Two Identities*

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Björner and Welker studied homology of k-equal partition lattices and obtained interesting identities for $(n-1)!$. In this article, simple combinatorial proofs of these identities are discussed.

Introduction

The following identities hold for all integers $n \geq 2$,

$$(n-1)! = \sum_{t=1}^{\lfloor \frac{n}{2} \rfloor} \left\{ \sum_{\substack{j_1+j_2+\ldots+j_t=n;\\ j_i \geq 2, \forall i}} \frac{(n-1)!}{(j_1-1)!(j_2)!\cdots(j_t)!} \prod_{k=1}^{t} (j_k - 1) \right\} \tag{1}$$

and

$$(n-1)! = \sum_{t=1}^{\lfloor \frac{n}{2} \rfloor} (t-1)! \left\{ \sum_{\substack{0=i_0 \leq i_1 \leq \ldots \leq i_t;\\ i_t = n-2t}} \prod_{j=0}^{t-1} (n-2j-i_j-1)(j+1)^{i_{j+1}-i_j} \right\}. \tag{2}$$

The inner sum in (1) is carried over all tuples $(j_1, j_2, \ldots, j_t)$ of positive integers ≥ 2 such that $j_1+\ldots+j_t = n$, while the inner sum in (2) is carried over all tuples $(i_1, \ldots, i_t)$ of non-negative integers such that $0 = i_0 \leq i_1 \leq \ldots \leq i_t = n-2t$. These interesting identities are implicit in the works of Björner and Welker [1], and Björner and Wachs [2] on partition lattices.

In this article, we shall discuss simple combinatorial proofs of these identities.

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Permutations, multinomial coefficients, poset, order complex, partition lattice.

1. Permutations and Multinomial Coefficients

In this section, we introduce notations and the basic concepts required to understand these identities and their proofs.

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The number $n! = n(n-1)...(2)(1)$ is called factorial n. Stirling showed that $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ gives a good approximation to $n!$ for large positive integer n. Here e is the base of natural logarithm.

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of non-negative integers. For $\alpha, \beta \in \mathbb{N}$, with $\alpha \leq \beta$, the subset

$$[\alpha, \beta] = \{a \in \mathbb{N} : \alpha \leq a \leq \beta\},$$

is called an *integer interval*. If $n \geq 1$, we simply write $[n] = \{1, 2, \ldots, n\}$ for $[1, n]$. Further, the number of elements of a finite set A is denoted by $|A|$,

Definition 1.1. A bijective mapping $\sigma : [n] \longrightarrow [n]$ is called a *permutation* of $[n]$. Let $\mathfrak{S}_n$ be the set of all permutations of $[n]$.

A permutation $\sigma \in \mathfrak{S}_n$ is determined by the sequence $\sigma(1)$, $\sigma(2)$, $\ldots$, $\sigma(n)$. Let $\sigma(i) = \sigma_i \in [n]$. Then a convenient way to represent the permutation σ is by arranging $\sigma_1, \sigma_2, \ldots, \sigma_n$ in a line as $\sigma = \sigma_1\sigma_2\ldots\sigma_n$. This representation is called the *one-line notation* of σ. Thus, a permutation $\sigma \in \mathfrak{S}_n$ can be equivalently regarded as a rearrangement of the sequence $1, 2, \ldots, n$. For $n = 3$, we have

$$\mathfrak{S}_3 = \{123, 132, 213, 231, 312, 321\}.$$

Definition 1.2. For $n \geq 1$, the number $n(n-1)(n-2)\cdots(2)(1) = \prod_{i\in[n]} i$ is denoted by $n!$ (read as *factorial* n). By convention, $0! = 1$.

We verify that $|\mathfrak{S}_n| = n!$. In fact, using one-line notation of permutations, we see that there are n choices for the first position, $(n-1)$ choices for the second position and in general, $(n-i+1)$ choices for i^{th} position $(1 \leq i \leq n)$. Thus the number of permutations of $[n]$ is $n!$.

The inversion table of a permutation determines the permutation.

Since composition of two bijective mappings is again a bijective mapping, the composition of two permutations of $[n]$ is also a permutation of $[n]$. The set $\mathfrak{S}_n$ is a *group* under the composition operation. The group $\mathfrak{S}_n$ is a very important finite group. However, in this article, we have not made any use of the group structure of $\mathfrak{S}_n$.

Consider the set

$$\mathbf{Y}_n = \{(a_1, \ldots, a_n) \in \mathbb{N}^n : 0 \leq a_i \leq n - i \;\forall i\}.$$

Clearly, $\mathbf{Y}_n = [0, n-1] \times [0, n-2] \times \ldots \times [0, 0]$ is a Cartesian product of integer intervals. Since $|\mathfrak{S}_n| = |\mathbf{Y}_n| = n!$, one would like to have an explicit bijection between $\mathfrak{S}_n$ and $\mathbf{Y}_n$. One such bijection is given by *inversion table* (see [3]).

Definition 1.3. Let $\sigma = \sigma_1\sigma_2 \ldots \sigma_n \in \mathfrak{S}_n$. Let a_i be the number of entries j in the one-line notation of σ to the left of i such that $j > i$. In other words, if $\sigma_k = i$, then

$$a_i = |\{\sigma_s : s < k \text{ and } \sigma_s > \sigma_k = i\}|.$$

Then $I(\sigma) = (a_1, a_2, \ldots, a_n)$ is called the *inversion table* of σ.

