synchronization is not possible without oscillations. However, not all oscillators can be synchronized.

Here, we discuss in detail those oscillators in which synchronization can be seen. If we expect two or more physical objects to get synchronized over time, what are the fundamental properties those physical objects should hold? To answer this question, let us understand this phenomenon in a little bit more detail. The word synchronization comes from the Greek letters:

Syn = Common and ***Chronos*** = Time.

The word synchronization means sharing the standard time. However, when we talk about synchronization in the context of the theory of dynamical system, then the current time is shared by oscillations. Oscillation is chosen to some degree of repetitions.

Definition of Oscillator

An oscillator is a mechanical or electronic device which works on basic rules of swinging back and forth.

An oscillator is a mechanical or electronic device which works on basic rules of swinging back and forth, i.e., a periodic fluctuation between two objects based on changes in energy. Some typical applications of oscillators include: audio and video systems, alarms and buzzes, metal detectors, stun guns, inverters, ultrasonic devices, radio, television, and other communication devices.

The oscillating objects in Huygens' case are two pendulum clocks and are weakly coupled through the translation of the beam. Here it should be mentioned that not all types of oscillators can be synchronized. The only oscillators that can synchronize are characterised as self-sustained oscillators. Such oscillators can be maintained only in non-linear systems, and the concept of their oscillations is essential for an adequate description of synchronization. Here we provide a detailed explanation of self-oscillators.

Example of Self-sustained Periodic Oscillator

A famous paradigm of a self-oscillator is a mechanical pendulum clock.

A famous paradigm of a self-oscillator is a mechanical pendulum clock (*Figure* 3). Not so long ago, virtually every clock and watch made a tick-tock noise because it was utterly mechanical rather than electrical. Instead of the clock being powered by a battery,

Figure 3. Old-fashioned mechanical pendulum wall clock. (Image Source: explainthatstuff.com/how-pendulum-clocks-work.html)

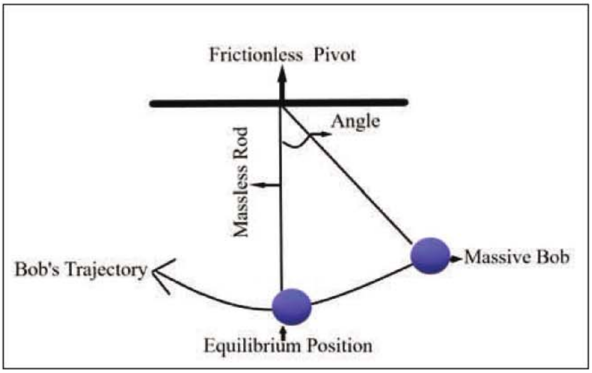

Figure 4. A simple pendulum consists of a mass m hanging from a string of length L and fixed at a pivot point. When displaced to an initial angle of θ^0 and released, the pendulum swings back and forth with the periodic motion. By applying Newton's second law for rotational systems, the equation of motion for the pendulum may be obtained.

it was wound up with a key, and there was a long swinging rod inside, called the pendulum that made sure that the whirring gears kept good time. So, how can those old-fashioned pendulums be modelled? Let us have a look.

To build the model, we define the following variables:

θ = Angle of pendulum,
L = length of rod,
T = tension in rod,
m = mass of pendulum,
g = gravitational constant.

We will derive the equation of motion for the pendulum using the rotational analogue of Newton's second law of motion about a fixed axis, which is:

$$\tau = I\alpha\,,$$

where
τ = net torque, I = rotational inertia, $\alpha = \theta''$ = angular acceleration.

