Hamilton's Optics

The Power of Wavefronts

Rajaram Nityananada

Building on work by Fermat and Huygens, Hamilton transformed the study of geometrical optics in his very first paper, presented when still in his teens. His 'characteristic function' was an analytical way to describe wavefronts, and in his hands a powerful tool to look at families of rays rather than isolated ones. His prediction of internal and external conical refraction in some crystals and its spectacular verification in a few months established his reputation among his contemporaries. This formulation of optics uncovered many general properties, not easy to see in the conventional method of tracing individual rays. The deepest outcome of his early optical work was a parallel view of the mechanics of particles, which played a fundamental role in the birth of quantum mechanics and continues to be the standard framework for classical mechanics up to the present time.

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Introduction

High school students are all exposed to geometrical optics – the reflection and refraction of rays of light according to the two well-known laws. These lead to analysis of mirrors, prisms, and lenses. While the experimental side of the subject has some charm – catching images on a screen in a darkened room, and peering at spectra through a telescope – the theory is rather uninspiring. One follows rules such as 'draw a ray through the focus and let it emerge parallel to the axis' or 'draw a ray through the optic centre of the lens and let it pass undeviated'. Images are located by intersections of these

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rays, and formulae derived for their size and location. This article introducing Hamilton's work in its historical context is intended as an antidote to the early dose that most of us have received. It is one of the best kept secrets of theoretical physics that there is a better way of doing geometrical optics, nearly two hundred years old. The ten volume *Course of Theoretical Physics* by Landau and Lifshitz has all of four pages devoted to this in Volume 2, but they at least capture the essence!

History

We begin with the historical background. It was known from ancient times that light travels in a straight line in a uniform medium such as air or water. This is clearly the path of shortest length between two points. Already in the first century AD (now called CE, for 'common era'). Heron, a Greek mathematician in Alexandria, stated that when light is reflected from a plane mirror, the usual law of reflection ensures that it takes the shortest path from source to receiver *via the mirror*(*Figure* 1a).(Mathematicians in those days were also engineers – Heron had an early version of a steam engine!) At a minimum, a first order change in the point of reflection C causes zero first order change in the path length (*Figure* 1b). (Usually, a function looks like a parabola near its minimum, so there is a second order change.) However, the reverse is not true. A zero first order change can also occur near a maximum, and reflection from a curved mirror provides an example (*Figure* 1c).

Fermat (1662) brought refraction into the same framework by suggesting that light travels more slowly in a medium like water, compared to air. This means that the path of least time is not the path of shortest length. It is advantageous to lengthen the portion of the path which is in air, so as to shorten the part in water where light travels more slowly. The speed of light in a medium of refractive index n is c/n; so the formula for the time

