Unearthing the Banach–Tarski Paradox

B Sury

Archimedes could move the earth
no matter – what the girth.
Banach–Tarski broke it in pieces,
rotated them and showed size increases.
How on earth? This is "anarth !"

Greeks used the method of cutting a geometric region into pieces and recombining them cleverly to obtain areas of figures like parallelograms. In such problems, the boundary is ignored. However, in our discussion, we will take every point of space into consideration. The human endeavour to compute lengths, areas, and volumes of irregular complicated shapes and solids created the subject of 'measure theory'. The paradox of the title can be informally described as follows. Consider the earth including the inside stuff. It is possible to decompose this solid sphere into finitely many pieces and apply three-dimensional rotations to these pieces such that the transformed pieces can be put together to form two solid earths! The whole magic lies in the word 'pieces'. The pieces turn out to be so strange that they cannot be 'measured'.

Sury likes interacting with talented high school students and writing for undergraduate students. At present, he is the National co-ordinator for the Mathematics Olympiad Programme in India.

The human endeavour to compute lengths, areas, and volumes of irregular complicated shapes and solids created the subject of measure theory. Kolmogorov's famous book that appeared in 1933 brought in the understanding that probability theory is best studied *via* measure theory. Measure theory itself was developed in various stages as demanded by Fourier analysis. It was found that Riemann's theory of integration was still deficient in treating integration of several reasonable functions. Lebesgue's work produced the optimal theory which was general enough and yet specific enough to derive many interesting applications. For instance, in the preface of his book, Kolmogorov says:

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After Lebesgue's publication of his investigations, the analogies between measure of a set and probability of an event, and between integral of a function and mathematical expectation of a random variable, became apparent.

"The purpose of this monograph is to give an axiomatic foundation for the theory of probability. The author set himself the task of putting in their natural place, among the general notions of modern mathematics, the basic concepts of probability theory – concepts which until recently were considered to be quite peculiar. This task would have been a rather hopeless one before the introduction of Lebesgue's theories of measure and integration. However, after Lebesgue's publication of his investigations, the analogies between measure of a set and probability of an event, and between integral of a function and mathematical expectation of a random variable, became apparent. These analogies allowed of further extensions; thus, for example, various properties of independent random variables were seen to be in complete analogy with the corresponding properties of orthogonal functions."

We discuss the following apparent paradox. Consider the earth including the inside stuff. We wish to decompose this solid sphere into finitely many pieces and apply three-dimensional rotations to these pieces such that the transformed pieces can be put together to form two solid earths or to form an earth of double the size. If this sounds ridiculous, we will show that this can actually be done! This goes under the name of Banach–Tarski paradox. The whole magic lies in the word 'pieces'. The pieces turn out to be so strange that they cannot be 'measured'. It is even more amazing that the Banach–Tarski paradox is not valid in dimension 2 but valid in all dimensions more than 3. We will talk a bit in the last section about a paradox that is valid in dimension 2.

To lead into the notions of what is measurable and what is not, we first discuss an example of a subset of real numbers whose 'length' cannot be measured. This was given by Vitali in 1903.

Example of a Non-Measurable Set

We try to keep the discussion as informal as possible without actually making a wrong mathematical statement. We suppose that the notion of Lebesgue-measure μ on subsets of real numbers has the following natural properties:

$\mu([a,b]) = b - a$ and $\mu(\sqcup_i E_i) = \sum_i \mu(E_i)$ when $\mu(E_i)$'s are welldefined.

On the set $[0,1]$ of real numbers, consider the identification of any two points which differ by a rational number. In this manner, the real numbers between 0 and 1 group themselves into distinct classes. Let us select one representative point $f(\overline{x})$ in $[0,1)$ corresponding to each class $\overline{x}$. This innocent-looking selection involves what is known as the 'axiom of choice'. Denote

$$E = \{f(\overline{x}) : x \in [0,1)\}.$$

As the rational numbers are countable, write a bijection

$$r : \mathbb{N} \to \mathbb{Q} \cap [0,1).$$

For each $n \geq 1$, we have a disjoint decomposition

$$E = \{x \in E : 0 \leq x < r(n)\} \sqcup \{x \in E : r(n) \leq x < 1\}.$$

Then, for all n, consider the sets

$$E_n^- = \{x \in [0, 1 - r(n)) : x + r(n) \in E\},$$

and

$$E_n^+ = \{x \in [1 - r(n), 1) : x - (1 - r(n)) \in E\}.$$

Note that $E_n = E_n^- \cup E_n^+$ can be identified with

$$E_n = \{x \in [0,1) : \{x + r(n)\} \in E\},$$

and that $\cup_n E_n = [0,1)$. We claim that E_n are pairwise disjoint sets for distinct n. If we prove these facts, it will follow that $\mu(E)$ cannot be defined; otherwise, $\mu(E_n) = \mu(E)$ for all n whereas, $\sum_n \mu(E_n) = \mu([0,1)) = 1$, which is absurd.