The rotational inertia about the pivot is $I = mL^2$. Torque can be calculated as the vector cross product of the position vector and the force. The magnitude of the torque due to gravity comes out to be

$$\tau = -mgL\sin(\theta).$$

So we have

$$-mgL\sin(\theta) = mL^2\alpha,$$

which simplifies to

$$\theta'' = -\frac{g}{L}\sin(\theta).$$

The above equation gives the motion for the simple pendulum. If the amplitude of angular displacement is small enough then the small-angle approximation ($\sin(\theta) \simeq \theta$) holds. Hence, the equation of motion reduces to the comparison of simple harmonic motion.

$$\theta'' = \frac{g}{L}\theta.$$

For the initial conditions $\theta(0) = \theta_0$ and $\dot{\theta} = \dot{\theta}_0$, the solution of second-order non-linear ordinary differential becomes

$$\theta(t) = \theta_0\cos(\sqrt{\frac{g}{l}}t + \dot{\theta}_0).$$

For numerical simulation, we use the fourth-order Runge-Kutta method. The set of parameter values have been chosen as g = 9.81 meter/sec^2, L = 1 meter. We plot solutions for four different initial conditions. The initial conditions $\theta(0) = (\theta_0, \dot{\theta}_0)$ are taken as $(\frac{\pi}{4}, 0)$, $(\frac{\pi}{2}, 0)$, $(\pi, 0)$ and $(\frac{2\pi}{3}, 0)$ for *Figures* 5(a) and 5(b)

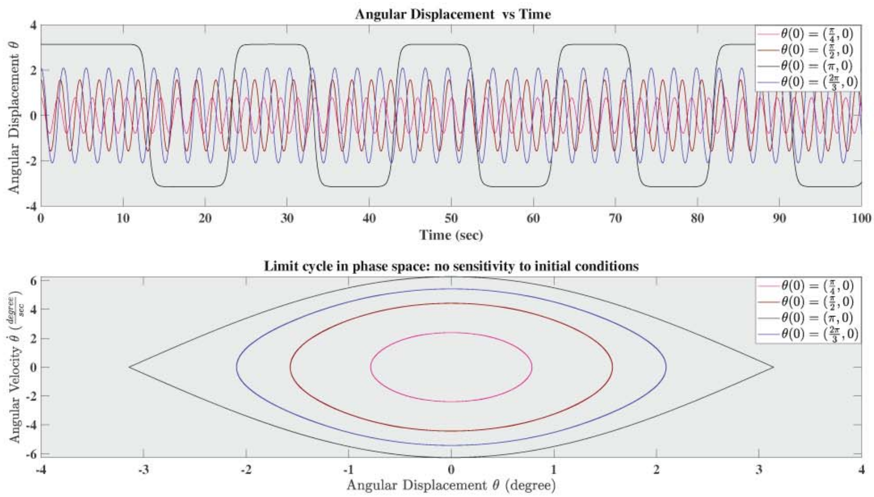

Figure 5. **(a)** Time series plot of angular displacement for four different initial conditions. **(b)** The phase portrait of the system for four different initial conditions.

respectively. These figures show the time plot and phase portrait for these four different initial conditions of the system. The periodic oscillations are represented by a closed curve in the phase space of the system. In *Figure* 5(a), it should be observed that the solution of the system is in periodic motion and in *Figure* 5(b), phase portrait of the system has come out as closed curve for all four different initial conditions. This easy phase space of the system does not change its qualitative behaviour for different initial conditions. From here, we can immediately determine a unique feature of the self-sustained periodic oscillator, i.e. if we push off the phase point from the attractor ('closed curve'), then phase point will return back to the attractor due to the absence of sensitive dependence on the initial condition. Here this 'closed curve' of the system is called as the 'limit cycle'.

> The periodic oscillations are represented by a closed curve in the phase space of the system.

We conclude that the self-sustained oscillations can be characterized by their image in phase space, i.e. by limit cycle. The limit cycle is the simplest attractor, in contrast to the notion of a 'strange attractor'. For a detailed explanation on the attractors of dynamical systems, readers should see [12]. The main features of self-sustained periodic oscillators are described as follows.

Main Features of Self-sustained Oscillators

- Its oscillations do not decay in time.
- Due to friction, it loses energy while oscillating, which means that self-sustained periodic oscillators possess dissipation.
  Explanation: For instance, in systems such as the clock shown above, energy is lost due to mechanical friction, and in electrical devices, dissipation is due to heat loss.
- It consumes energy (power) from outside. It means despite energy loss, in order to sustain oscillations, self oscillators need to feed on a source of energy.
  Explanation: In case of the pendulum clock, power is provided through lifting the load. When the load gradually comes down, it releases potential energy. This potential energy of the load transfers into the energy of oscillations.
- Power in-flux can be constant in time.
  Explanation: Power can arise in a non-oscillatory manner. For example, in the case of a clock, these self oscillators choose the amplitude and time-scale for their oscillations of initial conditions. After some time, the system settles down to the same pattern of oscillations automatically.
- It is persistent.
  Explanation: After perturbation, the swings spontaneously resume identically.

