In the Land of Convex Polygons*

Discrete Geometry of Polygons

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This article shares some insights and observations that we gained while exploring the world of discrete and computational geometry. It discusses several results related to polygons and gives some historical remarks. Mathematical parts of our discussion support the view that in the land of polygons, convexity alone tells a lot, and quadrilaterals act as our main guides whenever we visit there.


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Introduction

Modern science is by all measures a collaborative enterprise. It often requires active collaborations between individual scientists as well as scientific disciplines. Discrete and computational geometry is an example of such scientific field as it involves several mathematical disciplines including geometry, discrete mathematics, and computer science. In fact, its development is closely related to the developments in modern technology, although its roots can be traced back (at least) to the Renaissance.

In this article, we share some of our experiences of visiting this land through the lens of S Devadoss and J O'Rourke's book [5]. In particular, we focus on several results related to convex polygons.

Polygons are combinatorial objects as they can be defined by a finite sequence of (vertex) points. On the other hand, they constitute an endless source of geometrical studies since ancient times. Thus, they are indeed a fundamental subject of study in discrete geometry. As we discuss in section 1, some results on polygons

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are so natural that they receive recurrent interest. In general, widening our knowledge allows us to deepen it. In section 2, we demonstrate that this principle nicely applies to some classical results on polygons. As we discuss in section 3, the combination of geometry and combinatorics becomes challenging very soon.

Keywords
Convex polygon, discrete and computational geometry, geometric inequality.

Combination of geometry and combinatorics becomes challenging very soon.

In our discussions, we adopt the following notations. For any points X, Y and Z in the plane, XY denotes the line segment connecting X and Y, $|XY|$ is its length, and $\measuredangle XYZ$ is the angle between XY and YZ measured in the clockwise direction.

1. Rediscoveries on Polygons

We start with a discussion on the recent rediscovery of an old problem. The following problem was proposed by the Mongolian team participating in the International Mathematical Olympiad (IMO) in 1984.

Problem 1.1 (IMO 1984 Problem 5): Let p and d be the perimeter and the sum of diagonal lengths of a convex n-gon with $n > 3$. Prove that $n - 3 < \frac{2d}{p} < \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{n+1}{2} \right\rfloor - 2$.

It was created as a joint effort of two Mongolian mathematicians, Ts Dashdorj and R Gonchigdorj. According to Prof. Dashdorj's own account, at the last moment of their preparation for the IMO, they realized the need for one more problem to propose, and he asked his younger colleague to find lower and upper bounds for the ratio $\frac{d}{p}$, without much thinking. After few days, they were already discussing the solution, and the problem made it into the 25th IMO's six selected problems.

In 2007, this problem reappeared as an open conjecture in an article published in the *Journal of Global Optimization*; see Conjecture 1 in [1]. Accordingly, it was rediscovered and reproved in [1] and [10]. The original Mongolian solution is very similar to the one given in [10] as both the solutions rely on the analysis of quadrilaterals whose vertices are among the vertices of the polygon and two opposite vertices of the former are also opposite vertices of the latter. However, at the IMO, a contestant from the

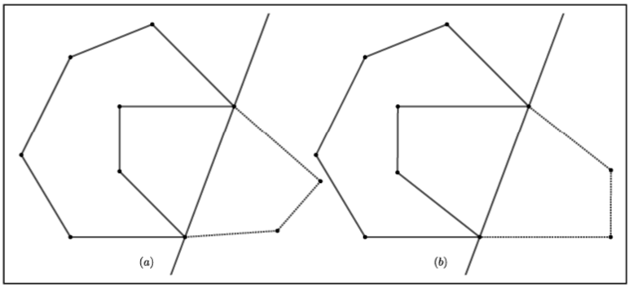

Figure 1. Flip (a) and flipturn (b).

Soviet team, Fedor Nazarov, gave an elegant solution which led him to earn the only special prize of the competition (see [13]).