Let us prove that $E_m \cap E_n = \emptyset$ for $m \neq n$. Suppose, if possible, $r(n) < r(m)$ but $x \in E_m \cap E_n$; then $\{x + r(m)\}, \{x + r(n)\} \in E$. However, their difference is $r(m) - r(n)$+integer. This is impossible as $0 \leq r(n) < r(m) < 1$ and E contains only one element from any class. Therefore, we have proved our claims, and E is not a Lebesgue-measurable set.

Remark.

In 1970, R M Solovay showed the existence of a model of set theory which does not have axiom of choice and in which every subset of $\mathbb{R}$ is Lebesgue-measurable!

The above construction using the axiom of choice was made by Vitali. In 1970, R M Solovay showed the existence of a model of set theory which does not have axiom of choice and in which every subset of $\mathbb{R}$ is Lebesgue-measurable!

1. Paradoxical Sets and Hausdorff Paradox

Free Groups

The discussion in the beginning tells us that we are interested in breaking sets into pieces and after transforming them, reassemble them. So, we need to make precise what the transformations alluded to are. We talk of groups of transformations in general and groups of rotations in three dimensions about an axis in particular. While talking of groups, the abstract notion of free groups is fundamental and leads to abstractly or combinatorially describe general groups.

A free group on two generators is a set of 'reduced words' that can be formed from two symbols a, b and their two more symbols denoted for convenience as a^{-1} and b^{-1} (see Combinatorial Group Theory, *Resonance*, November 1996). A reduced word is an expression when there are no two consecutive symbols of the form a and a^{-1} or b and b^{-1}. One can 'multiply' words by concatenating them – the order should be maintained; this is a nonabelian group. Different reduced words are considered different and the empty word is considered the identity element. While multiplying two words, x and y in the order $x.y$, one has to cancel any two symbols of the form a and a^{-1} or b and b^{-1} which appear adjacent. For example, $x = aba^{-1}$ and $y = a^2b^{-1}$ give the reduced word $xy = abab^{-1}$. The abstract group described in this manner is called the 'free group' on two generators.

The free group is *countable*; that is, it can be put in bijection with the set of integers. The free group also has natural avataars in many places.

For example, the two matrices $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ yield a free group as a group of matrices under matrix multiplication and inversion. This means that any two finite product expressions in A, B, A^{-1}, B^{-1} (where none of these 4 symbols appear adjacent to their inverses) give different matrices.

The group $SO(3)$ contains lot of free groups in two generators.

If a group G 'acts' as transformations on a set S (this means each $s \in S$ is carried by each element g of G into a 'new' point $g.s$ in a compatible manner; that is, $g_1.(g_2.s) = (g_1g_2).s$ and identity does not change any point). The paradoxical decompositions of sphere we will be discussing depend on the action by 'special orthogonal group' $SO(3)$, the rotation group of 3×3 real matrices A satisfying $A^t = A^{-1}$ and having determinant 1. The group $SO(3)$ contains lot of free groups in two generators. In fact, the set of pairs of matrices A, B in $SO(3)$ which generate a free group is 'dense' in the product $SO(3) \times SO(3)$. Let us see how to produce at least one free group as this will be crucial for the discussion.

The matrix of rotation by $cos^{-1}(1/3)$ *about the z-axis*

$$Z = \begin{pmatrix} 1/3 & -2\sqrt{2}/3 & 0 \\ 2\sqrt{2}/3 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and the matrix of rotation by $cos^{-1}(1/3)$ *about the x-axis*

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & -2\sqrt{2}/3 \\ 0 & 2\sqrt{2}/3 & 1/3 \end{pmatrix},$$

generate a free group.