Not every oscillator is self-sustained or an auto oscillator. Physicists are well-acquainted with forced and parametric rhythms, but typically not with self-oscillation, a property of specific dynamic systems that gives rise to a wide range of useful and destructive vibrations. In a self-sustained oscillator, the oscillation is itself regulated so that it acts in phase and causes negative damping which feeds the vibration energy; no external rate needs to be adjusted to the resonant frequency. A detailed theoretical description of self-oscillations can be seen in [13]. We compare self-sustained oscillators with other existing oscillators and list a few distinctions and similarities between self-sustained oscillators and other oscillators in *Table* 1.

Physicists are well-acquainted with forced and parametric rhythms, but typically not with self-oscillation, a property of specific dynamic systems that gives rise to a wide range of useful and destructive vibrations.

Sr. No.	Similarities between all oscillators	Differences between oscillators and self-sustained oscillators
1.	All oscillators have oscillations or rhythms.	The oscillators, which are not self-sustained, do not consume energy from outside i.e. such oscillators do not have power influx at all.
2.	Due to friction, all kind of oscillators lose energy while oscillating.	The oscillations of non self-sustained oscillator decay in time and eventually stop.
3.		Since all oscillations of non self-sustained oscillators stop eventually, we cannot speak of shape, time scale and amplitude of any oscillations of non self-sustained oscillator.

Table 1. Self-sustained oscillators vs. oscillators.

From the above discussion, it is clear that self-oscillation is a process where an entity preserves periodic motion on a non-periodic signal. It often occurs naturally, few examples being, heartbeat, sea waves, and leaves fluttering. We can verify that the oscillator is self-sustaining if we remove it from the environment and test whether it still oscillates. One can, therefore, separate a firefly from other insects, place it at a constant temperature, light, etc., and note that even if it is isolated, it nevertheless creates rhythmic flashes. An insect, a plant, or a human volunteer can be separated from others but is likely to still show patterns of daily activity. Hence, these oscillators can be classified as self-sustained oscillators. Signal-responsive materials allow the creation of artificial self-oscillators driven by different forms of energy, such as heat, light, and chemicals. Artificial self-oscillators show high potential for power generation and can be used in autonomous mass transport, and self-propelled micro-robotics applications [14]. In *Table* 2, we provide a list of some common examples of natural and man-made self-sustained oscillators.

Sr. No.	Naturally occurring self-sustained oscillators	Man-made self-sustained oscillators
1.	Ocean waves	Turbines
2.	Calcium oscillations inside a cell	Pendulum clock
3.	The firing of neurons	Lasers sources
4.	Predator-prey relationships	Quartz watches
5.	The pulsation of variable stars	Many musical instruments (including the human voice)
6.	Isolated frog‘s heart	Electronic generator (named after Vander-Pol)

Table 2. Natural vs. Artificial self-sustained oscillators

4. Basic Definition of Synchronization

Till now, we have seen that only self-oscillators can be synchronized. The three important parameters of self-sustained oscillators are:

1. Shape.
2. Amplitude.
3. Time-scale.

Synchronization occurs only when the time scale becomes either equal or rationally related.

Out of these three parameters, only time scale is relevant to synchronization. Synchronization occurs only when the time scale becomes either equal or rationally related. When we analyse synchronization, we usually ignore the shape and amplitude of the oscillations and pay attention only to the time scale which is a parameter that characterises how fast the quantity being observed roughly repeats a certain pattern. Synchronization is classically defined as:

“Synchronization is an adjustment of time scales of self-oscillations due to interactions.”

i.e., synchronization refers to the rhythmic adjustment of self-sustained periodic oscillators due to their weak interaction. This adjustment can be referred to as phase locking and frequency entrainment. In most cases, before the systems start to interact, they oscillate on different time scales. But as a result of some coupling between two or more systems, when systems start interacting, then their time scales start changing.