Clearly, the inversion table $I(\sigma) \in \mathbf{Y}_n$ for every $\sigma \in \mathfrak{S}_n$. Also, every element $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbf{Y}_n$ is the inversion table of a unique permutation in $\mathfrak{S}_n$. Construct a $\tau \in \mathfrak{S}_n$ as follows. If $n, n-1, \ldots, n-i+1$ have been inserted in one-line notation of τ, then insert $n-i$ so that there are exactly a_{n-i} elements to the left of $n-i$. Clearly, $I(\tau) = \mathbf{a}$. Thus the mapping $\sigma \mapsto I(\sigma)$ induces a bijection $I : \mathfrak{S}_n \longrightarrow \mathbf{Y}_n$. We illustrate inversion table construction with an example. Let $n = 6$ and $\sigma = 426153$. Then $I(\sigma) = (3, 1, 3, 0, 1, 0)$. On the other hand, let $\mathbf{a} = (2, 3, 3, 1, 1, 0) \in \mathbf{Y}_6$. Then $\tau = 641523 \in \mathfrak{S}_6$ such that $I(\tau) = \mathbf{a}$. In fact, the one-line notation of τ is obtained in the following sequence of steps,

$$6 \to 65 \to 645 \to 6453 \to 64523 \to 641523 = \tau.$$

We now describe binomial and multinomial coefficients.

Definition 1.4. Let $n, k \in \mathbb{N}$ such that $k \leq n$. The number $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is called a *binomial coefficient*.

We recall the binomial theorem,

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k, \quad \text{where } x \text{ a variable.}$$

Indeed, the binomial coefficient $\binom{n}{k}$ is the coefficient of x^k in the expansion of $(1+x)^n$. Further, $\binom{n}{k}$ has a combinatorial interpretation. For $n \geq 1$ and $0 \leq k \leq n$, the number of all k-element subsets of $[n]$ is precisely $\binom{n}{k}$. Thus, $\binom{n}{k} \in \mathbb{N}$.

The number of k-element subsets of $[n]$ is precisely the binomial coefficient $\binom{n}{k}$.

Definition 1.5. Let $r \geq 1$ and $n, k_1, k_2, \ldots, k_r \in \mathbb{N}$, such that $\sum_{i=1}^{r} k_i = n$. The number

$$\binom{n}{k_1, k_2, \ldots, k_r} = \frac{n!}{(k_1!)(k_2!)\cdots(k_r!)}$$

is called a *multinomial coefficient*. Clearly, $\binom{n}{k} = \binom{n}{k,\ n-k}$.

We have the multinomial theorem,

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{\substack{(k_1, k_2 \ldots, k_r) \in \mathbb{N}^r; \\ k_1 + k_2 + \ldots + k_r = n}} \binom{n}{k_1, k_2, \ldots, k_r} x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r},$$

where $x_1, x_2, \ldots, x_r$ are variables.

Definition 1.6. Let $n, r \geq 1$, and $k_1, k_2, \ldots, k_r \in \mathbb{N}$ such that $\sum_{i=1}^{r} k_i = n$. An r-tuple $(B_1, \ldots, B_r)$ of subsets of $[n]$ is called an *ordered disjoint decomposition* of $[n]$ of the type $(k_1, k_2, \ldots, k_r)$ if

1. $|B_i| = k_i$,
2. $B_i \cap B_j = \emptyset$ for $i \neq j$,
3. $B_1 \cup B_2 \cup \ldots \cup B_r = [n]$.

If, in addition, each $B_i \neq \emptyset$, then we say that the r-tuple $(B_1, B_2, \ldots, B_r)$ is an *ordered set-partition* of $[n]$ of the type $(k_1, k_2, \ldots, k_r)$.

Proposition 1.7. The number of ordered disjoint decompositions of $[n]$ of the type $(k_1, k_2, \ldots, k_r)$ is the multinomial coefficient $\binom{n}{k_1,\ k_2, \ldots, k_r}$.

An element $i \in [n-1]$ is called an ascent (or a descent) of a permutation $\sigma \in \mathfrak{S}_n$ if $\sigma(i) < \sigma(i+1)$ (respectively, $\sigma(i) > \sigma(i+1)$).

Proof. Let $(B_1, B_2, \ldots, B_r)$ be an ordered decomposition of $[n]$ of the type $(k_1, k_2, \ldots, k_r)$. Then $|B_i| = k_i$. The number of ways of choosing subset B_1 is $\binom{n}{k_1}$. If $B_1, \ldots, B_{i-1}$ for $i \geq 2$ have already been chosen, then B_i is a k_i-element subset of $[n] \setminus (B_1 \cup \ldots \cup B_{i-1})$. Thus the number of ways of choosing B_i is precisely $\binom{n-k_1-\ldots-k_{i-1}}{k_i}$. Hence, the number of ordered disjoint decompositions of $[n]$ of the type $(k_1, k_2, \ldots, k_r)$ is given by

$$\binom{n}{k_1}\binom{n-k_1}{k_2}\cdots\binom{n-k_1-\ldots-k_{r-1}}{k_r} = \binom{n}{k_1, k_2, \ldots, k_r}.$$

□

We now proceed to give proofs of identities (1) and (2).

2. Combinatorial Proofs of Two Identities

For $n \geq 2$, let $\mathbf{X}_n = \{\sigma \in \mathfrak{S}_n : \sigma(2) = n\}$. For $\sigma \in \mathbf{X}_n$, let $\hat{\sigma}$ be the permutation obtained from σ by deleting $n = \sigma(2)$. The mapping $\sigma \mapsto \hat{\sigma}$ induces a bijection between $\mathbf{X}_n$ and $\mathfrak{S}_{n-1}$. Clearly, $|\mathbf{X}_n| = |\mathfrak{S}_{n-1}| = (n-1)!$.

