Finite Amplitude Ocean Waves

Waves with Peaked Crests and Broad Troughs

K K Varma

When amplitudes are small, ocean waves are represented by sinusoidal functions. There are, however, situations when this treatment of waves is inadequate. In such situations, finite amplitude wave theories that deal with waves having peaked crest have to be used. In this article, selected features of a variety of such wave profiles are presented.

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1. Introduction

Waves are the undulations of the sea surface. The most commonly observed waves on ocean surface are those generated by wind forcing. In the beginning, small ripples appear on the sea surface and these grow further by extracting energy from prevailing winds. There are different ways in which waves are classified and one of them is based on its relative amplitude, as small amplitude waves and finite amplitude waves. This article provides a brief introduction to finite amplitude wave theories. Some of the general characteristics of waves as well as the importance of finite amplitude wave theories are touched upon.

2. Small Amplitude Waves

The topmost and the lowest levels of the waves are respectively called the crest and trough of the wave. The horizontal distance between successive identical points, say two crests, is the wavelength (L). Wave height (H) is the vertical distance between crest and trough. The vertical distance from mean position to crest or to trough is the amplitude (a), which in this case is half of wave

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As the waves progress, the water particle motion is either circular or elliptical.

height. The surface profile is given by

$$\eta = \mathrm{a}\sin(kx - \sigma t)\,\text{or,}$$

if there is phase shift by $\frac{\pi}{2}$, then,

$$\eta = \mathrm{a}\ \cos(kx - \sigma t)\,.$$

Here η is the departure from undisturbed level, a is the amplitude, k is wave number ($1/L$ or $2\pi/L$, L is wavelength), and σ is frequency ($1/T$ or $2\pi/T$, T is the period). The wave speed is given by L/T. This is an example of a sinusoidal wave. In the context of waves on the surface of fluid, these waves (also known as airy waves) are based on Bernoulli's equation for irrotational fluid motion

$$-\frac{\partial\varphi}{\partial t} + \frac{P}{\rho} + \frac{1}{2}(u^2 + w^2) + gz = 0\ , \qquad (1)$$

where φ is the potential function of flow, p is pressure, ρ is the density of fluid, u and w are the horizontal and vertical velocities, respectively, g is the acceleration due to gravity, and z is the depth. The higher order terms of velocities can be neglected for small amplitude waves and then the equation becomes

$$-\frac{\partial\varphi}{\partial t} + \frac{P}{\rho} + gz = 0 \qquad (2)$$

Hence, small amplitude waves are also called linear waves. Most of the aspects of the ocean waves can be explained by the small amplitude wave theory.

Let us now see the water particle motion due to waves. While wave energy is carried by the wave as it progresses forward, the water particles oscillate up and down. However, it is not merely an up and down movement. It is either circular or elliptical movement. If the depth of the water column is more than half of wavelength, then waves are known as deep water waves. In the case of such waves, particle motion is circular. On the other hand, if the depth of the water column is less than half of wavelength and more than 1/20 of wave length, they

