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On some explicit limiting distributions related to free multiplicative law of large numbers

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Abstract. Suppose $\mathbb{V}_\nu(\cdot)$ is the pseudo-variance function of the Cauchy–Stieltjes Kernel (CSK) family $\mathcal{K}_-(\nu)$ generated by a non degenerate probability measure ν on the positive real line. Denote by $\Phi(\nu)$ the law of large numbers for free multiplicative convolution given in [17]. An explicit expression of $\Phi(\nu)$ is given in [14] in terms of the pseudo-variance function $\mathbb{V}_\nu(.)$. In this paper, we give explicitly in terms of the pseudo-variance function $\mathbb{V}_\nu(.)$ the limiting distributions $\Phi(\nu^{\boxtimes t})$, $\Phi((D_{1/s}(\nu^{\boxplus s}))^{\boxtimes t})$, $\Phi((D_{1/r}(\nu^{\uplus r}))^{\boxtimes t})$ and $\Phi((\mathbb{B}_r(\nu))^{\boxtimes t})$ for $s > 1$, $t > 1$ and $r > 0$, where $\mathbb{B}_r(\nu) = (\nu^{\boxplus 1+r})^{\uplus \frac{1}{1+r}}$ and $D_c(\nu)$ denotes the dilation of measure ν by a number $c \neq 0$. Some examples of calculations of $\Phi(\nu^{\boxtimes t})$, $\Phi((D_{1/s}(\nu^{\boxplus s}))^{\boxtimes t})$, $\Phi((D_{1/r}(\nu^{\uplus r}))^{\boxtimes t})$ and $\Phi((\mathbb{B}_r(\nu))^{\boxtimes t})$ are given for probability measures ν of importance in noncommutative probability.

Keywords. Variance function; Cauchy–Stieltjes transform; probability measure.

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1. Introduction

The law of large numbers is well-known as a result that a sample average of independent identically distributed random variables with finite mean concentrates on the theoretical mean when a sample size is sufficiently large. As the analogous result on classical probability, the law of large numbers for free random variables was also established (see [18]). More precisely, for any probability measure μ on the real line, with mean a, we have $D_{1/n}(\mu^{\boxplus n}) \xrightarrow{w} \delta_a$ as $n \to +\infty$, where notation $\xrightarrow{w}$ means the weak convergence of sequences of probability measures, $D_c(\mu)$ is the push-forward of a measure μ by the mapping $x \longmapsto cx$ for $c \in \mathbb{R}$ and $\mu \boxplus \nu$ is called the free additive convolution. It represents the probability distribution of addition of free random variables X and Y distributed as probability measures μ and ν respectively. In particular, $\mu^{\boxplus n}$ is the n-th power of free additive convolution of μ.

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The law of large numbers for multiplication of (classically or freely) independent positive random variables are also considered. In classical probability, it is easy to formulate and investigate the law of large numbers of multiplication by considering the exponential mapping of those random variables. On the other hand, the non-commutativity of random variables appears more clearly in the multiplicative law of large numbers. The limit probability measure for the free multiplicative law of large numbers was proved by Tucci [21] for probability measures with bounded support. Haagerup and Möller [17] extended Tucci's result to probability measures with unbounded support and at the same time they gave a more elementary proof for the case of probability measures with bounded support. In contrast to the classical multiplicative convolution case, the limit measure for the free multiplicative law of large numbers is not a Dirac measure, unless the original measure is a Dirac measure. More precisely we have the following (see [17, Theorem 2]).

Theorem 1.1. *Let ν be a probability measure on $[0,\infty)$ and let $\phi_n : [0,\infty) \longrightarrow [0,\infty)$ be the map $\phi_n(x) = x^{1/n}$. Set $\alpha = \nu(\{0\})$. If we denote*

$$\mu_n = \phi_n(\underbrace{\nu \boxtimes \cdots \boxtimes \nu}_{n\ times}),$$

then μ_n converge weakly to a probability measure denoted by $\Phi(\nu)$ on $[0,\infty)$. If ν is a Dirac measure on $[0,\infty)$, then $\Phi(\nu) = \nu$. Otherwise, $\Phi(\nu)$ is the unique probability measure on $[0,\infty)$ characterized by

$$\Phi(\nu)\left(\left[0, \frac{1}{S_\nu(x-1)}\right]\right) = x \tag{1.1}$$

for all $x \in (\alpha, 1)$ and $\Phi(\nu)(\{0\}) = \alpha$. The support of the measure $\Phi(\nu)$ is the closure of the interval

$$(a,b) = \left(\left(\int_0^\infty x^{-1}\nu(dx)\right)^{-1}, \int_0^\infty x\nu(dx)\right),$$

where $0 \le a < b \le \infty$.