Let $\sigma \in \mathfrak{S}_n$. Let $A = [\alpha,\beta] \subseteq [n]$ with $\alpha < \beta$. Then the *restriction* $\sigma|_A$ of σ to A is the mapping $\sigma|_A \colon A \longrightarrow [n]$ given by $\sigma|_A(i) = \sigma(i)$ for $i \in A$.

Definition 2.1. Let $\sigma \in \mathfrak{S}_n$. An element $i \in [n-1]$ is called an *ascent* of the permutation $\sigma \in \mathfrak{S}_n$ if $\sigma(i) < \sigma(i+1)$. More generally, if $A = [\alpha,\beta] \subseteq [n]$ with $\alpha < \beta$, then $i \in [\alpha, \beta - 1]$ is called an *ascent* of $\sigma|_A$ if $\sigma(i) < \sigma(i+1)$.

Let $\sigma \in \mathbf{X}_n$. Then note that 1 is always an ascent of σ. Let $A = [\alpha,\beta] \subseteq [n]$ with $\alpha < \beta$. We say that A is a *maximal increasing run* of σ if the restriction $\sigma|_A \colon A \longrightarrow [n]$ is a strictly increasing function and $\sigma|_{A'}$ is not monotonic for any integer interval A' with $A \subsetneq A' \subseteq [n]$. Let $\{A_1, A_2, \ldots, A_r\}$ be the set of all maximal increasing runs of $\sigma \in \mathbf{X}_n$. Set $t_\sigma = \sum_{i=1}^{r} \lfloor \frac{|A_i|}{2} \rfloor$. Then $1 \leq r \leq t_\sigma \leq \frac{n}{2}$.

Definition 2.2. Let $\sigma \in \mathbf{X}_n$. Suppose there exists a strictly increasing sequence $1 = \ell_1, \ell_2, \ldots, \ell_t$ in $[n-1]$ such that

1. $\ell_{k+1} - \ell_k \geq 2$ for $1 \leq k \leq t$, where $\ell_{t+1} = n + 1$, and
2. $\sigma|_{[\ell_k,\, \ell_{k+1}-1]}$ has only one ascent at ℓ_k for $1 \leq k \leq t$.

On cleverly counting elements of a finite set in two ways, one gets an identity.

Then we say that σ has *t number of block ascents*.

If $\sigma \in \mathbf{X}_n$ has t number of block ascents, then $t = t_\sigma$. It is easy to show that every $\sigma \in \mathbf{X}_n$ has t number of block ascents for some t with $1 \leq t \leq \lfloor \frac{n}{2} \rfloor$. In fact, ℓ_t is the largest ascent of σ, i.e., $\ell_t \in [n-1]$ is the largest number such that $\sigma(\ell_t) < \sigma(\ell_t + 1)$. Also, ℓ_k is the largest ascent of the restriction $\sigma|_{[\ell_{k+1}-1]}$ of σ to $[\ell_{k+1} - 1]$ for $1 \leq k < t$.

Example: We illustrate the above notions with the help of an example. Let $n = 7$ and $\sigma = 2731465 \in \mathbf{X}_7$. Then $\hat{\sigma} = 231465 \in$

$\mathfrak{S}_6$. The maximal increasing runs of σ are $A_1 = [1,2]$ and $A_2 = [4,6]$. We see that $t_\sigma = 2$. The largest ascent of σ is $\ell_2 = 5$ as $\sigma(5) = 4 < 6 = \sigma(6)$. Further, $\sigma\mid_{[4]}= 2731$ has only one ascent at $\ell_1 = 1$. Clearly, σ has 2 block ascents. Let us take another element $\sigma' = 2713465 \in \mathbf{X}_7$. Then $\hat{\sigma'} = 213465 \in \mathfrak{S}_6$. The maximal increasing runs of σ' are $A'_1 = [1,2]$ and $A'_2 = [3,6]$. Thus $t_{\sigma'} = 3$. Now the largest ascent of σ' is $\ell'_3 = 5$. The restriction $\sigma'\mid_{[4]}= 2713$ and its largest ascent is $\ell'_2 = 3$. Since $\sigma'\mid_{[2]}= 27$, it has only one ascent $\ell_1 = 1$. Thus σ' has 3 block ascents.

We have $|\mathbf{X}_n| = (n-1)!$. On counting permutations in $\mathbf{X}_n$ according to the number of block ascents, we shall prove identity (1).

Theorem 2.3. For $n \geq 2$, the identity (1) holds.