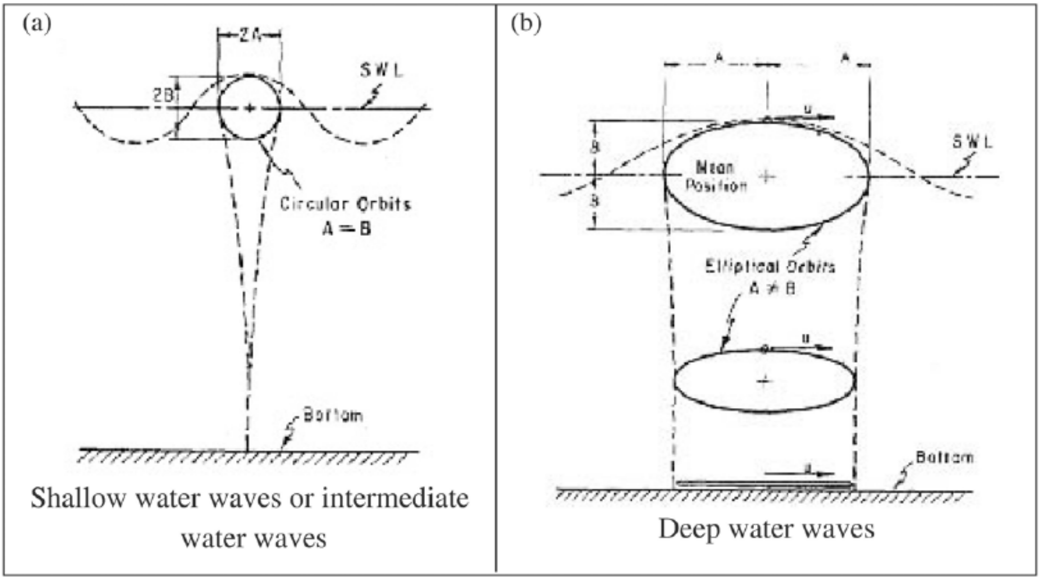

Figure 1. Motion of particles in small amplitude waves. (adapted from [1]). Radius of the circular motion of the deep water waves (a) decreases exponentially with depth. In (b) the ellipse becomes flatter as depth increases, and at the sea bottom, the movement is to and fro.

are known as intermediate or transitional water waves and if the depth of the water column is less than 1/20 of wavelength, they are called shallow water waves. In the case of both these waves, the particle motion is elliptical. Particle motions are shown in *Figure* 1.

The velocity of waves is generally referred to as wave celerity. For small amplitude waves, celerity, c is given by

$$c = \sqrt{\frac{g}{k} \tanh kh} \quad (3)$$

k is the wave number and h is the depth of the water column.

This equation gets simplified to $c = \sqrt{\frac{g}{k}}$ for deep water waves because tanh kh can be approximated as 1 for large h. For shallow water waves, celerity becomes $c = \sqrt{gh}$ as the hyperbolic term tends to kh and for intermediate water waves, (3) is to be used. Deep and intermediate water waves are dispersive as the velocity of these depends on wavelength. This is not the case with shallow water waves and they are nondispersive.

In the case of finite amplitude waves, the amplitude cannot be considered as small in comparison with wave length or water depth.

3. Finite Amplitude Waves

The basic feature of the finite amplitude waves is that the amplitude cannot be considered as small in comparison with either the wavelength or the depth of water column. Therefore, the higher order terms in (1) cannot be neglected and thus, they are also known as nonlinear waves. A consequence of considering the full form of (1) is that the wave profiles are not sinusoidal.

One of the main characteristics of the finite amplitude waves is that they do not have sinusoidal shape and instead have peaked crests and broad troughs. Also, the upward amplitude, i.e., towards the crest is more than the downward amplitude, i.e., towards the trough. The retention of nonlinear terms makes the solution of the equation difficult. As a result of this, there is no unique theory that is applicable to all depth regions from deep to very shallow water. It must be noted that in the case of small amplitude waves, a single theory is applicable irrespective of water depth. On the other hand, different finite amplitude wave types are applicable to waves at different depths. For deep water locations, Stokes wave or trochoidal wave is applicable. If water depth is less, it is Cnoidal wave; and in very shallow waters, solitary wave is found to be suitable (see *Table* 1). Another aspect is that different wave parameters gain importance in different depth regions. If the depth is more than half wavelength, then the important parameters are wavelength and wave steepness, given by H/L. In regions

Table 1. Applicability of different finite ampliude wave theories.
L, H and h are wavelength, wave height and water depth respectively.
* It must be noted that the depth limits are not exact and theories will overlap at the fringe regions.

Water depth in relation to wave length	Theories
$h > L/10$	Stokes and trochoidal
$L/10 > h > L/50$	Cnoidal
$h < L/50$	Solitary

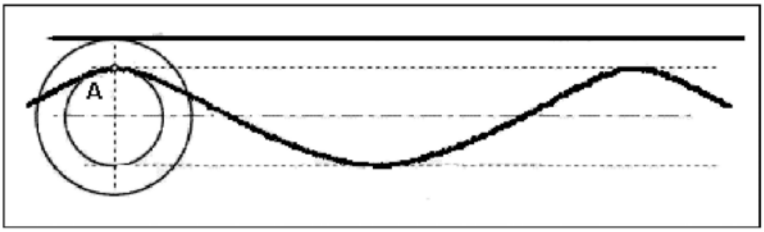

Figure 2. Trochoidal wave (adapted from [2]).The small circle is part of the large circle that rolls below the plane. The path traced by the point A has broad troughs and peaked crests.

with depth between half and 1/20 of wavelength, H/L and water depth, h are important, and in still shallower regions, the depth of water and H/h become important parameters. Some basic features of these three types of finite amplitude waves are now discussed.