Here, S_ν is the S-transform of ν, and $\boxtimes$ is the free multiplicative convolution which will be introduced in the next section (see [4] or [17] for more details).

On the other hand, in the framework of free probability and in analogy with the theory of natural exponential families, a theory of Cauchy–Stieltjes Kernel (CSK) families has been recently introduced. It is based on the Cauchy–Stieltjes kernel $1/(1-\theta x)$. For instance, the study of CSK families has been initiated in [5] for compactly supported probability measures. In [6], Bryc and Hassairi have extended the results established in [5] to allow probability measures with unbounded support. Many properties and characterizations of CSK families are also given in [7–9,12,13,15,16]. The results in [14] indicates the usefulness of CSK families for free probability. Theorem 3.1 given in [14] provides an interesting description of the free multiplicative law of large numbers $\Phi(\nu)$ in terms of the pseudo-variance function $\mathbb{V}_\nu$ of the CSK family generated by ν (see the next section for CSK families and the corresponding pseudo-variance functions). This description yields a number of explicit examples.

In this paper, we are interested in finding explicitly in terms of the pseudo-variance function $\mathbb{V}_\nu(.)$ the limiting distributions $\Phi(\nu^{\boxtimes t})$, $\Phi((D_{1/s}(\nu^{\boxplus s}))^{\boxtimes t})$, $\Phi((D_{1/r}(\nu^{\uplus r}))^{\boxtimes t})$ and $\Phi((\mathbb{B}_r(\nu))^{\boxtimes t})$ for $s>1$, $t>1$ and $r>0$, where $\mathbb{B}_r(\nu)=(\nu^{\boxplus 1+r})^{\uplus \frac{1}{1+r}}$, $\boxplus$ is the free additive convolution and $\uplus$ is the Boolean additive convolution (see the next section for these convolutions). Some examples of calculations of $\Phi(\nu^{\boxtimes t})$, $\Phi((D_{1/s}(\nu^{\boxplus s}))^{\boxtimes t})$, $\Phi((D_{1/r}(\nu^{\uplus r}))^{\boxtimes t})$ and $\Phi((\mathbb{B}_r(\nu))^{\boxtimes t})$ are given for probability measures ν of importance in non-commutative probability. Section 2 describes CSK families and their associated pseudo-variance function as a background, for the reader. Also some preliminaries about free additive convolution, Boolean additive convolution and free multiplicative convolution are given. Section 3 is devoted to the main result of the paper and Section 4 contains some examples.

2. Preliminaries and notations

2.1 *Cauchy–Stieltjes kernel families*

Here we recall a few features about CSK families. Our notations are the ones used in [10]. Let ν be a non-degenerate probability measure with support bounded from above. Then

$$M_\nu(\theta)=\int\frac{1}{1-\theta x}\nu(dx) \tag{2.1}$$

is defined for all $\theta\in[0,\theta_+(\nu))$ with $1/\theta_+(\nu)=\max\{0,\sup\operatorname{supp}(\nu)\}$. For $\theta\in[0,\theta_+(\nu))$, consider

$$P_{(\theta,\nu)}(dx)=\frac{1}{M_\nu(\theta)(1-\theta x)}\nu(dx).$$

The set of probability measures

$$\mathcal{K}_+(\nu)=\{P_{(\theta,\nu)}(dx);\,\theta\in(0,\theta_+(\nu))\}$$

is called the one-sided CSK family generated by ν.