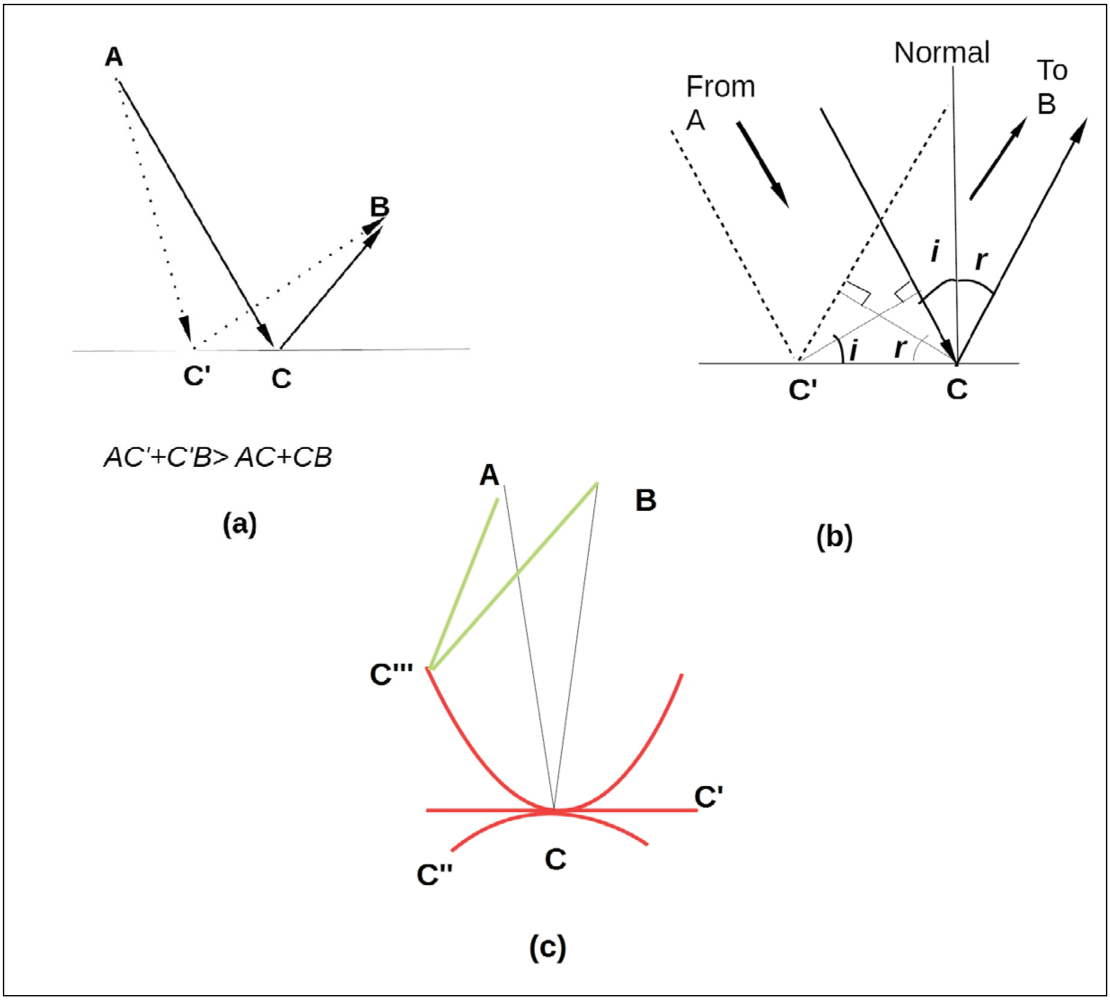

Figure 1a. Light reflected from a plane mirror follows the shortest path via the mirror.

Figure 1b. The law of reflection implies that the path length is `stationary' . This means that a first order change in the path causes zero first order change in the path length. This is shown in the blow-up of the region near the point of reflection. The first order difference between the path via C and the path via C′ is CC′(sin i – sin r) and this is zero when $i = r$.

Figure 1c. For a plane or a convex mirror, the path length is a minimum, but for a concave mirror of small radius, it can be a maximum. Three mirrors are shown in red, and have a common tangent, and hence a common normal, at C. Therefore, all three obey the law of reflection for the ray ACB. The alternative paths AC′B via the plane mirror, and AC′′B via the convex mirrror are longer than that via C. However, the path via C′′ on the concave mirror, shown in green, is clearly shorter -- which means that in this case, the ray follows the path of *maximum* time. Note that none of these is the case of image formation – that is dealt with in *Figure* 6.

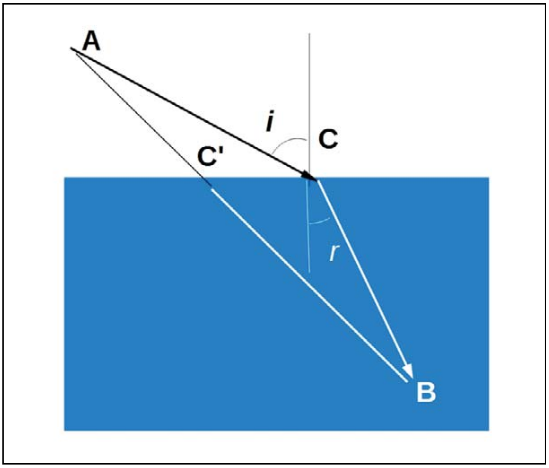

Figure 2. A light ray ACB going from air to water needs to travel straight in each medium but bend at the boundary so as to minimise the time taken. The shorter geometric path AC′B takes a longer time. (Think of the example of a a rescuer reaching a drowning person given in the text.) An argument similar to that in *Figure* 1 shows that the first order change in *optical path* (not geometric path!) is proportional to (sin i – n sin r). Setting this to zero gives Snell's law of refraction (which was stated by Ibn Sahl in Baghdad 600 years before Snell).

taken to travel a length $\mathrm{d}l$ is $\mathrm{d}t = n\mathrm{d}l/c$. It is usual to define 'optical path length' $\mathrm{d}P$ by the formula $\mathrm{d}P = n\mathrm{d}l$. Fermat's principle of 'least' time states that the quantity $P = \int_{\text{start}}^{\text{finish}} n\mathrm{d}l$ is stationary.

As a practical application, if you are on the bank of a river, then you should not run straight towards the drowning person you want to rescue. It is better to lengthen the part on land. This doesn't cost that much extra time, since you can run fast. Shortening your path in water gains quite a bit of time, given your slower swimming speed. The quickest path is given by the law of refraction. One interesting corollary of Fermat's principle is that light rays can be reversed – they take the same path whether going from A to B or vice versa. (We are assuming here that the speed has the same value for opposite directions of travel.)

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We now use Fermat's principle in a more interesting situation, the mirage. The refractive index is lower for hot air near the ground (say, a tar road) and higher as one goes up. The curved path taken by the ray has a longer distance, but a shorter time (i.e., shorter optical path) than the straight line, because it takes advantage of the higher speed – lower refractive index – near the ground (*Figure* 3a).