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Description of Synchronization in Dynamical Systems

To summarise, we can describe self-sustained oscillators as autonomous dynamical systems and their non-damped oscillations as self-sustained oscillations. A dynamical system can be defined as a system of one or more variables (self-oscillators) which evolve in time according to a given rule. In general, there are two types of dynamical systems.

- **Differential Equations:** Time is continuous (called flow)

$$\frac{dx}{dt} = f(x), x \in R^n.$$

- **Difference Equations (iterated maps):** Time is discrete (called cascade)

$$x_{n+1} = x_n, n \in \{0\} \cup N.$$

In the phase space, the self-sustained oscillators show bounded steady state behaviour. More precisely, we can say that the bounded region of the phase space corresponds to one of the two types of behaviour: a stable equilibrium point or a periodic or quasi-periodic oscillation. Many deterministic non-linear systems exhibit more complex invariant sets which act as attractors for their dynamics despite fixed-point solutions and limit cycles. Self-sustained chaotic oscillators produce such dynamics in phase space.

Self-sustained chaotic oscillators are also described by autonomous equations. Therefore, all instants of time are equivalent (that means self-sustained oscillators are time-independent). If we describe the oscillations of dissipative, self-sustained chaotic systems in the phase space, then we find that it does not correspond to such simple geometrical objects like a limit cycle any more, but rather to complex structures that are called 'strange attractors'. The pictures of the limit cycle (periodic), a torus(quasi-periodic), and strange attractor (chaotic) have been displayed in *Figure* 6.

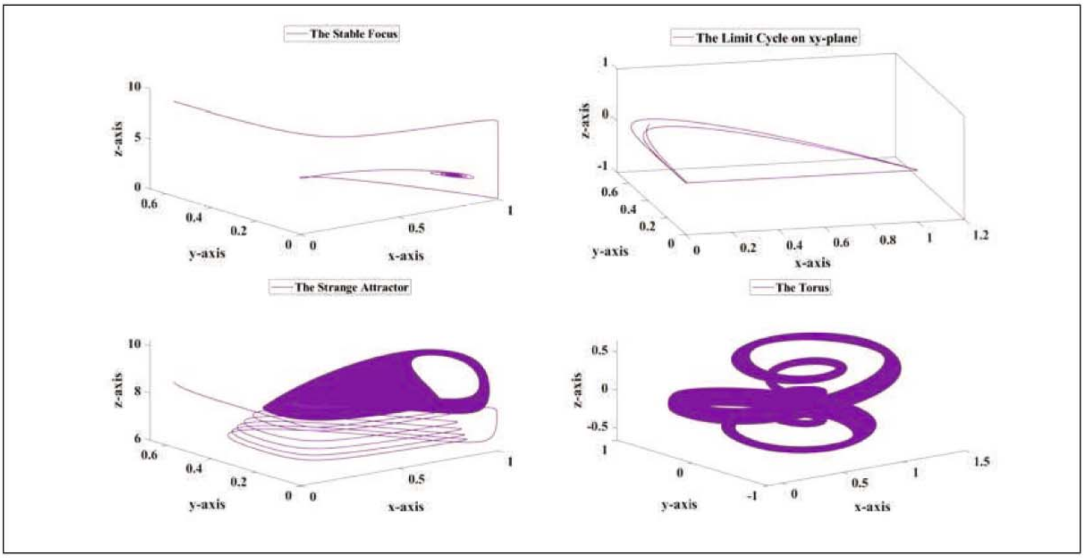

Figure 6. Image of self oscillators that can synchronize: **(a)** The point attractor, **(b)** The limit cycle, **(c)** The strange attractor, and **(d)** The torus.
(Figures **a–c** are reprinted from [15] and **d** is adapted from [16].)

We have shown the dynamics of a tri-trophic food chain model subjected with Refugia and Allee effect in prey species for subfigures (a), (b), and (c) of *Figure* (6) [15]. We refer the same set of parameters for stable focus, limit cycle and chaotic attractor which are mentioned in *Figure* (3) of the article [15]. For torus, we refer the model (1) of article [16] with initial condition (0.4999, 0.01, 0.234). Among these dynamics, the synchronization of periodic and chaotic oscillators have been widely studied. However, the synchronization of quasi-periodic oscillators has not been widely studied.