Proof. Let $\mathbf{X}_n(t)$ be the subset of $\mathbf{X}_n$ consisting of permutations having t number of block ascents. Then $\mathbf{X}_n$ has a disjoint decomposition

$$\mathbf{X}_n = \coprod_{t=1}^{\lfloor \frac{n}{2} \rfloor} \mathbf{X}_n(t).$$

We need only to show that

$$|\mathbf{X}_n(t)| = \sum_{\substack{j_1+j_2+\ldots+j_t=n;\\ j_i\geq 2,\forall i}} \frac{(n-1)!}{(j_1-1)!(j_2)!\cdots(j_t)!}\prod_{k=1}^{t}(j_k-1).$$

Let $\sigma \in \mathbf{X}_n(t)$. Then there is a sequence $1 = \ell_1 < \ell_2 < \ldots < \ell_t < n$ such that $\ell_{k+1} - \ell_k \geq 2$ and σ has exactly one ascent in $[\ell_k, \ell_{k+1} - 1]$ at ℓ_k for $1 \leq k \leq t$. Let $B_k = \{\sigma(r) : \ell_k \leq r < \ell_{k+1}\}$. Then $\pi_\sigma = (B_1, \ldots, B_t)$ is an ordered set-partition of $[n]$ associated to σ such that $n \in B_1$ and $|B_k| = \ell_{k+1} - \ell_k$. Also, $\sigma(\ell_k)$ is a chosen element of $B_k \setminus \{\max B_k\}$ and $\sigma(\ell_k + 1) < \sigma(\ell_k + 2) < \ldots < \sigma(\ell_{k+1} - 1)$, where $\max B_k$ is the largest element of B_k. On the other hand, an ordered set-partition $\pi = (B_1, \ldots, B_t)$ of $[n]$ with $n \in B_1, |B_k| = \ell_{k+1} - \ell_k$ and chosen element $b_k \in B_k\setminus\{\max B_k\}$ for $1 \leq k \leq t$ determines a unique permutation $\sigma \in \mathbf{X}_n(t)$ such that $\pi_\sigma = \pi$. In view of Proposition 1., the number of ways of

Consider the set $R_n = [0,n] \times [0,n]$. We have $|R_n| = (n+1)^2$. Let $D_k = \{(a,b) \in R_n : a + b = k\}$. Then

$$R_n = \coprod_{k=0}^{2n} D_k.$$

Also, $|D_k| = k + 1$ for $0 \leq k \leq n$ and $|D_k| = 2n - k + 1$ for $n \leq k \leq 2n$. Thus counting elements of R_n in two ways, we deduce the familiar identity $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$.

choosing an ordered set-partitions $\pi = (B_1, \ldots, B_t)$ of $[n]$ with $n \in B_1$ is given by the multinomial coefficient

$$\binom{n-1}{\ell_2-\ell_1-1, \ell_3-\ell_2, \ldots, n-\ell_t+1} = \frac{(n-1)!}{(\ell_2-\ell_1-1)!(\ell_3-\ell_2)!\cdots(n-\ell_t+1)!}.$$

Further, the number of ways of choosing each $b_k \in B_k \setminus \{\max B_k\}$ for $1 \leq k \leq t$ is given by $(\ell_2-\ell_1-1)(\ell_3-\ell_2-1)\cdots(n-\ell_t)$. Putting $j_k = \ell_{k+1} - \ell_k$ and summing over all possible sequences $(j_1, \ldots, j_t)$, we get $|\mathbf{X}_n(t)|$, as desired. □

Theorem 2.4. For $n \geq 2$, the identity (2) holds.

Proof. The identity (2) can be deduced from a modified counting of $\mathbf{X}_n(t)$. For each $\sigma \in \mathbf{X}_n(t)$, consider the associated ordered partition $\pi_\sigma = (B_1, \ldots, B_t)$. Choose a permutation $\rho \in \mathfrak{S}_t$ with $\rho(1) = 1$ such that $\max B_{\rho(1)} > \max B_{\rho(2)} > \ldots > \max B_{\rho(t)}$. Let $\overline{B}_{\rho(k)} = B_{\rho(k)} \setminus \{b_{\rho(k)}, \max B_{\rho(k)}\}$. Let i_k be the number of elements in $\bigcup_{r=1}^{k} \overline{B}_{\rho(r)}$ bigger than the max $B_{\rho(k+1)}$ for $k < t$ and $i_t = n - 2t$. The number of ways of choosing $\rho \in \mathfrak{S}_t$ with $\rho(1) = 1$ is $(t-1)!$. Also, the number of ways of choosing $B_{\rho(1)}, \ldots, B_{\rho(t)}$ for a given $0 = i_0 \leq i_1 \leq \ldots \leq i_t = n-2t$ is $\prod_{k=0}^{t-1}(n-2k-i_k-1)(k+1)^{i_{k+1}-i_k}$. Thus,

$$|\mathbf{X}_n(t)| = (t-1)! \sum_{0=i_0 \leq i_1 \leq \ldots \leq i_t = n-2t} \prod_{j=0}^{t-1} (n-2j-i_j-1)(j+1)^{i_{j+1}-i_j}.$$

This proves identity (2). □

We now give a more direct way to establish identity (2).

Let $\mathbf{Y}_{n-1} = \{\mathbf{a} = (a_1, \ldots, a_{n-1}) \in \mathbb{N}^{n-1} : 0 \leq a_i \leq n-i-1 \forall i\}$. Then $|\mathbf{Y}_{n-1}| = (n-1)!$. For $1 \leq t \leq \lfloor \frac{n}{2} \rfloor$ and $\mathbf{i} = (i_1, \ldots, i_t)$ with $0 = i_0 \leq i_1 \leq \ldots \leq i_t = n-2t$, set

$$\mathbf{Y}_{n-1}(\mathbf{i}) = \mathbf{Y}_{n-1}((i_1, \ldots, i_t)) = \prod_{k=1}^{n-1} I_k(\mathbf{i}),$$

where

$$I_k(\mathbf{i})=\begin{cases}[0,n-2j-i_j-2] & \text{if } k=(j+1)+i_j \text{ for } 0\le j\le t-1,\\ [n-j-k-1,n-k-1] & \text{if } i_{j+1}>i_j \text{ for } 0\le j\le t-1\\ & \quad\text{and } k\in[j+2+i_j,\, j+1+i_{j+1}],\\ [0,n-k-1] & \text{if } k\in[n-t+1,n-1].\end{cases}$$

Then $|\mathbf{Y}_{n-1}(\mathbf{i})| = \left(\prod_{j=0}^{t-1}(n-2j-i_j-1)(j+1)^{i_{j+1}-i_j}\right)(t-1)!$.