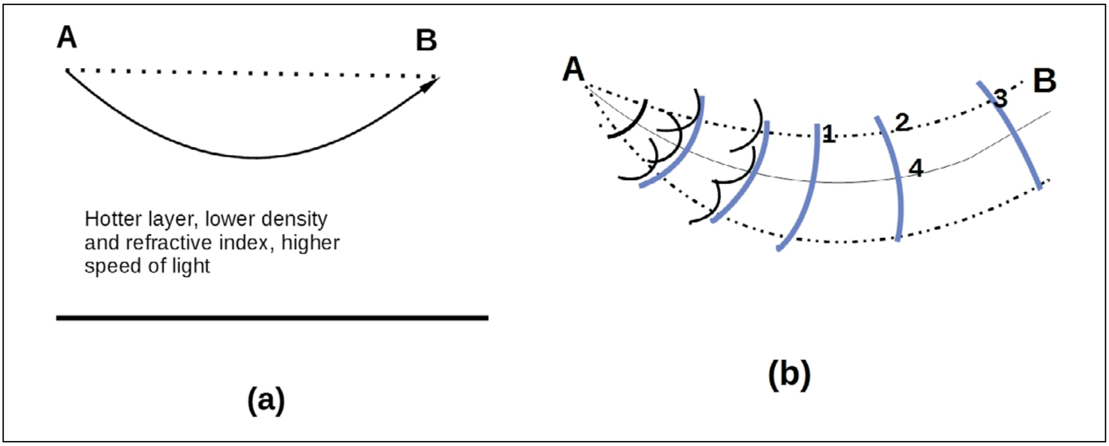

Figure 3a. Curved path of a ray in a medium where the refractive index increases with height above the ground. This bending gives rise to a mirage.

Figure 3b. Huygens construction for the same problem. The *n*-th wavefront represents the set of points which light reaches in the shortest time after *n* steps of duration Δ. The full construction is shown only for a few steps, and a small gap left between the wavelets and their common tangent, for clarity. Notice that we get all the rays starting from A, not just the one going through B. In this figure, in which the rays do not cross, it is clear that any other path from A to B will take a longer time than the one which takes the shortest time between each pair of wavefronts.

We now use the same mirage example to bring out Huygens, approach (1678). Interestingly, he used the idea of light waves but nowhere considered effects which depend on the wavelength, such as interference. His geometric construction for reflection and refraction is given in most high school texts. We show the construction for the same mirage problem, in *Figure* 3b .

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From the source, we first draw a spherical wave, which is our first wavefront – a set of points which can all be reached in the same time interval Δt. (We choose this interval to make this sphere, and the later ones, small enough so that the variation of the refractive index can be neglected over its radius, and we can use the speed at the centre.) Each point on this first spherical wavefront gives birth to a spherical wavelet, which consists of all those points which can be reached in $2\Delta t$. But we only keep that point on each wavelet where it touches a

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surface which is a common tangent to all the wavelets. (This is called the 'envelope' of the wavefronts.) Note that the wavelets correspond to the same time interval, which means that they have a larger radius in the region of lower refractive index. The rays are constructed by joining the source of each wavelet to the point where it touches the envelope of the whole family. Huygens construction shows that at any given point, the rays are perpendicular to the wavefronts, since the radius of a sphere is perpendicular to the tangent plane. It is clear from *Figure* 3b that we recover the same kind of curved path which Fermat's principle predicts.

The figure also shows that the construction is a clever way of implementing the Fermat principle. The rule for constructing the ray guarantees the shortest time from a point on one wavefront to the next. It is as if the wavelets are sent out to explore all possible paths, and the one of shortest time is chosen. In fact, it is unnecessary to draw the wavelets. We need to just draw rays, perpendicular to the wavefronts, of equal optical path length. Their end points give the new wavefront. This is closely related to a famous algorithm used by computer scientists for finding the shortest path starting from a given city to any other, in a group of cities interconnected by roads This was proposed in 1956 by Dijkstra, who, like Huygens, was Dutch, i.e., from the Netherlands (see suggeted reading). Here too the list of roads starting from each city at any stage is pruned so that we keep only the shortest routes to the next city, and this process is repeated.

Huygens' construction implies that the set of all rays starting from a point will have the property that one can draw a family of surfaces which are everywhere perpendicular to these rays. This may seem obvious when we draw a plane figure with an arbitrary set of rays (*Figure* 4a). But as *Figure* 4b shows, this is not true for a three-dimensional family of rays twisting around each

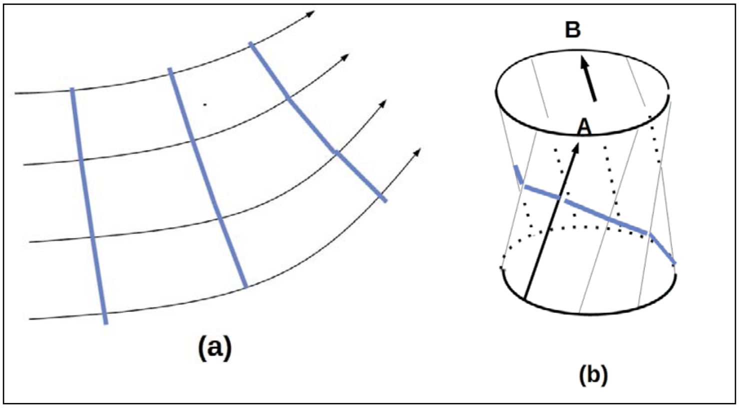

Figure 4a. A family of rays of light which follow general smooth curves in a plane (thick lines). There is no difficulty in constructing a family of wavefronts normal to them simply by drawing perpendiculars to the rays.

Figure 4b. A family of rays in three-dimensional space, which twist around each other. The familiar cane `modha' often used as a seat in India is an example (for those not familiar with this piece of furniture, it is a hyperboloid of one sheet). One can start constructing a surface normal to these rays by the same method, (thick blue lines). There are two blue lines from A to B, one going clockwise and the other anticlockwise. It is clear they will not meet the ray B at the same point – one is going up and the other down. The attempt to construct a wavefront perpendicular to all the rays fails.

other. The attempt to construct such a surface fails.