We wish to show that any reduced word w in Z, X, Z^{-1}, X^{-1} is a nontrivial (that is, not the identity) matrix. Now, we may conjugate w (that is, pre-multiply by a matrix M and post-multiply by M^{-1}) and assume that w ends in Z or Z^{-1}. Evidently, it suffices to show that such a w is not the identity matrix. We shall do this by proving inductively that if n is the length of the word w, then the first column of w is necessarily of the form $\frac{1}{3^n}\begin{pmatrix} a \\ b\sqrt{2} \\ c \end{pmatrix}$, where a, b, c are integers such that 3 does not divide b (in particular, it is not 0). This is clear when $n = 1$ because $a = 1, b = \pm 2, c = 0$. Assume that the assertion for all words of length n for some $n \geq 1$. Let w have length $n + 1$. Then, $w = gw_n$ where w_n has length n and $g = Z^{\pm 1}$ or $X^{\pm 1}$. Now,

$$w_n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^n} \begin{pmatrix} a_n \\ b_n\sqrt{2} \\ c_n \end{pmatrix},$$

where a_n, b_n, c_n are integers with 3 not dividing b_n. So,

$$w \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{3^n} g \begin{pmatrix} a_n \\ b_n\sqrt{2} \\ c_n \end{pmatrix} = \frac{1}{3^{n+1}} \begin{pmatrix} a_{n+1} \\ b_{n+1}\sqrt{2} \\ c_{n+1} \end{pmatrix},$$

where (respectively, for $g = Z^{pm1}$ or $X^{\pm 1}$):

$$a_{n+1} = a_n \mp 4b_n, b_{n+1} = b_n \pm 2a_n, c_{n+1} = 3c_n\,,$$

or

$$a_{n+1} = 3a_n, b_{n+1} = b_n \mp 2c_n, c_{n+1} = c_n \pm 4b_n.$$

Evidently, $a_{n+1}, b_{n+1}, c_{n+1}$ are integers. We only need to verify that b_{n+1} is not a multiple of 3. Depending on which matrix w_n begins with, there are many cases and it is left as an exercise to verify that this always holds.

The Cantor–Schroeder–Bernstein theorem says that if there is an injective map from a set A to a set B and one from B to A, then there is a bijection between them.

Suppose G acts on a set S. A subset P of S is said to be *paradoxical* (for this action) if there exists pairwise disjoint subsets $A_1, \cdots, A_r$, and pairwise disjoint subsets $B_1, \cdots, B_s$ of P and elements $g_1, \cdots, g_r, h_1, \cdots, h_s$ in G such that

$$P = \bigcup_{i=1}^{r} g_i(A_i) = \bigcup_{j=1}^{s} h_j(B_j).$$

By replacing the A_i's and B_j's by smaller sets, we may suppose that the sets $g_i(A_i)$'s are pairwise disjoint and that the sets $h_j(B_j)$'s are pairwise disjoint. Moreover, by the Cantor–Schroeder–Bernstein theorem, one may even assume that P is the union A_i's and the B_j's.

An example is the free group $G = F(a, b)$ on two generators a, b acting on itself by left multiplication. Indeed, a paradoxical decomposition is

$$F(a, b) = w(a) \cup aw(a^{-1}) = w(b) \cup bw(b^{-1}),$$

where $w(a)$ is the set of reduced words starting with a etc. Indeed, this is almost clear excepting that we need to concern ourselves with the empty word which is the identity in the group. We have

$$F(a, b) = W_1 \sqcup W_2 \sqcup W_3 \sqcup W_4,$$

where
$W_1 = w(a) \cup \{I, a^{-1}, a^{-2}, \cdots\}$,
$W_2 = w(a^{-1} \setminus \{a^{-1}, a^{-2}, \cdots\}$,
$W_3 = w(b), W_4 = w(b^{-1}$ and

$$F(a, b) = W_1 \cup aW_2 = W_3 \cup bW_4.$$

Free Actions for Paradoxical Decomposition

If a free group F acts freely on some set S (that is, no nontrivial element of F fixes any point of S), then S admits a paradoxical decomposition. Indeed, (by the axiom of choice) one can choose one point of S for each orbit of G on S and the paradoxical decomposition of F evidently yields a paradoxical decomposition on S. If the action is transitive (that is, there is only one orbit), one does not need to use the axiom of choice. To see how we get a paradoxical decomposition, consider a subset P of S such that each point of P corresponds to a unique F-orbit in S. Then, a paradoxical decomposition $F = \bigcup_{i=1}^{r} g_i A_i = \bigcup_{j=1}^{s} h_j(B_j)$ for F gives a paradoxical decomposition

$$S = \bigcup_{i=1}^{r} g_i(\hat{A}_i) = \bigcup_{j=1}^{s} h_j(\hat{B}_j),$$

Obtaining a paradoxical decomposition of the sphere needs the axiom of choice: we need to CHOOSE one point for each orbit.

where $\hat{A}_i = \bigcup_{g \in A_i} g(P)$ for all $i \leq r$ and $\hat{B}_j = \bigcup_{h \in B_j} h(P)$ for all $j \leq s$.