Denote $k_\nu(\theta)=\int xP_{(\theta,\nu)}(dx)$ the mean of $P_{(\theta,\nu)}$. According to [6, pp. 579–580], the map $\theta\mapsto k_\nu(\theta)$ is strictly increasing on $(0,\theta_+(\nu))$. It is given by the formula

$$k_\nu(\theta)=\frac{M_\nu(\theta)-1}{\theta M_\nu(\theta)}. \tag{2.2}$$

The image of $(0,\theta_+)$ by the function $k_\nu(.)$ is denoted by $(m_0(\nu),m_+(\nu))$ and it is called the (one-sided) domain of means of the family $\mathcal{K}_+(\nu)$. This provides a new parametrization of the family $\mathcal{K}_+(\nu)$ by the mean. In fact, let ψ_ν be the inverse of k_ν, and writing for $m\in(m_0(\nu),m_+(\nu))$, $Q_{(m,\nu)}(dx)=P_{(\psi_\nu(m),\nu)}(dx)$, we obtain

$$\mathcal{K}_+(\nu)=\{Q_{(m,\nu)}(dx);\,m\in(m_0(\nu),m_+(\nu))\}.$$

Now let

$$B=B(\nu)=\max\{0,\sup\operatorname{supp}(\nu)\}=1/\theta_+(\nu)\in[0,\infty). \tag{2.3}$$

Then it is proved in [6] that the bounds $m_0(\nu)$ and $m_+(\nu)$ of the one-sided domain of means $(m_0(\nu),m_+(\nu))$ are given by

$$m_0(\nu)=\lim_{\theta\to0^+}k_\nu(\theta)\quad\text{and}\quad m_+(\nu)=B-\lim_{z\to B^+}\frac{1}{G_\nu(z)}, \tag{2.4}$$

with $B = B(\nu)$, and $G_\nu(z)$ is the Cauchy transform of ν given by

$$G_\nu(z) = \int \frac{1}{z-x}\nu(dx). \tag{2.5}$$

for $z \in \mathbb{C}^+ = \{x + iy \in \mathbb{C};\ y > 0\}$.

Note that one may define the one-sided CSK family for a measure ν with support bounded from below. This family is usually denoted by $\mathcal{K}_-(\nu)$ and parameterized by θ such that $\theta_- < \theta < 0$, where θ_- is either $1/A(\nu)$ or $-\infty$ with $A = A(\nu) = \min\{0, \inf \operatorname{supp}(\nu)\}$. The domain of means for $\mathcal{K}_-(\nu)$ is the interval $(m_-(\nu), m_0(\nu))$ with $m_-(\nu) = A - 1/G_\nu(A)$.

If ν has compact support, the natural domain for the parameter θ of the two-sided CSK family $\mathcal{K}(\nu) = \mathcal{K}_+(\nu) \cup \mathcal{K}_-(\nu) \cup \{\nu\}$ is $\theta_- < \theta < \theta_+$.

We come now to the notions of variance and pseudo-variance functions. The variance function

$$m \mapsto \mathcal{V}_\nu(m) = \int (x-m)^2 Q_{(m,\nu)}(dx) \tag{2.6}$$

is a fundamental concept in the theory of CSK families as presented in [5]. Unfortunately, if the measure ν does not have a first moment which is, for example, the case for a free 1/2-stable law, then all probability measures in the CSK family generated by ν have infinite variance. This fact has led Bryc and Hassairi [6] to introduce a notion of pseudo-variance function $\mathbb{V}_\nu(\cdot)$ defined by

$$\mathbb{V}_\nu(m) = m\left(\frac{1}{\psi_\nu(m)} - m\right). \tag{2.7}$$

If $m_0(\nu) = \int x d\nu$ is finite, then (see [6]) the pseudo-variance function is related to the variance function by

$$\mathbb{V}_\nu(m) = \frac{m}{m - m_0}\mathcal{V}_\nu(m). \tag{2.8}$$

In particular, $\mathbb{V}_\nu(.) = \mathcal{V}_\nu(.)$ when $m_0(\nu) = 0$.