As we revisit this problem in the context of discussions in section 4 we discovered that convexity condition is not needed to prove the upper bound. To see this, we shall define a *diagonal* of a simple polygon as a segment connecting its two non-adjacent vertices. Given any simple non-convex polygon, we can transform it into a convex polygon by a finite sequence of operations known as flips and flipturns (see *Figure* 1), such that each step preserves the perimeter of the polygon while increases the total diagonal sum d (see [7]). This result is known as the Erdős–Nagy theorem and it has its own history of rediscoveries which seems more intricate than the one we just told (see [4]). In the light of this result, we can state the following result:

Theorem 1.2 *Let p and d be the perimeter and the sum of diagonal lengths of a simple n-gon with $n > 3$. Then,*

(a) $\frac{2d}{p} < \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{n+1}{2} \right\rfloor - 2,$

(b) $0 < \frac{2d}{p}$ for $n = 4$,

(c) $1 < \frac{2d}{p}$ for $n \geq 5$

and these bounds are best possible.

Proof. As discussed above, (a) follows from Problem 1.1 and the Erdős–Nagy theorem. To see that this bound cannot be improved, take a cyclic convex n-gon such that $\left\lfloor \frac{n}{2} \right\rfloor$ of its vertices are clustered around a point on the circumscribing circle, and the

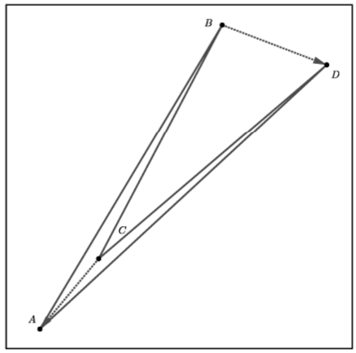

Figure 2. A simple quadrilateral where d can be made arbitrarily small.

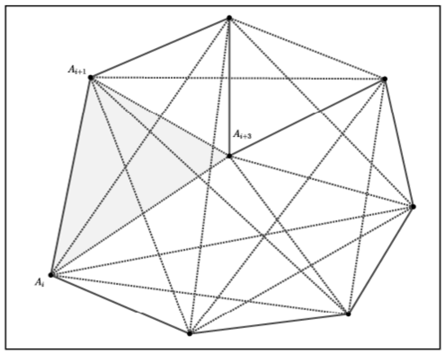

Figure 3. A simple 8-gon where the sum of dashed diagonals is greater then the perimeter.

remaining $\left\lfloor \frac{n+1}{2} \right\rfloor$ vertices are clustered around the point antipodal to this point. In this way, we can make p arbitrarily close to $4r$, where r is the radius of the circumcircle, and d arbitrarily close to $2r(\left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{n+1}{2} \right\rfloor - 2)$.

As for (b), it is clearly true that $0 < \frac{2d}{p}$ for $n = 4$. To see that this bound cannot be improved, we refer to the construction in *Figure* 2.

Let us now turn to (c). Assume $n \geq 5$ and let $A_1, ..., A_n$ be the vertices of P_n. Triangle inequality implies $|A_iA_{i+1}| < |A_iA_{i+3}| + |A_{i+1}A_{i+3}|$ for all $i \in \{1, 2, ..., n\}$ where the subindices are taken on modulo n. By adding these n inequalities, we obtain $2d \geq \sum_{i=1}^{n}(|A_iA_{i+3}|+|A_{i+1}A_{i+3}|) > p$, which gives the desired inequality; see *Figure* 3.

To see that this bound is best possible, construct a simple polygon resembling the chess queen symbol and flatten it so that $2d$ gets

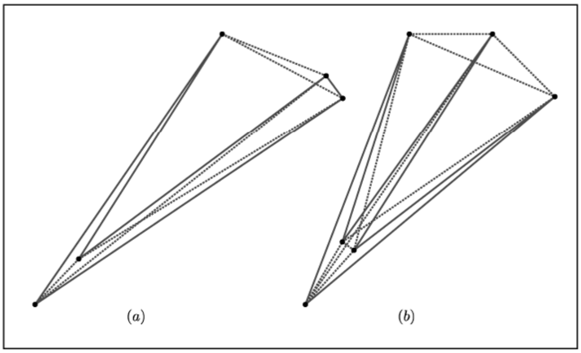

Figure 4. A simple 5-gon (a) and 6-gon (b) where $2d$ can be made arbitrarily close to p.

arbitrarily close to p; see *Figure* 4.