The different oscillatory moves are characterised by different attractors in the phase space of the system. Now, from the above discussion, it is clear that we can describe synchronization in dynamical systems with this transition between attractors which occur through bifurcations. Attractors and bifurcation can be detected by analysing the phase portraits. Thus when the parameters of the system change and the transition occurs from no synchronization to synchronization through a precise mechanism, then the behaviour of time scales can be explained through the bifurcation taking place in the joint phase space of the interacting system.

5. Types of Synchronization

In the previous section, we saw that synchronization has been seen only in self-sustained oscillators, and these oscillators can be periodic or chaotic. Various synchronization schemes have been discovered on the basis of the type of oscillators. Here we name a few of them.

1) Periodic Self-sustained Oscillator

Based on external forcing and coupling (unidirectional coupling or bi-directional coupling in coupled oscillators), there are mainly three types of synchronization mechanisms. The three mechanisms are as listed below:

Based on external forcing and coupling, there are mainly three types of synchronization mechanisms.

(a) Phase or frequency locking: For two self oscillators with phases ϕ_1 and ϕ_2, the phenomenon of phase-locking can be defined in terms of coincidences of phases. In mathematical notation, it can be described as

$$n\phi_1 - m\phi_2 = constant,$$

where (n, m) is a pair of integers and designates the ratio of oscillation periods. A weaker notion for synchronization is given by

$$|n\phi_1 - m\phi_2| \leq constant.$$

This definition allows systems with bounded variation in phase to be defined as synchronized. We refer to this as 'frequency locking' since relative phase may vary.

The biological mechanisms through which oscillators are coupled and can become synchronized differ from situation to situation. Based on the phase difference, the phase synchronization can be of many types such as:

- Hybrid synchronization
- Anti-synchronization
- Identical synchronization

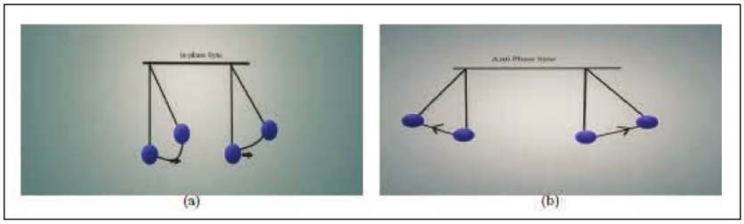

Figure 7. **(a)** Synchronized pendulums with phase difference zero **(b)** Anti-synchronized pendulums with phase difference 180.

In some cases, the oscillators are in anti-phase with one another, i,e., their phases differ by 180 degrees. If phases of the oscillators differ by zero-degree, then we say that oscillators are completely synchronized.

In some cases, the oscillators are in anti-phase with one another, i,e., their phases differ by 180 degrees. If phases of the oscillators differ by zero-degree, then we say that oscillators are completely synchronized. *Figure* (7) shows the image of two synchronized and anti-synchronized oscillators.

Based on weak or strong forcing and unidirectional or bi-directional coupling, forced synchronization falls into three categories and these categories are listed as

1. Phase approximation for weak forcing
2. Higher order phase locking or frequency locking
3. Moderate and strong forcing

(b) Suppression of natural dynamics.
(b) Homo-clinic mechanism.

In the case of periodic self-sustained oscillators, we provide the mathematical definition of phase and frequency locking and suggest the book *Synchronization: A Universal Concept in Non-linear Sciences* [17] for interested readers.

2) Chaotic Self-sustained Oscillator

Before discussing the theory of synchronization of chaos, we will look at the basic definition and proprieties of chaos. One of the most important achievements of non-linear dynamics within the last few decades was the discovery of complex, chaotic motion rather than a simple oscillator. It can be defined as follows:

'Chaos is a long-term aperiodic behaviour of a deterministic system that exhibits SDIC.'