It is therefore a special property of rays which start from a point and obey the standard laws of reflection and refraction, and hence Fermat's principle, that such surfaces exist. Huygens tells us how to find them – they are the wavefronts given by his construction. This property of rays – of being normal to wavefronts – is sometimes known as Malus' law, rather unfairly as we will see below.

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Huygens made brilliant use of his construction to explain the phenomenon of double refraction in calcite crystals. He had to postulate two wavefronts, one spherical and the other a spheroid. A spheroid is the surface obtained by rotating an ellipse about one of its axes.(see *Box* 1, particularly *Figure* B for some more details). He could not identify the physical cause of this split personality of light, this had to wait for more than a hundred years.

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Huygens did not realise he was dealing with transverse waves. In the early nineteenth century, Malus, in France, discovered the polarisation of light. A transverse wave in three-dimensional space has two possible independent directions of vibration, each of which can have a different refractive index, and hence travel differently in a crystal. (This is unlike sound waves in air or water waves which have only one such mode of vibration.)

It is very interesting that the same Malus gave an analytical proof of the result that rays starting from a point source and reflected from a single mirror will be perpendicular to a surface. This was more than a hundred years after Huygens! He then considered the case when these rays are reflected from a second mirror, tried to find such a surface perpendicular to them analytically, and gave up. Posed in this way, the mathematical problem is quite complex. However, there is is no difficulty in applying Huygens construction to find such a surface normal to the rays, after any number of reflections or refractions. We can thus say that these two great students of light are tied at one game each.

Hamilton's Theory of Systems of Rays

In 1824, a nineteen year old Hamilton presented an early version of his theory of systems of rays to the Royal Irish Academy but it did not appear in printed form in the proceedings. Three years later, he presented his more general theory and this time it was published and recognised. We outline his far-reaching ideas using a very simple example. This is an optical system in which all the rays lie in the x–z plane, and make small angles with the z-axis (such situations are called 'paraxial'). The system can still consist of many refracting surfaces, as long as the angles of the rays with the z-axis remain small. The geometry is shown in *Figure* 5. A ray travels from the point A with co-ordinates (x, z) to the point B on the other side of the optical system with co-ordinates

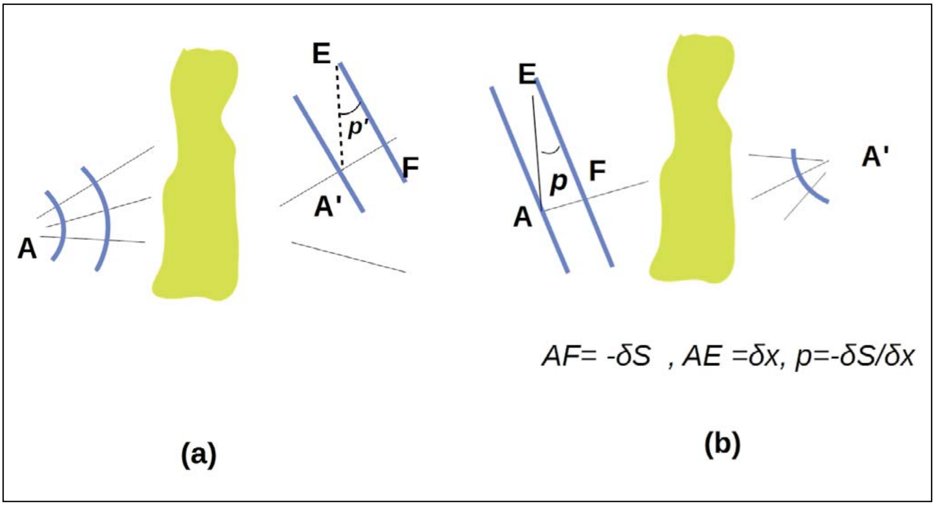


***Figure* 5.** Hamilton's characteristic function.

(a) Surfaces of constant S as a function of x' keeping x fixed. Two wavefronts corresponding to values S and $S+\delta S$ are shown near A', and their perpendicular separation is δS since the refractive index is unity. The ratio of this to the vertical separation, $\delta x'$, is the sine of the angle p'.

(b) A similar argument using S as a function of x at fixed x' shows that $S/\delta x$ is $-p$.

(x', z') obeying Fermat's principle. Both A and A' are in a medium of refractive index 1. The ray takes off from A at a small angle p to the z-axis and arrives at A' at an angle p'. The total optical path along this ray, including the factor of refractive index for the portion inside a lens or prism, is denoted by $S(x, x', z, z')$.