2. Smallest Enclosing Rectangles

We now state and prove another result which we (re)discovered while visiting the land of computational geometry. One of the problems discussed in this field is to find a minimum area enclosing rectangle of a polygon. The following result plays a key role in designing algorithms to fulfil the task.

Theorem 2.1: (Exercise 2.7 in [5], Theorem 2 in [6]) *The rectangle of minimum area enclosing a convex polygon has a side collinear with one of the edges of the polygon.*

Strictly speaking, Theorem 2 in [6] concerns with n-gons ($n \geq 4$), and quadrilaterals play a decisive role in its elegant proof. However, it must be well-known that triangles have the property given in Theorem 2.1 (see also discussions below). Notably, this result holds if we replace the rectangle by a triangle and it has a three dimensional analogue; see [9] and [11].

Within the context of Theorem 2.1, the following result was conjectured during one of our seminars. We then found an easy proof using Theorem 2.1 and a problem solved in Prasolov's problem book [12].

Theorem 2.2 *Let P_n be a convex n-gon whose area is S_n and S_R be the area of the smallest area rectangle containing it. Then $\frac{1}{2} \leq \frac{S_n}{S_R}$ for all $n \geq 3$. Moreover, equality holds if and only if $n = 3$*

and this bound is best possible for $n \geq 4$.

Proof. Problem 20.23 in [12] shows that there exists an enclosing rectangle of P_n with an area of at most $2S_n$. Its proof is based on the following argument. Let AB be the longest segment among the sides and diagonals of P_n. Draw two parallel lines which are perpendicular to AB, one passing through A and the other through B (see *Figure* 7). Then, because of our choice of AB, P_n is contained in the region bounded between these lines. Draw two other lines parallel to AB which support (i.e. are tangent to) P_n. Then the rectangle formed by the intersection of these four lines, which we denote by R', has the desired property. This result directly implies that $S_R \leq S_{R'} \leq 2S_n$, where $S_{R'}$ is the area of R'.

Take $\triangle ABC$ and assume AB is the longest side. Then, R' as described above for $\triangle ABC$ has an area of $2S_{ABC}$; see *Figure* 5(a). Take any rectangle R'' which contains $\triangle ABC$ and with area of $S_{R''}$. We claim that $S_{R'} = 2S_{ABC} \leq S_{R''}$. In order to see this, we may assume that A, B, C are located on the sides of R'' in such a way that at least one of them is a vertex of R'', as otherwise R'' can be made smaller. We are then confront with the following possibilities.

- Thee vertices of $\triangle ABC$ overlap with vertices of R''; then $\triangle ABC$ is right and $2S_{ABC} = S_{R''}$.

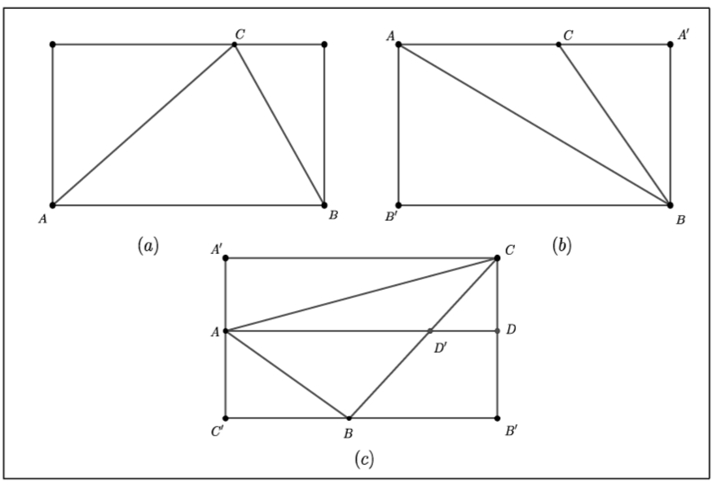

Figure 5. Rectangles containing $\triangle ABC$.