3.1 *Trochoidal Wave*

This type of wave derives its the name from its shape. Trochoid is the shape traced by a point on a circle as the circle rolls over or under a surface (*Figure* 2). In the figure, the rolling circle and the point on it are shown. If the distance of the point from the centre of the circle decreases, the shape tends to be sinusoidal, while if it increases to the radius of the circle, the crest becomes too pointed. It may be noted that these waves are also called Gerstner waves (after F Gerstner who propounded the theory behind these waves).

The height of the crest is $\frac{H}{2} + \frac{\pi H^2}{4L_0}$ and the depth of trough is $\frac{H}{2} - \frac{\pi H^2}{4L_0}$, where L_0 is deep water wavelength and H is wave height. From the above, it is clear that the vertical distance from the mean level to the crest is more than half wave height, but the vertical distance from mean level to trough is less than half wave height. Though this theory gives an exact solution, it does not satisfy the irrotationality condition.

3.2 *Stokes Wave*

This wave has a broad trough and somewhat peaked crest (*Figure* 3a). Stokes presented a method for the solution of the above nonlinear equation of velocity

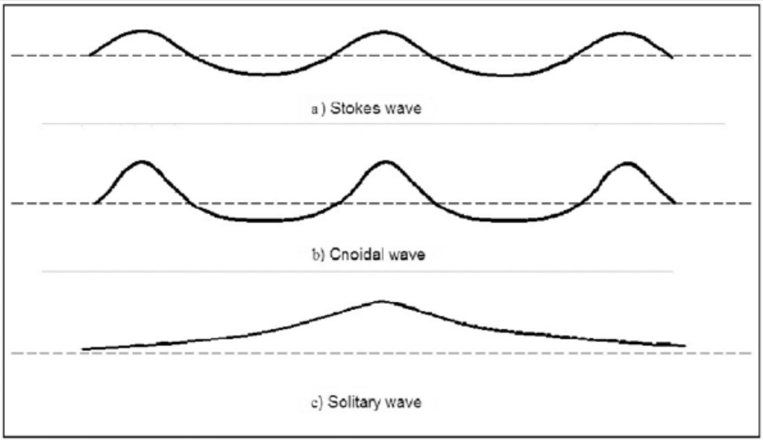

Figure 3. Different finite amplitude wave profiles (adapted from [1]).

potential. Hence, these are known as Stokes waves. The equation is solved as a series, using perturbation method. It is assumed that the solution can be represented in terms of a power series expansion of a small quantity and this quantity is related to wave steepness, which is the ratio of wave height to wavelength. The sum up to the nth order term is the nth order solution. Solution is, therefore, not unique and as the order increases, the resulting wave becomes more peaked. The solution satisfies the irrotationality condition. It may be noted that the method requires a number of numerical calculations, which increases as the order of the solution increases. Though Stokes presented the second order solution, subsequent researchers have worked out higher order solutions.

The first order solution gives results for wave profile and wave celerity that are similar to those for small amplitude waves. In the case of second order solution, though the expression for wave celerity does not change, the wave profile is complicated and is given by

$$\eta = \frac{H}{2}\cos\left(kx - \sigma t\right)$$

$$+\frac{\pi}{8}\frac{H^2}{L}\frac{\cosh kh(2 + \cosh 2kh)}{(\sinh kh)^3}\cos 2\left(kx - \sigma t\right). \quad (4)$$

3.3.1 *Stokes Drift*: In the case of Stokes waves, there is a net transport of water as the wave passes. At this stage, it is necessary to consider the particle motion as presented earlier for small amplitude waves, i.e., the water particles return to their original position at the end of the passage of a sinusoidal wave. In other words, particle motion is in closed circles in deep water waves. However, on examining the horizontal and vertical components of particle motion for Stokes wave, it can be seen that the horizontal component has a non-periodic term. Hence, the circle is not a closed one and there is net transport of water in the direction of progress of wave. This is known as Stokes drift. One of the expressions for Stokes drift presented by Weigal [3] is

There will be a net transport of water due to the Stokes wave.

$$u = \left(\frac{\pi H}{T}\right)\left(\frac{\pi H}{L}\right) \mathrm{e}^{\frac{4\pi Z}{L}}\,, \tag{5}$$

where, H, T and L are height, period and length of the wave, respectively; and z is the depth at which drift is estimated. The depth axis is positive upwards. Thus, the drift decreases rapidly with the depth. It can be said that the Stokes drift at surface is about 1% of the wind speed that generates the waves.