Example: We illustrate the above construction in the case $n = 5$. The possible values of t are $t = 1, 2$. For $t = 1$, $\mathbf{i} = (i_1) = (3)$ and the subset $\mathbf{Y}_4((3)) = \{(a_1, 2, 1, 0) \in \mathbf{Y}_4\}$ has 4 elements. For $t = 2$, the sequence $\mathbf{i} = (i_1, i_2)$ has two possible values; namely, $(0, 1)$ and $(1, 1)$. We see that the subset $\mathbf{Y}_4((0, 1)) = \{(a_1, a_2, a_3, 0) \in \mathbf{Y}_4 : a_2 \le 1\}$ has 16 elements, and the subset $\mathbf{Y}_4((1, 1)) = \{(a_1, 2, 0, 0) \in \mathbf{Y}_4\}$ has 4 elements. Further, $\mathbf{Y}_4$ is a disjoint union of these subsets and $24 = 4 + 16 + 4$.

This example motivates the following alternate proof of Theorem 2.

Proof. Given $\mathbf{a} = (a_1, \ldots, a_{n-1}) \in \mathbf{Y}_{n-1}$, we proceed to show that there exists a unique $\mathbf{i} = (i_1, \ldots, i_t)$ for some $1 \le t \le \lfloor \frac{n}{2} \rfloor$ with $0 = i_0 \le i_1 \le \ldots \le i_t = n - 2t$ such that $\mathbf{a} \in \mathbf{Y}_{n-1}(\mathbf{i})$. Set $i_0 = 0$. If $a_2 = a_{2+i_0} \in [0, n-4]$, then put $i_1 = i_0$. Otherwise, $a_2 \in [n-3, n-3] = \{n-3\}$. Let r_1 be the largest number such that $a_{1+j} \in [n-j-2, n-j-2]$ for all $j \in [1, r_1]$. Then put $i_1 = i_0 + r_1 = r_1$. Suppose $(i_1, \ldots, i_s)$ has already been obtained with desired properties. If $i_s = n - 2s$, then set $t = s$ and $\mathbf{i} = (i_1, \ldots, i_s)$. Otherwise $i_s < n - 2s$. If $a_{s+2+i_s} \in [0, n-2s-i_s-4]$, then put $i_{s+1} = i_s$. On the other hand, if $a_{s+2+i_s} \in [n-2s-i_s-3, n-s-i_s-3]$, then choose the largest integer r_{s+1} such that $a_{s+1+i_s+j} \in [n-2s-i_s-2-j, n-s-i_s-2-j]$ for all $j \in [1, r_{s+1}]$. In this case, $i_{s+1} = i_s + r_{s+1}$. This completes the inductive step. Hence, $\mathbf{Y}_{n-1}$ has a disjoint decomposition

$$\mathbf{Y}_{n-1} = \coprod_{t=1}^{\lfloor \frac{n}{2} \rfloor} \left(\coprod_{0=i_0 \le i_1 \le \ldots \le i_t = n-2t} \mathbf{Y}_{n-1}((i_1, \ldots, i_t)) \right).$$

This proves identity (2). □

The mapping $\sigma \mapsto I(\hat{\sigma})$ induces a bijection between $\mathbf{X}_n$ and $\mathbf{Y}_{n-1}$. For $1 \leq t \leq \lfloor \frac{n}{2} \rfloor$, set $\mathbf{Y}_{n-1}(t) = \{I(\hat{\sigma}) : \sigma \in \mathbf{X}_n(t)\}$. Clearly, $\mathbf{Y}_{n-1}$ is a disjoint union of $\mathbf{Y}_{n-1}(t)$ and identity (1) can also be proved from a counting of $\mathbf{Y}_{n-1}$ in two ways.

We ask the following question.

Question: Is it possible to prove identity (1) from a counting of $\mathbf{Y}_{n-1}$ in two ways, without using a bijection between $\mathbf{Y}_{n-1}$ and $\mathbf{X}_n$ (or $\mathfrak{S}_{n-1}$)?

Finally, for readers having some knowledge of algebraic topology, we describe a connection between partition lattices and the identities (1) and (2).

3. Partition Lattices

In this section, we illustrate some combinatorial and topological aspects of partition lattices. For unexplained topological terms and concepts, we refer to Bredon [4] and Hatcher [5].

Definition 3.1. A relation $\leq$ on a non-empty set P is called a *partial ordering* on P, if

1. $\leq$ is reflexive (i.e. $a \leq a$ for all $a \in P$),
2. $\leq$ is antisymmetric (i.e. $a \leq b$ and $b \leq a$ in $P \Rightarrow a = b$),
3. $\leq$ is transitive (i.e. $a \leq b$ and $b \leq c$ in $P \Rightarrow a \leq c$).

The set P with a partial ordering $\leq$ is called a *partially ordered set* (or *poset*).

Suppose $a \leq b$ in a poset P. We say that b covers a if there does not exist $c \epsilon P$ such that $a < c < b$. A finite poset P is determined by its cover relation.

A non-empty subset of a poset is also a poset under the induced partial ordering.