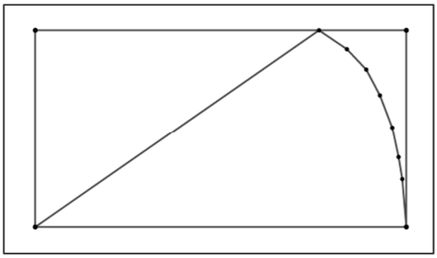

Figure 6. A convex 8-gon which is close to a triangle.

- Exactly two vertices of $\triangle ABC$ overlap with two adjacent vertices of R''; see *Figure* 5(a). Then $\triangle ABC$ has a side which overlaps with that of R'' and $2S_{ABC} = S_{R''}$.

- Exactly two vertices of $\triangle ABC$ overlap with two non-adjacent vertices of R''. This situation is in *Figure* 5(b), in which case $S_{R''} = 2S_{AA'B} > 2S_{ACB}$.

- Exactly one vertex of $\triangle ABC$ overlaps with a vertex of R''. This situation is reflected in *Figure* 5(c) where R'' is $A'CB'C'$ and $AD \perp CB'$, in which case $S_{R''} = S_{A'CDA} + S_{DB'C'A} = 2(S_{ACD} + S_{ADB'}) > 2(S_{ACD'} + S_{AD'B}) = 2S_{ABC}$.

This establishes our claim, which then implies that an equality in Theorem 2.2. holds when $n = 3$. On the other hand, for any $n \geq 4$, there is a convex n-gon arbitrarily close to a triangle which it contains (see *Figure* 6). Thus, the bound is best possible.

To see that equality is possible only if $n = 3$, first notice that we can identify the following three possibilities when $n \geq 4$.

- AB, the longest segment chosen, is a side of P_n,

- AB is a diagonal of P_n and $n \geq 5$, and

- AB is a diagonal of P_n and $n = 4$.

It is easy to see that the only case with $S_{R'} = 2S_n$, where R' is as defined above, is the last possibility (see *Figure* 7). In other cases, we then have $S_{R'} < 2S_n$, thus $S_R \leq S_{R'} < 2S_n$.

Consider the last possibility; see *Figure* 7. Since AB is a diagonal, none of the two sides of R' which are parallel to AB can be collinear with any side of $ACBD$. Moreover, this is also true for the remaining two sides of R'. To see this, assume that the side of R' passing through A is collinear with AD. Then $\measuredangle BAD = \frac{\pi}{2}$, which implies $|BD| > |AB|$, which is a contradiction. Thus, the situation in *Figure* 7 is generic, i.e. no side of R' is collinear with a side of $ACBD$. Then by Theorem 2.1 there exists another rectangle containing $ACBD$ whose area is less than $S_{R'}$. Thus, in the exceptional case, we have $S_R < S_{R'} = 2S_n$.

3. Geometry of Non-intersection

In a recent paper [8], C Huemer and P Pérez–Lantero studied intersection behaviour of set of disks with diameters as the sides of a convex n-gon. They called such disks '*side disks*' and showed that their intersection graph, a graph whose vertices represent the disks and any two vertices are connected if and only if their associated disks intersect, is planar for all $n \geq 3$; see Theorem 4 in [8].

Discrete geometry, naively interpreted, is a synthesis of classical (Euclidean) geometry with graph theory.

We found this result rather interesting for the following two reasons. First, its main content is consistent with our naïve interpretation of discrete geometry as a synthesis of classical Euclidean geometry with graph theory. Second, the geometric aspect of this result referred to an unusual situation as it pins down to investigating the 'disjointness' of planar convex figures. In order to give some insights on this aspect, we now discuss the case of convex quadrilaterals, P_4, which was not covered in [8]. The main finding is as follows.