The group $SO(3)$ acts on the unit sphere S^2. We already saw that $SO(3)$ contains an abundance of copies of the free group on two generators, and the latter admits paradoxical decompositions. We may take the aid of a free subgroup of $SO(3)$ to push a paradoxical decomposition for almost the whole of the unit sphere. However, at this point we will require the axiom of choice.

Corollary (Hausdorff Paradox).
There exists a countable subset C of S^2 such that $S^2 \setminus C$ is paradoxical for the $SO(3)$-action.
Proof. Each nontrivial element of the group $SO(3)$ fixes two points (the points where the axis of the rotation meets the sphere). Consider a free group F in $SO(3)$ (one was produced above). Then, the set C of points of S^2 fixed by some nontrivial element of F is countable as F is countable. Hence, the action of F on the complement of C in S^2 is free, and we have a paradoxical decomposition on this complement by the above discussion.

2. Banach–Tarski Paradox

The paradox referred to in the title involves paradoxical decompositions of the solid unit ball in $\mathbf{R}^3$. Before looking at the solid ball, we show that the whole of the sphere S^2 and the above $S^2 \setminus C$ have equivalent decompositions.

Let G act on a set S. Subsets A and B of S are said to be *equi-decomposable* if there are pairwise disjoint subsets $A_1, \cdots, A_n$ of A with $A = \cup_i A_i$ and elements $g_i \in G$ such that $g_1(A_1), \cdots, g_n(A_n)$ are pairwise disjoint subsets of B with $B = \cup_i g_i(A_i)$.
Equi-decomposability is an equivalence relation.

We shall prove the following surprising consequence for the usual action of the rotation group $SO(3)$ on the unit sphere S^2:

The unit sphere S^2 is equi-decomposable under $SO(3)$-action to $S^2 \setminus C$ for any countable subset C. Further, the sphere S^2 is paradoxical for $SO(3)$ – that is, S^2 can be dissembled into finitely *many disjoint pieces so that after applying rotations, the pieces can be reassembled to form two copies of* S^2.

Note that the latter statement above follows from the former one on applying the previous corollary.

The idea underlying the proof of the former assertion can be described as follows. As C is countable, we can certainly find line l that avoids C. We consider rotations around the axis l. If T is the collection of angles $\theta \in [0, 2\pi]$ so that for some point c in the countable set C, the rotation around l by angle θ repeated n times takes c to another point of C for some n. Evidently, the collection T of all such angles is countable. Let us avoid these countably many angles and FIX any $\theta \in [0, 2\pi)$ with $\theta \notin T$. This means that the rotation ρ_θ has the property that $\rho_\theta^m(C)$ and $\rho_\theta^n(C)$ are disjoint for any $m \neq n$. In other words,

$$C \sqcup \rho_\theta(C) \sqcup \rho_\theta^2(C) \sqcup \cdots ,$$

is a disjoint union; call this union $\tilde{C}$. Then, obviously $\tilde{C}$ and $\rho_\theta(\tilde{C})$ are equi-decomposable, and $\rho_\theta(\tilde{C})$ does not intersect C. This gives the fact that $S^2 = \tilde{C} \sqcup (S^2 \setminus \tilde{C})$ is equi-decomposable with $\rho_\theta(\tilde{C}) \sqcup (S^2 \setminus \tilde{C}) = S^2 \setminus C$.

When we talk about the Banach–Tarski paradox, we consider the solid ball in three dimensional space, and the problem with the action of $SO(3)$ is that it fixes the origin. Therefore, along with rotations, we shall allow translations also. In this case, we can finally prove what we mentioned in the beginning:

Banach–Tarski Paradox. *Using rotations and translations, any solid ball (of any radius) in* $\mathbf{R}^3$ *has a paradoxical decomposition. In other words, the unit solid ball can be broken into finitely many disjoint pieces such that they can be reassembled after rotations and translations into two solid unit balls.*

Indeed, consider the (solid) unit ball D^3 in $\mathbf{R}^3$ centered at the origin. As we know that S^2 is $SO(3)$-paradoxical, there are pairwise disjoint subsets $A_1, \cdots, A_r, B_1, \cdots, B_s$ of S^2 and elements $g_1, \cdots, g_r, h_1, \cdots, h_s$ in $SO(3)$ such that

$$S^2 = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_r \sqcup B_1 \sqcup B_2 \sqcup \cdots \sqcup B_s$$