Let $(P, \leq)$ be a poset. We write $a < b$ in P, for $a \leq b$ and $a \neq b$. A subset S of P is called a *chain* if for any $a, b \in S$, either $a \leq b$ or $b \leq a$. We say that P is a *finite poset* if P has only finitely many elements. A chain $S = \{a_1, a_2, \ldots, a_s\}$ in a finite poset P can be represented by arranging its elements in an increasing order, say $a_{j_1} \prec a_{j_2} \prec \ldots \prec a_{j_s}$. In this case, we say that S is a *chain* of P of *length* $s-1$. A chain S is called a *maximal chain* of P, if there does not exists a chain S' of P such that $S \subsetneq S'$. A finite poset P is called a *graded poset of rank n*, if all maximal chains of P have the same length n.

To every finite poset P, we can associate a simplicial complex $\Delta(P)$.

Definition 3.2. Let V be a non-empty finite set. A collection $\mathcal{K}$ of subsets of V is called an *abstract simplicial complex* on the vertex set V, if

1. $\{v\} \in \mathcal{K}, \ \forall\, v \in V$,
2. $F \in \mathcal{K}$ and $G \subseteq F \Rightarrow G \in \mathcal{K}$.

Let $\mathcal{K}$ be a simplicial complex on V. Every $F \in \mathcal{K}$ is called a *face* of $\mathcal{K}$, and dimension of the face F is $\dim(F) = |F| - 1$. The 0-dimensional faces $\{v\}, v \in V$ are called *vertices*, while 1-dimensional faces are called *edges* of $\mathcal{K}$. The empty set $\emptyset$ is always a face of $\mathcal{K}$ of dimension -1. The dimension of the simplicial complex $\mathcal{K}$ is given by $\dim(\mathcal{K}) = \sup\{\dim(F) : F \in \mathcal{K}\}$, where $\sup(A)$ denotes the supremum of the set A.

A face F of a simplicial complex $\mathcal{K}$ is called a *maximal face* if there does not exist any face $F' \in \mathcal{K}$ with $F \subsetneq F'$. A maximal face of the simplicial complex $\mathcal{K}$ is called a *facet*. The simplicial complex $\mathcal{K}$ is completely determined by its facets. If $F_1, F_2, \ldots, F_t$ are all the facets of $\mathcal{K}$, then we say that $\mathcal{K} = \langle F_1, F_2, \ldots, F_t \rangle$ is generated by the facets $F_1, F_2, \ldots, F_t$. Further, $\mathcal{K}$ is called a *pure simplicial complex*, if all its facets have the same dimension.

A hollow tetrahedorn **T** and the unit sphere $\mathbf{S}^2$ are the same topological objects.

A 'nice' topological space $||\mathcal{K}||$ can be associated to an abstract simplicial complex $\mathcal{K}$. The topological space $||\mathcal{K}||$ is called the *geometric realization* of $\mathcal{K}$, while the abstract simplicial complex $\mathcal{K}$ is called a *triangulation* of $||\mathcal{K}||$. We conveniently use the term simplicial complex for an abstract simplicial complex as well as for its geometric realization.

Example: Let v_1, v_2, v_3, v_4 be four non-coplanar points in the 3-dimensional space $\mathbb{R}^3$. Let $F_1 = \{v_1, v_2, v_3\}, F_2 = \{v_1, v_2, v_4\}, F_3 = \{v_1, v_3, v_4\}, F_4 = \{v_2, v_3, v_4\}$ and $\mathcal{K}$ be the abstract simplicial complex generated by F_i $(1 \leq i \leq 4)$. Then $\mathcal{K}$ is a 2-dimensional pure simplicial complex on the vertex set $V = \{v_1, v_2, v_3, v_4\}$. The geometric realization $||\mathcal{K}||$ of $\mathcal{K}$ is the hollow tetrahedron $\mathbf{T}$ with vertices v_i $(1 \leq i \leq 4)$ having the 2-dimensional triangular faces spanned by these vertices. The vertices, edges and faces of $\mathbf{T}$ correspond to 0-dimensional, 1-dimensional and 2-dimensional faces of $\mathcal{K}$, respectively. Since the hollow tetrahedron $\mathbf{T}$ is homeomorphic to the unit sphere $\mathbf{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ in $\mathbb{R}^3$, the abstract simplicial complex $\mathcal{K}$ is a triangulation of the 2-dimensional unit sphere $\mathbf{S}^2$. Now, let $F_5 = \{v_3, v_4\}$ and $\mathcal{K}' = \langle F_1, F_5 \rangle$. Then $\mathcal{K}'$ is a 2-dimensional non-pure simplicial complex on the vertex set V. The geometric realization $||\mathcal{K}'||$ of $\mathcal{K}'$ consists of the 2-dimensional triangular face spanned by v_1, v_2, v_3 and the line segment joining v_3 and v_4.

Definition 3.3. Let P be a finite poset and let $\Delta(P)$ be the set of all chains of P. Then $\Delta(P)$ is called the *order complex* of P.

Since a subset of a chain of P is also a chain of P, the order complex $\Delta(P)$ is a simplicial complex on the vertex set P. The facets of $\Delta(P)$ are the maximal chains of P. Thus $\Delta(P)$ is a pure simplicial complex of dimension n if and only if P is a graded poset of rank n.