In the above definition of chaos, SDIC stands for sensitive dependence on the initial condition. Because of this property, chaos is also referred to as the 'butterfly effect'. It signifies that a small change in the initial conditions (butterfly wings up vs. down) could lead to entirely different trajectories. Lorentz chaotic system is the first chaotic system observed by a meteorologist. In 1963, Ed Lorentz published his famous work, where a strange attractor was found in numerical experiments in the context of studies of turbulent convection [18]. This publication led to a boom in the field of non-linear science. Since then, many chaotic systems have been discovered and studied by researchers. Famous examples among them are the Lorentz System, Rossler System, and Chua System. The book *The First Course in Chaotic Dynamical Systems: Theory and Experiment*, can be followed by the readers for a better understanding of chaotic dynamical systems [19]. Chaotic dynamical systems and chaos control is a theme that has been widely developed in the last few years. The study of systems with this kind of behaviour is well-documented. In 1998, Pecora and Caroll explained the phenomenon of synchronization of chaos [8]. The field of synchronization of coupled chaotic systems gained popularity and momentum after the work of Pecora and Carrol. In the present day, scientists realise that chaotic behaviour can be observed in experiments and in computer models of response from all fields of science. Although chaotic systems have SDIC (sensitive dependence on initial conditions), they are susceptible to synchronization. Chaotic synchronization deals with the possibility of two or more chaotic systems oscillating in a synchronized way, or it can be said that synchronization of chaos refers to the tendency of two or more chaotic systems which are coupled together to undergo closely related motion.

Chaos is also referred to as the 'butterfly effect'. It signifies that a small change in the initial conditions (butterfly wings up vs. down) could lead to entirely different trajectories.

Synchronization of chaos can be achieved by adjusting a given property of chaotic systems to collective behaviour.

- Synchronization of chaos can be achieved by adjusting a given property of chaotic systems to collective behaviour.
- Synchronization of chaos can be achieved by introducing a coupling term between the two chaotic systems and providing a controlling feedback in one that will eventually cause its trajectory to converge to that of the other, and then remain synchronized

with it.

There are mainly four types of chaos synchronization as listed below:

1. Complete synchronization (strong coupling)
2. Generalised synchronization
3. Phase synchronization (weak coupling)
4. Lag synchronization

Because of its potential applications in various areas, this field has invited tremendous research attention. In the literature, various synchronization schemes, such as variable structure control, parameters adaptive control, O G Y method, observer-based control, time-delay feedback approach, backstepping design technique and so on, have been successfully applied to achieve chaos synchronization.

The above mentioned four types of synchronization of chaos have their sub-types. From this broad category of synchronization, we opt for the most straightforward kind of synchronization of chaos. Further, we will discuss complete synchronization for chaotic self-sustained oscillators. Details about other types of synchronization of chaos can be learnt from the article 'Tutorial and review on the synchronization of chaotic dynamical systems' [20]. Here, we only discuss the most common types of synchronization of periodic and chaotic oscillators. In the next section, the mathematics behind forced synchronization of periodic oscillators and complete synchronization of coupled oscillators have been discussed in detail.

6. Study of Most Common Types of Synchronization

Forced Synchronization in Periodic Self-sustained Oscillators

It is the simplest case of synchronization. This type of synchronization occurs when external force has been applied through coupling. Coupling can be done by applying force to single or more than one self-sustained oscillators of a dynamical system, two dynamical systems or more than two dynamical systems. The coupling in coupled dynamical systems can be unidirectional or bi-directional.

Let us understand it in a two-dimensional continuous dynamical system. Suppose force is applied to one direction only. Note that a single self oscillator is an autonomous dynamical system and can be defined in mathematical notation as:

$$\frac{dx}{dt} = f(x).$$

A forced dynamical system becomes a non-autonomous dynamical system and can be defined in mathematical notation as:

$$\frac{dx}{dt} = g(x, v(t)).$$

Here $v(t)$ is periodic force such that $g(x, 0) = f(x)$. Let us have a look at a problem on synchronization of periodically forced Vander-Pol oscillator.