One can similarly obtain a paradoxical decomposition on the unit circle using $SO(2)$ but it needs infinitely many pieces.

$$= \bigcup_{i=1}^{r} g_i(A_i) = \bigcup_{j=1}^{s} h_j(B_j).$$

Consider the scaled subsets $\tilde{A}_i = \{uA_i : 0 < u \leq 1\}$ and $\tilde{B}_j = \{uB_j : 0 < u \leq 1\}$ of $D^3 \setminus \{(0,0,0)\}$. They clearly give a paradoxical decomposition of $D^3 \setminus \{(0,0,0)\}$. Finally, we show that the solid unit ball D^3 is equi-decomposable to the latter space that misses the origin. To show this, fix any line l not passing through origin and containing the point $(0.5, 0, 0)$ say. Then, a rotation ρ about l having infinite order defines a set

$$D := \{\rho((0,0,0)), \rho^2((0,0,0)), \rho^3((0,0,0)), \cdots\}$$

that does not contain the origin. As D and $\rho(D)$ are equi-decomposable, and the latter misses the origin, we have the equi-decomposability of $D^3 = D \sqcup (D^3 \setminus D)$ and $D^3 \setminus \{(0,0,0)\} = \rho(D) \sqcup (D^3 \setminus D)$.

3. Paradoxes in Dimension 2

We saw that the group $SO(3)$ was used to obtain a paradoxical decomposition of the unit sphere S^2. If we use exactly the same method, it is possible to obtain a paradoxical decomposition of the unit circle S^1 using the plane rotation group $SO(2)$; however, we need to break into infinitely many pieces. Here is the proof.

Consider two polygons congruent if one of them can be broken into finitely many polygonal pieces and after transforming by isometries, ignoring boundaries and can be reassembled to form the other polygon. Then, two polygons are congruent if and only if they have the same area.

On the unit circle, if a point is rotated (clockwise or anticlockwise) by an angle that is a rational multiple of 360 degrees, call the new point to be equivalent to it. By the axiom of choice, we can choose one point in each equivalence class; call this chosen set S. As the rotations by rational multiples of 360 degrees can be enumerated as $\rho_1, \rho_2, \cdots, \rho_3, \cdots$ the sets $\rho_1(S), \rho_2(S), \rho_3(S), \cdots$ give a countably infinite disjoint decomposition of the circle S^1. As any two $\rho_i(S)$ and $\rho_j(S)$ are rotationally equivalent, we may look at the disjoint union of all the 'even' ones $\rho_{2i}(S)$ and individually rotate them to get all the $\rho_j(S)$'s. One may do this for the 'odd' ones $\rho_{2i+1}(S)$'s also. In this manner, we obtain

$$S^1 = \sqcup_{i=1}^{\infty} \rho_i(S) = \sqcup_{i=1}^{\infty} g_{2i}\rho_{2i}(S) = \sqcup_{j=1}^{\infty} h_{2j+1}\rho_{2j+1}(S),$$

for some rotations g_{2i}, h_{2j+1}.

4. On Dissection of Polygons

In the beginning of the article, we referred to the Greek way of computing areas of polygonal regions by dissection and reassembling. If we consider two polygons to be congruent if one can be broken into finitely many polygonal pieces that can be transformed by isometries and reassembled (ignoring boundaries) to form the other polygon. Clearly, congruent polygons must have the same area. It is a very beautiful result due to Bolyai and Gerwien that the converse is also true! In other words,

Dehn showed that a regular tetrahedron is not congruent (using dissections only into polyhedra) to a cube.

Two polygons are congruent by dissection (defined as above) if they have the same area.

It is instructive to do this for a triangle and a square.

In three dimensions this is not possible. Max Dehn showed that a regular tetrahedron is not congruent to a cube (allowing dissections into polyhedra only). However, we do note from the Banach-Tarski paradox that if we allow more weird pieces in the dissection, such a congruence is possible (even when the volumes are different).

Address for Correspondence
B Sury
Stat-Math Unit
Indian Statistical Institute
8th Mile Mysore Road
Bangalore 560 059 India.
Email: surybang@gmail.com

Suggested Reading

[1] Stan Wagon, *The Banach–Tarski Paradox*, Cambridge University Press, 1999.