We now proceed to describe a *lattice*. Let $(P, \leq)$ be a poset and $x, y \in P$. An element $z \in P$ is called the *supremum* $\sup\{x, y\}$ of x and y, if $x \leq z, y \leq z$ and whenever $x \leq z', y \leq z'$ for any $z' \in P$, then $z \leq z'$. Also, an element $w \in P$ is called the *infimum* $\inf\{x, y\}$ of x and y, if $w \leq x, w \leq y$ and whenever $w' \leq x, w' \leq y$ for any $w' \in P$, then $w' \leq w$.

A lattice P is said to be a bounded lattice if the least element $\hat{0}$ and the largest element $\hat{1}$ exist in P.
Every finite lattice is bounded.

Definition 3.4. A poset $(P, \leq)$ is called a *lattice* if $\sup\{x, y\}$ and $\inf\{x, y\}$ exist in P for every pair (x, y) of elements $x, y \in P$.

We write $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$ and call them *join* and *meet* of x and y, respectively. For a finite lattice P, the $\sup(S)$ and $\inf(S)$ exists in P for every subset $S \subseteq P$. In particular, $\inf(P) = \hat{0}$ is the least element of P while $\sup(P) = \hat{1}$ is the largest element of P. Set $\bar{P} = P \setminus \{\hat{0}, \hat{1}\}$.

If P is a finite poset such that $\hat{0} \in P$ (or $\hat{1} \in P$), then the order complex $\Delta(P)$ has the homotopy type of a point, i.e., the geometric realization $||\Delta(P)||$ is contractible. Hence, for a finite lattice P, the order complex $\Delta(\bar{P})$ of $\bar{P} = P \setminus \{\hat{0}, \hat{1}\}$ has more interesting topological properties than the order complex $\Delta(P)$ of P.

We now give a few examples of finite lattices.

1. Boolean Lattices: Let $\mathcal{P}[n] = \{A : A \subseteq [n]\}$ be the *power set* of $[n]$. Then $\mathcal{P}[n]$ is a poset with respect to the partial ordering $\leq$ induced by inclusion, i.e. $A \leq B$ in $\mathcal{P}[n]$ if $A \subseteq B$. Since every maximal chain in $\mathcal{P}[n]$ is of length n, it is a graded poset of rank n. Further, the poset $\mathcal{P}[n]$ is a lattice as $A \vee B = A \cup B$ and $A \wedge B = A \cap B$ for any $A, B \in \mathcal{P}[n]$. Also, $\hat{0} = \emptyset$ and $\hat{1} = [n]$. The lattice $\mathcal{P}[n]$ is an example of a *Boolean lattice*. We have $|\mathcal{P}[n]| = 2^n$. On counting elements of $\mathcal{P}[n]$ according to their cardinality, we get the identity

$$2^n = \sum_{i=0}^{n} \binom{n}{i}.$$

2. Partition Lattices: Let Π_n be the set of all (unordered) set-partitions of $[n]$. If $(B_1, B_2, \ldots, B_k)$ is an ordered set-partition of $[n]$, then the collection $\{B_1, B_2, \ldots, B_k\}$ is called a *set-partition* of $[n]$ with k-blocks. Let $\mathcal{B} = \{B_1, \ldots, B_k\}$, $C = \{C_1, \ldots, C_\ell\} \in \Pi_n$. We say that $\mathcal{B}$ is a *refinement* of C, if each block B_i of $\mathcal{B}$ is contained in some block C_j of C. Clearly, Π_n is a poset under the partial ordering $\leq$ induced by refinement i.e., $\mathcal{B} \leq C$ in Π_n if $\mathcal{B}$ is a refinement of C. The poset Π_n is a graded poset of rank $n - 1$. Also, $\mathcal{B} \wedge C$ is the set partition of $[n]$, whose blocks are non-empty intersection $B_i \cap C_j$ of the blocks of $\mathcal{B}$ and C. The join $\mathcal{B} \vee C$ is obtained in a finite sequence of steps. Set $\mathcal{B}_0 = \mathcal{B}$. Let $\mathcal{B}_1$ be the set-partition of $[n]$ obtained by merging all the blocks B_j of $\mathcal{B}_0$ that intersect C_1 non-trivially (i.e., $B_j \cap C_1 \neq \emptyset$).

A lattice P is said to be distributive lattice if join $\vee$ distributes over meet $\wedge$, and vice-versa. The partition lattice $\prod_n$ is non-distributive for $n \geq 3$.

Clearly, $\mathcal{B}_0 \leq \mathcal{B}_1$ and the block C_1 of C is contained in one block of $\mathcal{B}_1$. If $\mathcal{B}_{i-1}$ has already been obtained, then in the ith step, the set-partition $\mathcal{B}_i$ is obtained by merging all blocks of $\mathcal{B}_{i-1}$ that intersect C_i non-trivially. We have $\mathcal{B} \leq \mathcal{B}_\ell, C \leq \mathcal{B}_\ell$ and $\mathcal{B} \vee C = \mathcal{B}_\ell$. This shows that Π_n is a lattice called the *partition lattice* of $[n]$. Also, $\hat{0} = \{\{1\}, \{2\}, \ldots, \{n\}\}$ and $\hat{1} = \{[n]\}$. We illustrate partial ordering, join and meet in the partition lattice Π_n for $n = 8$. We use compact notation 58|14|6|2|3|7 to denote the set-partition $\mathcal{B} = \{\{5, 8\}, \{1, 4\}, \{6\}, \{2\}, \{3\}, \{7\}\} \in \Pi_8$. Let $C = 146|2|58|37$ and $\mathcal{D} = 584|26|3|17$. Since each block of $\mathcal{B}$ is contained in a block of C, we have $\mathcal{B} \leq C$. Also, $\mathcal{B} \not\leq \mathcal{D}$ as the block $\{1, 4\}$ of $\mathcal{B}$ is not contained in any block of $\mathcal{D}$. We have $\mathcal{B} \wedge \mathcal{D} = 58|4|2|6|1|7|3$. In order to obtain $\mathcal{B} \vee \mathcal{D}$, we see that $\mathcal{B}_1 = 5814|6|2|3|7$ is obtained by merging the blocks $\{5, 8\}$ and $\{1, 4\}$ of $\mathcal{B}$. The blocks $\{5, 8\}$ and $\{1, 4\}$ have non-empty intersection with the block $\{5, 8, 4\}$ of $\mathcal{D}$. Similarly, $\mathcal{B}_2 = 5814|26|3|7$ is obtained by merging $\{2\}$ and $\{6\}$. Also, $\mathcal{B}_3 = \mathcal{B}_2$ and $\mathcal{B}_4 = 58147|26|3$. Hence, $\mathcal{B} \vee \mathcal{D} = 58147|26|3$.