Periodically Forced Vander-Pol Oscillator: We consider the two-dimensional Vander-Pol oscillator with parameter μ. The Vander-Pol oscillator is a self-sustained oscillator which has a limit cycle if the setting $\mu > 0$. To model the simplest case, we apply an external force into the second equation with coupling strength B. We opt for external force in the form of cosine with amplitude 1 and frequency ω.

$$\begin{aligned} \dot{x_1} &= x_2, \\ \dot{x_2} &= \mu(1 - {x_1}^2)x_2 - v^2 x_1 + B\cos\omega t. \end{aligned} \tag{1}$$

We plot the phase portrait of the Vander-Pol oscillator in MATLAB using Runge-Kutta fourth order technique. We choose initial condition (x_1, x_2) as $(1.1, 1.41)$ and values of parameter are taken as $\mu = 2$, $v = 10$, $B = 0$, $\omega = 9.790$ and $\mu = 100$, $v = 10$, $B = 890$, $\omega = 9.790$ for non-forced and forced systems respectively.

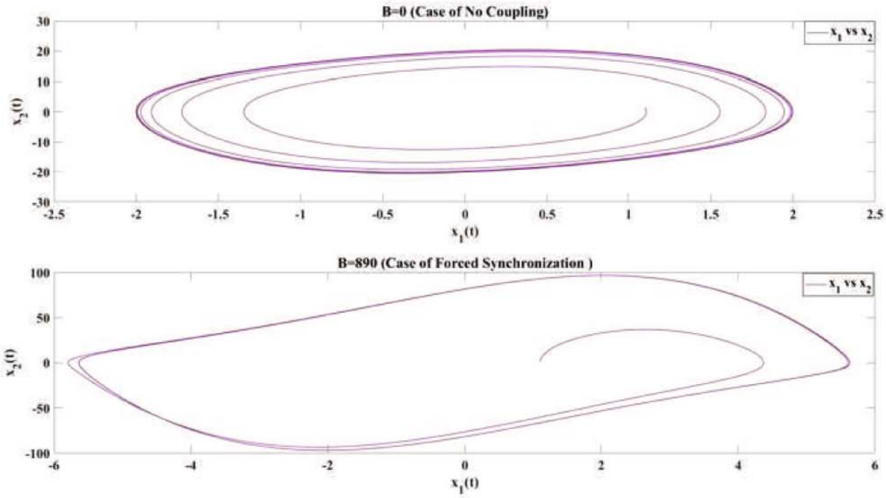

Figure 8. Phase portraits of Vander-Pol system for **(a)** non-forced oscillators, **(b)** forced oscillators.

From *Figure* (8), we can see that the system has a limit cycle for no coupling $B = 0$. But as a result of forcing, the limit cycle which existed in the original unforced order no longer exists at the same location and in identical form in phase space. It means the original system and forced system have different rhythms. If for the right choice of coupling strength B, well-timed beats produce another limit cycle for a frequency equal to or rationally equal to the forcing frequency, then only forced synchronization can be achieved. To understand this, the readers are suggested to read 'Forced synchronization of coupled oscillators' [21].

Unlike periodic oscillations, it is not essential to distinguish between self-sustained and forced systems for phase synchronization of chaos.

Complete Synchronization in Chaotic Self-sustained Oscillators

Unlike periodic oscillations, it is not essential to distinguish between self-sustained and forced systems for phase synchronization of chaos. Contrary to phase synchronization of chaos, complete synchronization can be observed in any chaotic systems; not necessarily autonomous. The phenomenon of complete synchronization is not close to classical synchronization of periodic oscillations. Complete synchronization means the suppression of differences in coupled identical systems. Therefore, this effect cannot be described as entrainment or locking; it is closer to the onset of symmetry. To understand the synchronization of a chaotic system, we consider two dynamical systems – master and slave. The master system is the original system, and the slave system is the controlled system which we want to synchronize with the master system.

Consider the master system as:

$$\dot{x} = F(x), \quad x \in R^n \tag{2}$$

$$F(x) = (F_1(x(t)), F_2(x(t)), \ldots, F_n(x(t))) \in R^n, \textit{and} \;\; F(0) = 0.$$

The corresponding slave system will be:

$$\dot{y} = F(y) + u, \quad y \in R^n, \tag{3}$$

u is the controller to be designed. Let the synchronization error state be defined as:

$$e = y - x, \quad e \in R^n. \tag{4}$$

So that the error dynamics is given as follows

$$\dot{e} = F(y) - F(x) + u, \quad e \in R^n. \tag{5}$$

The synchronized coupled chaotic system means that the trajectory of one of the systems will converge at the same values as the other. The master system (2) synchronizes the slave system (3), if and only if, the following condition is satisfied, i.e.

$$\lim_{t\to\infty} \|e(t)\| = 0.$$

Based on the error between the variables of master and slave systems, there are generally three forms of synchronizations as follows:

1. Exact synchronization : $y(t) = x(t)$.
2. Asymptotic synchronization : $\lim_{t\to\infty} \|y(t) - x(t)\| = 0$.
3. Approximate synchronization : $\lim_{t\to\infty} \|y(t) - x(t)\| \leq \epsilon_0$.