Theorem 3.1. *Let P_4 be a convex quadrilateral. Then*

(a) *The number of disjoint pairs among its four side disks is at most one,*

(b) *If P_4 has orthogonal diagonals, then this number is zero, and*

(c) *If P_4 is such that any three of its side disks intersect, then it has orthogonal diagonals.*

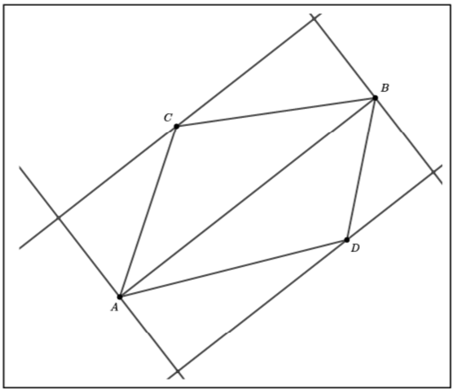

Figure 7. The case where $S_{R'} = 2S_n$.

Proof. The following result is well known, and can be used in solving some classical geometric extremum problems.

Lemma 3.2. *In any quadrilateral, the sum of the lengths of two opposite sides is at least twice the distance between the midpoints of the remaining two sides.*

For a proof, see p.130 in [2]. Suppose $ABCD$ is a convex quadrilateral with more than one disjoint pairs of side disks. Since a quadrilateral has only two pairs of non-neighbouring side disks, this implies that the number of disjoint pairs is two and the disjoint pairs must be the disks corresponding to the opposite sides of $ABCD$.

Denote the midpoints of its sides as E, F, G, H, which are the centers of the side disks (see *Figure* 8). Notice that AD-disk and BC-disk are disjoint if and only if $|HF| > \frac{|AD|+|BC|}{2}$. Similarly, AB-disk and CD-disk are disjoint if and only if $|EG| > \frac{|AB|+|CD|}{2}$. On the other hand, by Lemma 3.2 we have $\frac{|AD|+|BC|}{2} \geq |EG|$ and $\frac{|AB|+|CD|}{2} \geq |HF|$. By combining these four inequalities, we get $|HF| > |EG|$ and $|EG| > |HF|$, which is a contradiction. This proves Theorem 3.1 (a).

Let $A_1B_1C_1D_1$ be a convex quadrilateral with orthogonal diagonals, i.e. $A_1C_1 \perp B_1D_1$, and let O be their intersection point. Then $\triangle A_1OB_1$, $\triangle B_1OC_1$, $\triangle C_1OD_1$, and $\triangle D_1OA_1$ are right triangles. Thus, O belongs to all four side disks, and this proves Theorem 3.1 (b).

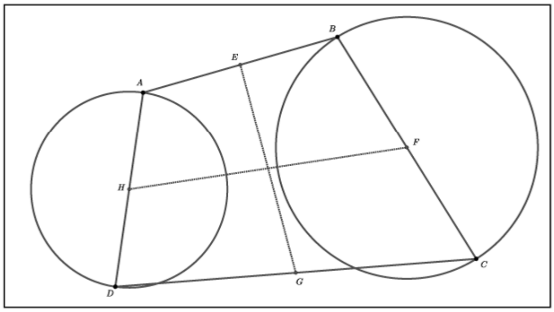

Figure 8. A quadrilateral with a disjoint pair of side disks.

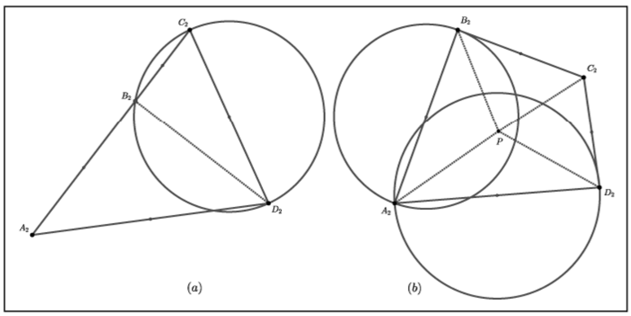

Figure 9. When four side disks intersect.