3.3 *Cnoidal Wave*

If the depth decreases, the theories mentioned earlier are not the appropriate ones to be used. It is the cnoidal wave theory that is more applicable. In this theory, Jacobian elliptical function (cn) is used. Hence, this is called as cnoidal, a word analogous to sinusoidal. Jacobian elliptical function is periodic and its modulus lies between 0 and 1, both inclusive. The shape of the cnoidal wave is characterized by very flat troughs and peaked crests (*Figure* 3b). The wave profile is given by

$$\eta = z_t + H\mathrm{cn}^2[2K(k)\left(\frac{2\pi}{L}x - \frac{2\pi}{T}t\right), k]\,. \tag{6}$$

Here, cn is the Jacobian elliptical function, $K(k)$ is the elliptical integral of first order with modulus k, z_t is the

Solitary wave is a single wave crest alone, that translates forward.

height of trough from sea bottom and H is the wave height. When the modulus k is zero, the wave profile becomes sinusoidal. At the other extreme, i.e., when k is 1, the period will be ∞. However, a small reduction in k gives finite period (for example, if $k = 0.9999$, period becomes 7π).

3.4 *Solitary Wave*

For even shallower conditions, the limiting case of infinite wavelength is approached. This is the limiting case of cnoidal wave mentioned earlier and then cnoidal wave becomes a solitary wave. All the waves that have been considered earlier are oscillatory in nature. Solitary wave, on the other hand, is translatory. The profile of this consists of only a crest and the entire wave will be above mean level (*Figure* 3c). Thus, the wave translates forward. The wave profile is

$$\eta = h + \operatorname{sech}^2\left[\sqrt{\frac{3}{4}\frac{H}{h^3}}(x - ct)\right] . \tag{7}$$

Here, h is the depth of the water column and c is the velocity of propagation.

This kind of wave was first observed by John Scott Russell in 1834. Russell was an engineer and naval architect. On observing a boat being pulled rapidly by horses through a narrow channel, Scott saw that the water that accumulated just in front of the boat rose as a single wave crest and moved forward without change of shape or reduction of speed. He called it a 'wave of translation'. This chance observation occurred in a channel in Edinburgh and was subsequently verified by studies in a wave tank.

4. Further Development of Theories

Stokes and cnoidal wave theories were propounded around the middle and second half of 19th century. Though solutions of orders higher than the second give more

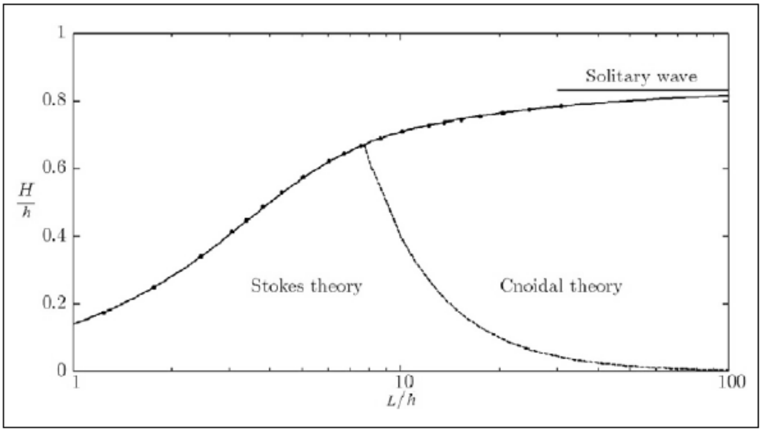

Figure 4. The regions of possible wave theories.
Adapted from [5].
Courtesy: J D Fenton.