The generating measure ν is uniquely determined by the pseudo-variance function $\mathbb{V}_\nu$. In fact, if we set

$$z = z(m) = m + \frac{\mathbb{V}_\nu(m)}{m}, \tag{2.9}$$

then the Cauchy transform satisfies

$$G_\nu(z) = \frac{m}{\mathbb{V}_\nu(m)}. \tag{2.10}$$

We now recall the effect on a CSK family of applying an affine transformation to the generating probability measure. Consider the affine transformation $\varphi : x \longmapsto (x - \epsilon)/\sigma$, where $\sigma \neq 0$ and $\epsilon \in \mathbb{R}$ and let $\varphi(\nu)$ be the image of ν by φ. In other words, if X is a random variable with law ν, then $\varphi(\nu)$ is the law of $(X - \epsilon)/\sigma$, or $\varphi(\nu) = D_{1/\sigma}(\nu \boxplus \delta_{-\epsilon})$, where $D_r(\mu)$ denotes the dilation of measure μ by a number $r \neq 0$, that is, $D_r(\mu)(U) = \mu(U/r)$. The point m_0 is transformed to $(m_0 - \epsilon)/\sigma$. In particular, if $\sigma < 0$, the support of the measure $\varphi(\nu)$ is bounded from below so that it generates the left-sided family $\mathcal{K}_-(\varphi(\nu))$. For m close enough to $(m_0 - \epsilon)/\sigma$, the pseudo-variance function is

$$\mathbb{V}_{\varphi(\nu)}(m) = \frac{m}{\sigma(m\sigma + \epsilon)}\mathbb{V}_\nu(\sigma m + \epsilon). \tag{2.11}$$

In particular, if the variance function exists, then $\mathcal{V}_{\varphi(\nu)}(m) = \frac{1}{\sigma^2}\mathcal{V}_\nu(\sigma m + \epsilon)$.

2.2 *Free additive convolution*

Denote by $\mathcal{M}$ (respectively by $\mathcal{M}_+$) the set of Borel probability measures on $\mathbb{R}$ (respectively on $\mathbb{R}_+$). For $\nu \in \mathcal{M}$, the free cumulant transform $\mathcal{R}_\nu$ is a function analytic in a neighborhood of zero (see [3]) and it is defined by

$$\frac{1}{G_\nu(z)} = z - \mathcal{R}_\nu(G_\nu(z)). \tag{2.12}$$

The free additive convolution of the probability measures μ, $\nu \in \mathcal{M}$ is a uniquely defined probability measure $\mu \boxplus \nu$ such that

$$\mathcal{R}_{\mu\boxplus\nu}(z) = \mathcal{R}_\mu(z) + \mathcal{R}_\nu(z). \tag{2.13}$$

A probability measure $\nu \in \mathcal{M}$ is $\boxplus$-infinitely divisible, if for each $n \in \mathbb{N}$, there exists $\nu_n \in \mathcal{M}$ such that

$$\nu = \underbrace{\nu_n \boxplus \cdots \boxplus \nu_n}_{n \text{ times}}.$$

Let $\nu^{\boxplus t}$ denote the t-fold free additive convolution of ν with itself. In contrast to classical convolution, this operation is well-defined for all real $t \geq 1$ (see [19]), and we have

$$\mathcal{R}_{\nu^{\boxplus t}}(z) = t\mathcal{R}_\nu(z). \tag{2.14}$$

The probability measure ν is $\boxplus$-infinitely divisible if its free additive convolution power $\nu^{\boxplus t}$ is well-defined for all real $t > 0$.

Our interest in the free additive convolution power stems for two important properties: its effect on the pseudo-variance function and the subordination function. The action of the free additive convolution power on the pseudo-variance function is given in [6, Proposition 3.10]. More precisely, it was shown that for $t > 0$ such that $\nu^{\boxplus t}$ is defined and for m close enough to $m_0(\nu^{\boxplus t}) = tm_0(\nu)$,

$$\mathbb{V}_{\nu^{\boxplus t}}(m) = t\mathbb{V}_\nu(m/t). \tag{2.15}$$

Concerning subordination function, Belinschi and Bercovici [2] used subordination results in order to show that $\nu^{\boxplus t}$ has no continuous singular part if $t > 1$, and that the density of its absolutely continuous part is locally analytic. In fact, there exists an injective analytic map (called subordination) $w_t : \mathbb{C}^+ \longrightarrow \mathbb{C}^+$ such that $G_{\nu^{\boxplus t}}(z) = G_\nu(w_t(z))$ for $z \in \mathbb{C}^+$. Furthermore, we have

$$w_t(z) = z/t + \frac{(1-1/t)}{G_{\nu^{\boxplus t}}(z)} \tag{2.16}$$

and $H_t(w_t(z)) = z$, where

$$H_t(z) = tz + \frac{(1-t)}{G_\nu(z)}. \tag{2.17}$$

For more details about subordination function, see [2, Theorem 2.5].