Let $A_2B_2C_2D_2$ be a convex quadrilateral such that any three of its side disks intersect. Then, by the celebrated Helly theorem, all four side disks intersect. Assume that one of the vertices, perhaps B_2, is in the intersection of the four side disks. Since B_2 is in both C_2D_2-disk and D_2A_2-disk, $\measuredangle C_2B_2D_2 \geq \frac{\pi}{2}$ and $\measuredangle D_2B_2A_2 \geq \frac{\pi}{2}$. On the other hand, $\measuredangle C_2B_2D_2 + \measuredangle D_2B_2A_2 = \measuredangle C_2B_2A_2 \leq \pi$, where the last inequality follows from convexity. Then, these three inequalities are equalities and A_2C_2 and B_2D_2 intersect at B_2 orthogonally (see *Figure* 9 (a)).

Let $P \notin \{A_2, B_2, C_2, D_2\}$ be a point in the intersection of four side disks. Since P is in A_2B_2-disk, $\measuredangle A_2PB_2 \geq \frac{\pi}{2}$ and similarly, $\measuredangle B_2PC_2 \geq \frac{\pi}{2}$, $\measuredangle C_2PD_2 \geq \frac{\pi}{2}$, $\measuredangle D_2PA_2 \geq \frac{\pi}{2}$ (see *Figure* 9(b)). But the sum of these four angles is 2π, which implies that each inequality is an equality. Since $\measuredangle A_2PB_2 = \measuredangle B_2PC_2 = \frac{\pi}{2}$, points A_2, P, C_2 are collinear, and so are points B_2, P, D_2. Thus, the diagonals of $A_2B_2C_2D_2$ intersect at P orthogonally, and this proves Theorem 3.1(c).

Remarks: The task to prove Theorem 3.1(a) was suggested by the authors as a problem for the 2017 Mongolian Student Mathematical Olympiad for the category of first year students. One contestant gave a short proof using complex numbers. One should also notice that convexity assumption is not necessary for this result to hold. Instead of Theorem 3.1(c), one could try to prove the converse of Theorem 3.1(b), which is a stronger statement. However, there are quadrilaterals with no disjoint pair of side disks and non-orthogonal diagonals; for example, the right trapezoid with vertices at $(0, 0)$, $(10, 0)$, $(10, 7)$, $(5, 7)$. One can also prove that the union of four side disks of a convex quadrilateral covers it; see Problem 3.17 in [14].

Driven by these motivations, we worked on similar problems and ended up with two research papers. In one of them, we investigated disks associated with a spherical great polygon. The latter is a circle partitioned into $n \in \mathbb{N}$ arc segments by n points marked on it, which we denote by C_n. If we identify each marked point as a vertex and each arc connecting two subsequent vertices which does not pass over any other vertex as a side, C_n is a spherical polygon with vertices on a great circle; hence the name 'spherical great polygon'. For each side of C_n, there is a unique disk centered on its midpoint and passing through its end points by the boundary. We call these disks as 'side disks' of C_n. We proved the following result in [3].

Theorem 3.3. *For any $n \geq 3$, the number of disjoint pairs among side disks of C_n is between $\frac{(n-2)(n-3)}{2}$ and $\frac{n(n-3)}{2}$. Moreover, these bounds are best possible and intersection graph of these disks is a subgraph of a triangulation of a convex n-gon.*

In its proof, a quadrilateral, C_4, played a key role once more. We also studied the intersection pattern of side squares of a convex n-gon. What is interesting is that for $n \geq 5$, they behave just like the side disks of a spherical great n-gon, but their behaviour differs in the baseline case of $n = 4$.

4. Conclusion

It seems that every pupil of science needs to remember, if possible every morning, the following three questions:

- What is known?
- What is open?
- What is current?

Our discussions in the first and second sections are more concerned with the first two questions, while third section is concerned with the third question. It should be highlighted that Theorem 1.2 is obtained as a synthesis of an IMO problem with the Erdős–Nagy theorem while Theorem 2.2 is obtained as a synthesis of a result in [6] with a result in [12]. On the other hand, the proof of Theorem 3.3 given in [3] synthesises geometry with graph theory. This pattern suggests the following guiding principle: In a scientific study, a breadth allows for a depth. So, widen it!

Acknowledgement

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