The important property of this function S is the following. By differentiating with respect to x', we can find the angle p' and by differentiating with respect to x, we can find the angle p. In symbols,

$$\frac{\partial S}{\partial x'} = p' \ , \ \frac{\partial S}{\partial x} = -p.$$

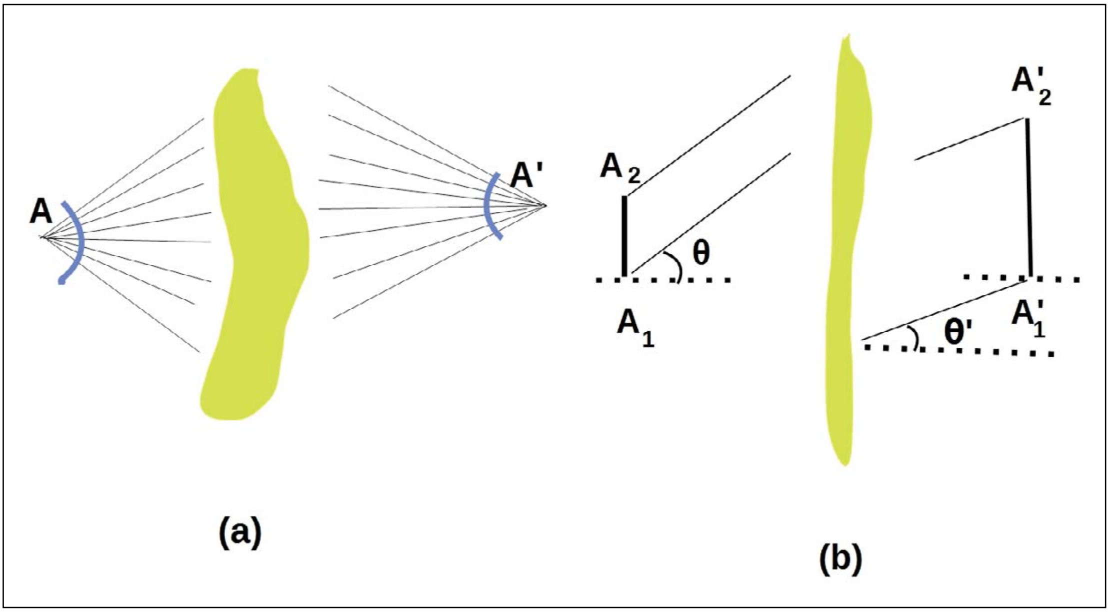
These equations follow from the geometry of rays and wavefronts and are explained in *Figure* 5. These equations are central to Hamilton's thinking and he called S, the characteristic function.

We will bring out some of the consequences of such a general picture below. Let us consider the case of perfect imaging, where all rays from one point on the object converge to the corresponding point on the image. We are not assuming the angles to be small.

Figure 6a gives the pictorial proof that the optical path from a point on the object to the corresponding point on the image is constant for all rays. The wavefronts are clearly expanding spheres near the object A′, and shrinking spheres near the image A. The optical path from one wavefront to another is fixed for all rays. The general statement made earlier that there is a unique ray between A and B does not apply in this exceptional case of imaging. In fact there is a continous family of

***Figure* 6a.** Because the optical path between a small wavefront around A to a small one at B must be fixed, we get the following result in the limit that they shrink to zero. When an image is formed, the optical path has a constant value when calculated along any ray.

***Figure* 6b.** The sine condition for imaging a small object perpendicular to the axis of the optical system. The optical system itself is the coloured patch, and the regions near the object and the image are blown up. Rays leaving at an angle θ and arriving at an angle θ' are sketched. Note that this is a case of non-inverted imaging. The fact that the difference in optical path is the same for all rays leads to the sine condition.



such paths, all of which are stationary. If you try to prove this for a lens using the old fashioned ray optics approach, you will find it takes much more effort and calculation; especially for large angles you will find that it is not at all this obvious.

Another important and useful property of perfect imaging systems follows from the existence of wavefronts, independent of their form. *Figure* 6b shows a small object A_1A_2, perpendicular to the axis, which is imaged to $A'_1A'_2$. We have just seen that all rays from A_1 to A'_1 have equal optical paths, and all rays from A_2 to A'_2 also have equal optical paths. This means that the difference of optical paths between a pair of rays emerging on the object side at an angle θ and reaching the image side at an angle θ' has to be constant, independent of θ

This implies $A_1A_2 \sin\theta - A'_1A'_2 \sin\theta' = \text{constant}$.

We now assume a system symmetric about the z-axis, so that the ray at $\theta = 0$ emerges at $\theta' = 0$, hence this constant is zero. We thus have the strong condition $A_1A_2/A'_1A'_2 = \sin\theta'/\sin\theta$. The sines of the angles have to be in a constant ratio, given by the magnification, for all rays. The paraxial version of this was given by Lagrange earlier.

A microscope objective, or camera lens may image one point perfectly, by suitable design of the refracting surfaces. But to image even two adjacent points perpendicular to the axis perfectly, one needs to satisfy this further condition! This goes by the name of the Abbe sine condition, named after the great optician, Ernst Abbe of Jena in Germany, who is also famous for his formula for the resolving power of a microscope. You can be sure that this condition went into the design of the cameras which all mobile phones have – they need to have large angles to collect enough light and focus it in such a short space!

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There is great interest nowadays in collecting solar energy. One needs to achieve a high temperature to make a solar furnace or a solar steam generator for an electrical power plant. It is then important to concentrate all the collected energy into as small an area as possible. The designers of solar concentrators need to therefore use as large values of $\sin\theta'$as possible to make $A_1'A_2'$ as small as possiible, Advanced ideas of optics based on Hamilton's formulation are very useful in this very practical and important problem.

Epilogue

Hamilton's work is much more general than the simple examples given above. From the start, he allowed for rays in three dimensions, with the speed of light dependent on direction as in a crystal, and possibly on position, as in our mirage problem. He derived a partial differential equation satisfied by S and exhibited the solution in some simple cases – this programme was taken much further by the German mathematician Jacobi (see the article on Hamilton's life and work in this issue). His grand plan was to convert the geometry of earlier work on optics into algebra and analysis –most of his papers do not carry a single figure. In this, he was surely inspired by the *Mecanique Analytique* of Lagrange, which had no figures. He describes that book as a poem – like good poetry, it demands imagination from its readers!