We now discuss some combinatorial properties of Π_n. The cardinality $|\Pi_n| = \text{Bell}(n)$ is called the nth *Bell number*. The number of set-partitions of $[n]$ having exactly k-blocks is called the *Stirling number* $S(n, k)$ of the second-kind. We have $S(n, 1) = S(n, n) = 1, S(n, n-1) = \binom{n}{2}$ and $S(n, 2) = 2^{n-1} - 1$. Also, there is a recurrence relation

$$S(n, k) = S(n-1, k-1) + kS(n-1, k) \quad \text{for} \quad n \geq k > 1.$$

This recurrence relation is proved as follows. Each $\mathcal{B} = \{B_1, \ldots, B_{k-1}\} \in \Pi_{n-1}$ with $(k-1)$-blocks gives a unique $\{B_1, \ldots, B_{k-1}, \{n\}\} \in \Pi_n$ with k-blocks. Thus the number of set-partitions of the form $\{B_1, \ldots, B_{k-1}, \{n\}\} \in \Pi_n$ is precisely $S(n-1, k-1)$. Also, for each $C = \{C_1, \ldots, C_k\} \in \Pi_{n-1}$ with k-blocks, we can associate exactly k different set-partitions $C_1, \ldots, C_k \in \Pi_n$ (with k-blocks), where C_i is the set-partition obtained from C by replacing the block C_i in C with $C_i \cup \{n\}$. Thus the number of $\mathcal{D} = \{D_1, \ldots, D_k\} \in \Pi_n$ such that $D_i \neq \{n\}$ for all i is precisely $kS(n-1, k)$.

The number of set-partitions of $[n]$ having k-blocks is called the Stirling number $S(n, k)$ of the second kind.

On counting elements of Π_n according to the number of blocks, we get

$$\text{Bell}(n) = \sum_{k=1}^{n} S(n, k).$$

3. k-equal Partition Lattices: Let $2 \leq k \leq n$ and let

$$\Pi_{n,k} = \{\mathcal{B} = \{B_1, \ldots, B_r\} \in \Pi_n : \text{either } |B_i| = 1 \text{ or } |B_i| \geq k, \ \forall i\}.$$

Then $\Pi_{n,k}$ is also a poset under the partial order induced from Π_n. Further, $\Pi_{n,k}$ is a lattice. The lattice $\Pi_{n,k}$ is called the *k-equal partition lattice*. Clearly, $\Pi_{n,2} = \Pi_n$. For $k > 2$, $\Pi_{n,k}$ need not be a graded lattice. In $\Pi_{6,3}$, the maximal chains

$$1|2|3|4|5|6 \prec 1|2|3|456 \prec 1|2|3456 \prec 1|23456 \prec 123456,$$

and

$$1|2|3|4|5|6 \prec 1|2|3|456 \prec 123|456 \prec 123456,$$

are of different lengths.

Let $n \geq 3$. Consider the order complex $\Delta(\bar{\Pi}_n)$ of the poset $\bar{\Pi}_n = \Pi_n - \{\hat{0}, \hat{1}\}$. Since all maximal chains of $\bar{\Pi}_n$ are of length $n-3$, the order complex $\Delta(\bar{\Pi}_n)$ is a pure simplicial complex of dimension $n-3$. Using a notion of shellability for pure simplicial complexes, it can be shown that $\Delta(\bar{\Pi}_n)$ is shellable and has the homotopy type of a wedge (sum) of $(n-1)!$ spheres of dimension $n-3$. Thus the top (reduced) Betti number of $\Delta(\bar{\Pi}_n)$ is precisely $(n-1)!$. Björner and Wachs introduced a notion of shellability for non-pure simplicial complexes and showed that the order complex $\Delta(\bar{\Pi}_{n,k})$ of the poset $\bar{\Pi}_{n,k} = \Pi_{n,k} - \{\hat{0}, \hat{1}\}$ is also shellable (see [2] (Theorem 6.1)). Therefore, it is possible to calculate the (reduced) Betti numbers of $\Delta(\bar{\Pi}_{n,k})$. Since $\Pi_n = \Pi_{n,2}$, two different ways of calculating the top (reduced) Betti number of $\Delta(\bar{\Pi}_n)$ give rise to identities (1) and (2).

$\Delta(\bar{\Pi}_n)$ has the homotopy type of a wedge of $(n-1)!$ spheres of dimesnion $n-3$.

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Suggested Reading

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