7. Applications of Synchronization in Various Fields

Now that we have learnt the mechanism behind synchronization phenomenon, let us discuss some applications. Objects or entities that are not individually very powerful may work together to create a significant amount of power. This property can apply to different types of scenarios if skilfully implemented. For example, in a circuit that connects several Josephson junctions (each of which generates a small amount of current based on quantum-mechanical principles), if the junctions are linked in a manner conducive to synchronization, a large current can be allowed to flow. Synchronization or phase-locking, appear in a large variety of systems such as neural networks, lasers, charge density waves, Josephson junction arrays, heart/breathing systems, and population of flashing fireflies. The phenomenon is expected to be exploited for the treatment of Parkinson's disease, signal processing or optomechanical systems.

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Some of the applications of synchronization in diverse scientific fields such as biology, neuroscience, engineering, computer science, economics and social sciences are as follows:

1. **Biological Systems and Neuroscience**
   Examples: Disease dynamics, Population dynamics, Genetic networks, Circadian rhythms, Cortical networks, Neuronal networks.

2. **Physical Systems**
   Examples: Clocks, Dynamics of coherent structures in spatially extended systems.

3. **Defence Applications**
   Examples: Secure communications, New tunable radiation sources.

4. **Computer Science and Engineering**
   Examples: Data mining, Parallel/Distributed computation, Consensus problems, Wireless communication networks, Power-grids.

5. **Social Science and Economy**
   Examples: Opinion formation, Finance, World Trade Web.

For a more detailed explanation of these applications of synchronization in complex networks, see [22].

8. Conclusion

Through the present review, we have highlighted the basic principles of synchronization. Since traditionally, synchronization has been based upon periodic signals, in a classical context, it can be defined as the mechanism by which a group of individuals going through periodic motion begins moving through same parts of their cycles at the same time. The synchronization of non-linear and/or spatially extended processes have been extensively studied over the past decades and scientists from different fields have assigned it various names. These names are influenced by the classical definition of synchronization. It has been referred to as phase locking, phase trapping, frequency pulling, frequency locking, etc. The intrinsic rhythms exhibited by dynamical systems have attracted interest in a wide range of fields. However, till now, there is no single unified concept that clearly covers all the well-known examples of synchronization phenomenon together.

The intrinsic rhythms exhibited by dynamical systems have attracted interest in a wide range of fields. However, till now, there is no single unified concept that clearly covers all the well-known examples of synchronization phenomenon together.

Since this phenomenon occurs widely in Nature, one may ask, how can we potentially grasp them together? We can find its solution by discovering new mathematical tools. There is no denying that the strength of mathematics lies in its universality. Whether we are thinking about cells or circuits, it is possible to study each subject in the same way as long as the basic mathematical framework is the same. Although the work in mathematics is advancing quite slowly, when a specific problem is solved, the effect of the solution on neighbouring fields is unmistakable. A lot of researchers have shown their curiosity in this area since the groundbreaking work of Kuramoto, Pecora, and Caroll. In the last two decades, it has been realised that chaotic signals can also be used for synchronization. Though the phenomenon is quite recent in the theory of non-linear dynamical systems and has not been completely explored yet. We expect this article to raise interest in the scientific and engineering communities. We believe that the exploration of new mathematical tools, capable of combining the information provided by specific dynamics with the whole process towards synchronization, should be the focus of intensive research if we aim to provide a general theory of synchronization processes in complex networks. As the field of dynamic systems with many degrees of freedom is still evolving, the inclusion of new powerful mathematical tools are eagerly awaited by researchers in physics, biology, ecology, engineering, and other fields.

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