In this, he was surely inspired by the *Mecanique Analytique* of Lagrange, which had no figures. He describes that book as a poem – like good poetry, it demands imagination from its readers!

We have kept to the last the most important outcome of Hamilton's early insights into geometrical optics. He was already aware of a principle similar to that of Fermat. This is known as Maupertuis' principle (1743). It states that the motion of particles of a given energy minimises a quantity called the action, given by $\int v dl$, the integral of the velocity with respect to distance along the path. The velocity at any point on the path is to be calculated using the law of conservation of kinetic plus potential energy. (It was probably Euler who gave first clear statement and applications to planetary motion soon after Maupertuis.) This seems to be contradict Fermat's principle for light which has velocity in the denominator, some people to doubt whether light really travels more slowly in a medium – Newton himself had said it travels faster. This was not setted experimentally till Foucault's direct measurement of the speed of light in water in 1850. Hamilton was unaffected by such issues, since he conceived of these as two different minimum principles, governing two different kinds of systems, and his mathematical development proceeded on similar lines in both cases. He was able to define a function S for the mechanics of particles by the same method of calculating the action as a function of the end points, along the curve on which it is stationary. He thus laid the foundations for what we call 'Hamiltonian mechanics' today, which is now an impressive edifice. His way of looking at mechanics has all but superseded Newton's for any advanced work. For more details, refer to the article on Hamilton's life and work in this issue. Unlike his optics, Hamilton's mechanics is part of the standard advanced physics curriculum and there are several excellent introductions to the subject. We can now explain the strange choice of the symbol p for the angle. The space derivative of the action turns out to be the momentum in particle mechanics, and p is the standard symbol for that.

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Hamilton's formulation of mechanics proved to be extraordinarily useful in the birth of quantum theory in the first quarter of the twentieth century, 1900–1925. The connection to quantum theory is a whole subject in itself, so we can just give a hint as a parting offering to the reader. For this we go back to the underlying reason for the Fermat principle and the Huygens construction. This was brought out by Fresnel in 1816. The number of wavelengths contained in the geometric path is given by $\int \frac{ds}{\lambda_0/n}$, where λ_0 is the wavelength in vacuum and n

Can we interpret the Maupertuis--Euler action $\int v\,dl$ as the number of wavelengths? Yes, after quantum mechanics!

is the refractive index. This is clearly proportional to $\int n dl$, the optical path. When the optical path is stationary for a given ray, it means that a large number of paths close to this ray also have nearly the same value for the optical path. A function varies very slowly around a stationary point with only second order changes. Those paths which differ by less than say half a wavelength interfere constructively, and the others cancel. (This is confirmed by actual calculation!) So paths for which the number of wavelengths is stationary are the most important ones, even in a full wave theory. These are the rays of geometrical optics.

Can we interpret the Maupertuis–Euler action $\int v dl$ as the number of wavelengths? Yes, after quantum mechanics! The de Broglie wavelength of material particles is given by the famous formula $\lambda = h/mv$. When we divide by the wavelength, the velocity comes to the numerator! The quantity being made stationary, in both geometric optics and the classical limit of quantum theory, is the same – the phase. If Hamilton had the least hint of the wave behaviour of matter, maybe he would have written down the corresponding wave equation, which is exactly what Schrodinger did, a hundred years later.

Suggested Reading

[1] The Feynman Lectures on physics, now available free on the caltech website, has an elementary discussion of Fermats principle and the principle of least action, www.feynmanlectures.caltech.edu, Volume 1, chapter 27 and Volume 2, chapter 19

[2] For a modern treatment of conical refraction, see the paper by Berry, Jeffrey and Lunney, in Proceedings of the Royal Society A, volume 462, page 1629 (2006), the link is http://rspa.royalsocietypublishing.org/content/royprsa/462/2070/1629.full.pdf

[3] For Hamiltonian mechanics see the article by N Mukunda in this issue and references therein.

[4] For Dijkstras algorithm and the wavefront interpretation, see the book *Algorithms* by Dasgupta, Papadimitrion and Vazirani.

[5] The 'Raman spot' is described in the paper by Raman, Rajagopalan, and Nedungadi, Proceedings of the Indian Academy of Sciences A volume 14, page 221, (1941) link at http://www-old.ias.ac.in/j_archive/proca/14/3/221-227/viewpage.html

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Box 1. A Brief History of Conical Refraction

When seventeenth century sailors returned from a cold country with reports of seeing double, it would have been easy to infer that this was the result of extra doses of something strong to keep warm. It is thus to the credit of Bartholin in Denmark (1669) that he studied the crystals they brought back, now known as calcite (a form of $CaCO_3$) and discovered a new optical phenomenon. You can see for yourself from the picture in *Figure* A that one sees two images whereas an ordinary glass slab would show one image. The two images are seen at different apparent depths, and hence correspond to different values of the refractive index for the two kinds of rays. One of them is seen to obey the standard law of refraction, and is hence called ordinary. But the other ray truly deserves its name – extraordinary. Even at normal incidence, this ray is refracted away from the normal.