accurate results, these involve cumbersome calculations. However, in the second half of the 20th century, several studies on theoretical and laboratory aspects of higher order waves have been carried out. One of the important developments is the Fourier method of solution. In this method, the full nonlinear equation is solved. The Fourier coefficients are obtained numerically with the help of a computer program. The main advantage is that the entire depth region from deep water to very shallow water is covered in the solution. The regions of applicability of different theories can be delineated with better accuracy (see *Figure* 4). In the figure, the closed circles are from Williams [4] and the curve is fitted by Fenton [5]. The fitted curve in this figure indicates the maximum H/h (wave height/water depth) ratio, beyond which the wave breaks. Wave breaking occurs when the wave becomes unstable and the crest tumbles forward. In the deep water region, h is large and therefore L/h ratio is small. Then the wave breaks as the ratio of wave height to wavelength rises to 0.141. At the left-hand end of the figure, the fitted curve corresponds to this situation. On the other hand, as the depth decreases, wave-breaking limit is reached according to the ratio H/h, which increases with decrease in depth. For solitary wave (at the right-hand end of figure), the limiting value of this ratio is 0.83.

5. Why are Finite Amplitude Waves Important?

The study of finite amplitude waves is important as these theories have different applications. One aspect is the waves in the generating area. As mentioned earlier, the wind waves are generated by the transfer of energy of winds blowing over the sea. As the waves grow in size, the gravity tries to restore the sea surface back to normal condition. Gravity being the restoring force, these are known as gravity waves. The small amplitude and finite amplitude waves considered here are gravity waves. At the regions of wave generation, waves of different characteristics will be present simultaneously giving a chaotic appearance to the sea surface. Finite amplitude waves, such as the higher order Stokes waves, which break when they become unstable, will also be present here. This occurs when the breaking condition for deep water waves, mentioned earlier, is reached and then white caps appear on the sea surface. Sharp-crested waves that are always present in the wave-generating area during strong winds can be seen in *Figure* 5. In order to understand the transfer mechanism of the energy of wind to waves and between different waves and also to delineate the wave generation processes, the finite amplitude wave theories are very important.

Figure 5. Waves in the generating area (NOAA).
Source:Commons.wikimedia.org/wiki/file/00810.jpg.

Another aspect is the designing of marine structures. These structures are necessary for extraction of oil and natural gas, minerals, deep sea nodules, etc. Similarly, harnessing wave energy or for harvesting wind energy, by establishing wind mills in coastal waters also requires the construction of suitable structures. Use of the appropriate nonlinear wave theory depending on the area of application, gives better estimates of wave forcing, wave breaking, etc. Wave-induced forcing needs to be estimated according the appropriate wave theory for the reliable design of marine structures. Using wave data and information on water depth, diagrams such as *Figure* 4 can be used to decide which wave theory is appropriate.

6. Conclusions

Finite amplitude waves, which are also known as nonlinear waves, have peaked crests and flat troughs. These types of waves are also present on the sea surface, along with linear small amplitude waves. Depending on the relative depth of the water column, different finite amplitude wave theories become suitable.

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Suggested Reading

[1] U S Army Corps of Engineers, *Coastal Engineering Manual*, Part II, Chapter 1. pp.II-1.1-1.121, 2008.

[2] W Bascom, Waves and Beaches, *The dynamics of the ocean surface*, Doubleday, Garden City, N Y, 268 p.196.

[3] R L Wiegel, *Oceanographical Engineering*, Prentice Hall Inc. pp.11–132, 1964.

[4] J M Williams, Limiting gravity waves in water of finite depth, *Phil. Trans. Roy. Soc. London*, Vol.A31, pp.139–188, 1981.

[5] J D Fenton, Nonlinear wave Theories, *The Sea*, Vol.9, Ocean Engineering Science, Eds. B Le Méhauté and D M Hanes, New York, Wiley, pp.1–17, 1990.

[6] R J Dean and P A Eeagleson, Finite amplitude waves, *Estuary and coastline hydrodynamics*, Ed. A T Ippen Mc Graw-Hill book company, pp.93–132, 1966.

[7] B Kinsman, *Wind Waves, their generation and propagation on ocean surface*, Prentice Hall Inc, NJ, p. 675, 1965.

[8] J D Fenton, Coastal and ocean engineering http://johndfenton.com/Lectures/Coastal-and-Ocean-Engineering/Coastal-and-Ocean.pdf

[9] J D Fenton and M M Rienecker, A Fourier method for solving nonlinear water--wave problems: application to solitary-wave interactions, *J. Fluid Mech*., Vol.118, pp.411–443, 1982.