The ordinary ray can of course be understood using the same Huygens construction with a spherical wavelet as in glass or water. It was Huygens' inspiration that the extraordinary ray in calcite could be explained by a wavelet in the form of an oblate spheroid (*Figure* B). The ordinary and extraordinary wavelets touch on one unique axis, called the optic axis . It corresponds to an axis of threefold symmetry of calcite, very visble in the external form of the crystal in *Figure* A. Along this one axis, the ordinary and extraordinary rays travel with the same speed. Hence the name uniaxial for the family of crystals behaving in this way – quartz is another example.

We can think of the Huygens wavelet as the surface made up by the set of points reached in a unit time by rays travelling in different directions. Of course, this shape can be scaled up or down depending on the time interval we are interested in. This is is called the ray surface, depicting ray speed as a function of direction. In calcite, this surface has two sheets, with the extraordinary sheet lying outside the ordinary sheet. This means that in any direction (except the optic axis) the extraordinary ray travels more rapidly; it has a lower refractive

A

B

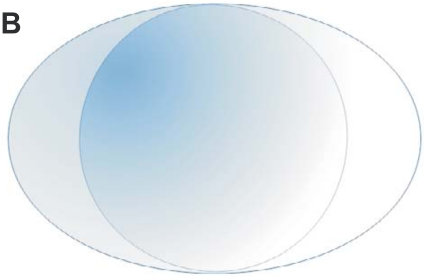


***Figure* A .** A crystal of calcite placed on a picture of butterflies. The doubling of the image is clearly visible. (Courtesy: Raman Research Institute, Photo credit: Sushila Rajagopal)

***Figure* B.** The two sheets of the ray surface for calcite. Ordinary is the sphere and extraordinary is the oblate spheroid touching this inner sphere along the optic axis.

Box 1. Continued...

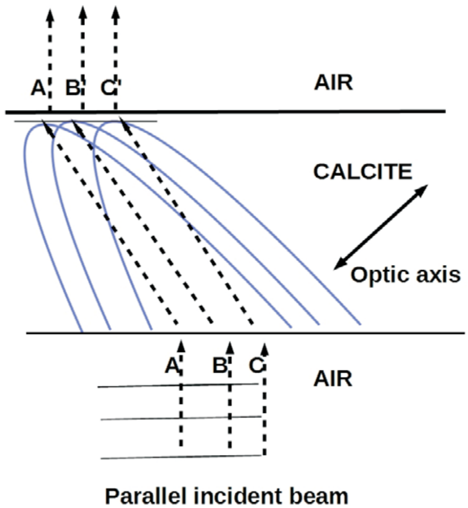
Box 1. Continued...

index. Quartz exhibits the opposite property – the ordinary ray has a higher speed. *Figure* C shows a slab of calcite, cut so that the optic axis is inclined to the normal. The incident light has wavefronts parallel to the crystal face. Huygens' construction for the extraordinary ray shows that it is deviated, even though the wavefront tangent to it remains parallel to the crysal face. Once it emerges from the crystal into air, the ray loses its extraordinary property and emerges along the normal again!

Calcite is not the only crystalline form of $CaCO_3$. Another form, aragonite, is usually found as a mass of individual smaller crystals (see *Figure* D). This crystal has a lower symmetry than calcite. There are still two ray speeds for any direction, and two sheets to the ray surface. Its shape is more complicated and was worked out by Fresnel. The important feature for our immediate purpose is that there are now *two* directions where the the velocities are equal, see *Figure* E. This more general class of crystal is called biaxial, and aragonite is an example.

One might think that finding this surface completed the theory of the propagation of light in biaxial crystals. Since neither sheet of the ray surface is spherical, neither ray obeys Snell's law and the details can be quite complicated.

C



D

***Figure* C.** A parallel beam (plane wavefronts) incident normally on a slab of calcite. Note that the optic axis is not normal to the surface. The extraordinary Huygens wavelets (ellipticity exagerrated) are in blue, and the corresponding rays AA', BB', CC' are shown as dashed arrows. Note that while the wavefront in the crystal is still plane and parallel to the incident wavefront, the rays are inclined.They regain the original direction, with a lateral displacement, once they re-enter air.

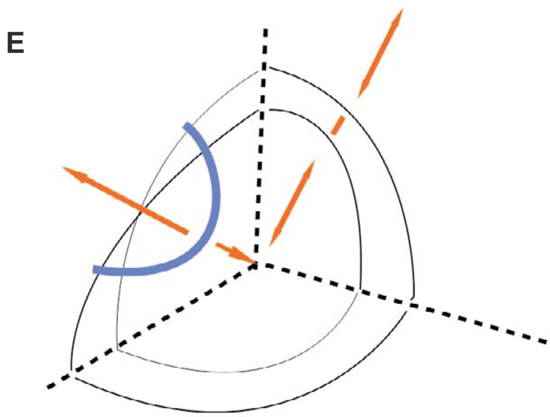
***Figure* D.** Aragonite. (Courtesy John Zander, Wikimedia Commons)
https://upload.wikimedia.org/wikipedia/commons/1/19/Aragonite_Mineral_Macro.JPG

Box 1. Continued...

Box 1. Continued...

Hamilton uncovered a beautiful and new physical phenomenon hidden in this complexity. In a uniaxial crystal, the two sheets of the ray surface touch and have a common tangent plane. The contact between the two sheets at the optic axis is quite different for a biaxial crystal. Along each optic axis, the sheets touch at a single point of contact which is the vertex of a double cone (*Figure* E). In any plane through the optic axis, this would be a pair of lines, interscting along the optic axis. We now focus our attention on propagation near one of the two optic axes. *Figure* F shows the rays and wavefronts in one of the planes through this optic axis.

The figure shows the very special case of propagation in such a crystal which Hamilton discovered. A plane wave incident in just the right direction on the slab comes out as a cone inside the crystal, not just two rays. The figure shows only two because it is restricted to one plane, but there is such a pair in every plane passing through the optic axis. On emerging, these rays become parallel again and form a cylinder. A proper explanation would be far too long for this box. An elementary geometric explanation based on Huygens wavelets is given in *Figure* F. Of course, Hamilton did much better – indeed, he created the mathematics



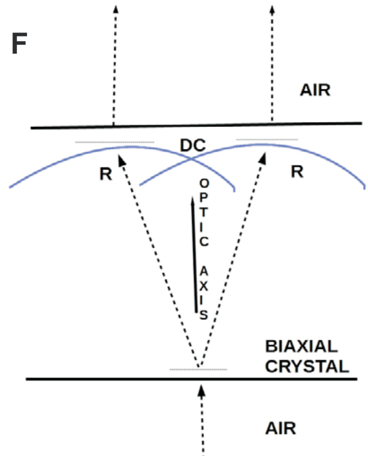


***Figure* E.** One eighth of the biaxial ray surface, in the region where the three co-ordinates are all positive. The rest of the surface can be reconstructed by symmetry. Note that neither sheet is spherical though there are circular sections in each of the x–y, y–z, and z–x planes. The conical meeting of the two sheets along the optic axis (shown in red) is apparent. The circular ring in blue, discovered by Hamilton, is where a common tangent plane would touch the surface.. A general surface would not have such a ring so it is a nontrivial geometric property. This is the ring of rays produced by internal conical refraction – see *Figure* F

Figure F. A schematic view of internal conical refraction. As in *Figure* B, a plane wave is incident normally on a crystal plate. The figure shows a section through the optic axis. Unlike *Figure* B, only one Huygens wavelet is shown in blue, bringing out both the double cone DC and the ring RR surrounding it which a single plane can touch. The common tangent plane appears to have two separate parts in the figure, . However, in three dimensions, these are opposite sides of a single annular wavefront (see *Figure* E). These correspond to a continuous cone of rays, which emerge on the other side of the slab as a cylinder.

Box 1. Continued...

Box 1. Continued...

needed. The crucial step was to introduce a new surface, At each point on the ray surface, a vector is constructed normal to it. This will naturally be perpendicular to any plane wavefront tangent to this wavelet at this point. The length of this vector is chosen to be inversely proportional to the speed of advance of that plane wavefront. He called the set of all such vectors the 'surface of normal slowness'. This is precisely the definition of the wave vector used in condensed matter, fluid and plasma physics. Hamilton gave us k-space!

This novel mathematics would not have convinced the larger community of scientists – the prediction needed to be verified.

By remarkable good fortune Hamilton had just the right contemporary, in Trinity College, Dublin. Lloyd, five years his senior, was a fine experimenter, and is well known to students of optics for his mirror version of the Young experiment, which gives two slits for the price of one. It was a struggle to find even a partially transparent crystal of aragonite, and a clear spot on that crystal, and then to search for the right incident direction. The angle of the cone in aragonite is very small, about one degree, which required a fine beam, just 0.2 millimetres wide. Lloyd overcame these obstacles and was able to exhibit the predicted behaviour, which is called 'internal conical refraction', within just two months. The theory and experiment were presented trimuphantly at a meeting of the British Association for the Advancement of Science in 1833 and made an instant impact.

Many interesting developments followed. Internal conical refraction corresponds to one plane wavefront, many rays. Hamilton predicted a dual phenomenon, external conical refraction. Two pinholes are placed to ensure that there is just one ray along the optic axis inside the crystal. *Figure* E shows that exactly at the vertex of the double cone, there are an infinite number of normals, themselves forming a cone. This means many possible wavefronts, all tangent to the ray surface, and all travelling along the optic axis. These wavefronts have different normals but travel together so long as they are in the crystal. On being liberated at the exit, they go their separate ways and now form a diverging cone. To see this phenomenon, called external conical refraction, one has to supply all these wavefronts at the input of the crystal. This calls for a beam focused at the entrance pinhole. Lloyd was able to demonstrate this as well. Poggendorf (1839) observed that the ring had a dark gap in the middle. This was not explained till the work of Voigt (1905) which gave a deeper analysis of the intensity distribution far away from the crystal, keeping in mind that we always deal with rays occupying a finite solid angle. In 1941, Raman, Nedungadi, and Rajagopalan, working at the Indian Institute of Science, Bengaluru, found a new phenomenon in the region near the crystal, which they explored with a microscope! When the incoming beam was made really narrow, a bright spot appeared inside the ring. This effect could even be used to image a small object behind the crystal. But all this is another story – see Suggested Reading. Conical refraction is visually and mathematically beautiful. It is a great lesson in the interplay between rays and waves which has much wider implications – for example, in the semiclassical limit of quantum mechanics.