2.3 *Boolean additive convolution*

The definition of Boolean additive convolution is based on the notion of K-transform (see [20]). For $\nu \in \mathcal{M}$, the K-transform of ν is defined by

$$K_\nu(z) = z - \frac{1}{G_\nu(z)}, \quad \text{for } z \in \mathbb{C}^+. \tag{2.18}$$

For probability measures $\mu, \nu \in \mathcal{M}$ their Boolean additive convolution is the probability measure $\mu \uplus \nu$ defined by

$$K_{\mu\uplus\nu}(z) = K_\mu(z) + K_\nu(z), \quad \text{for } z \in \mathbb{C}^+. \tag{2.19}$$

A probability measure $\nu \in \mathcal{M}$ is infinitely divisible in the Boolean sense if for each $n \in \mathbb{N}$, there exists $\nu_n \in \mathcal{M}$ such that

$$\nu = \underbrace{\nu_n \uplus \cdots \uplus \nu_n}_{n \text{ times}}.$$

Note that all probability measure ν on $\mathbb{R}$ are $\uplus$-infinitely divisible, see [20, Theorem 3.6].

The action of the Boolean additive convolution power on the pseudo-variance function is given in [11, Theorem 2.3], that is, for $t > 0$ and for $m > m_0(\nu^{\uplus t}) = tm_0(\nu)$ close enough to $m_0(\nu^{\uplus t})$, it was shown that

$$\mathbb{V}_{\nu^{\uplus t}}(m) = t\mathbb{V}_\nu(m/t) + m^2(1/t - 1). \tag{2.20}$$

Note that Belinschi and Nica [1] have defined for $t \geq 0$, the mapping

$$\begin{aligned} \mathbb{B}_t : \mathcal{M} &\to \mathcal{M} \\ \nu &\mapsto (\nu^{\boxplus(1+t)})^{\uplus \frac{1}{1+t}}. \end{aligned}$$

The pseudo-variance function of the CSK family generated by $\mathbb{B}_t(\nu)$ is given in [11]. More precisely, it was shown that if $\nu \in \mathcal{M}$ is a probability measure with support bounded from above, then for $m > m_0(\nu) = m_0(\mathbb{B}_t(\nu))$ close enough to $m_0(\nu)$, we have

$$\mathbb{V}_{\mathbb{B}_t(\nu)}(m) = \mathbb{V}_\nu(m) + tm^2. \tag{2.21}$$

2.4 *Free multiplicative convolution*

Let $\nu \in \mathcal{M}_+$ such that $\alpha = \nu(\{0\}) < 1$ and consider the function

$$\Psi_\nu(z) = \int_0^{+\infty} \frac{zx}{1 - zx} \nu(dx), \qquad z \in \mathbb{C} \setminus \mathbb{R}_+ \tag{2.22}$$

The function Ψ_ν is univalent in the left half-plane $i\mathbb{C}^+$ and its image $\Psi_\nu(i\mathbb{C}^+)$ is contained in the disc with diameter $(\nu(\{0\}) - 1, 0)$. Moreover, $\Psi_\nu(i\mathbb{C}^+) \cap \mathbb{R} = (\nu(\{0\}) - 1, 0)$. Let $\chi_\nu : \Psi_\nu(i\mathbb{C}^+) \longrightarrow i\mathbb{C}^+$ be the inverse function of Ψ_ν. Then the S-transform of ν is the function

$$S_\nu(z) = \chi_\nu(z)\frac{1+z}{z}. \tag{2.23}$$

The product of S-transforms is an S-transform. For ν_1 and $\nu_2 \in \mathcal{M}_+$, their free multiplicative convolution is the probability measure $\nu_1 \boxtimes \nu_2$ which is defined by

$$S_{\nu_1 \boxtimes \nu_2}(z) = S_{\nu_1}(z) S_{\nu_2}(z).$$

We say that the probability measure $\nu \in \mathcal{M}_+$ is infinitely divisible with respect to $\boxtimes$ if for each $n \in \mathbb{N}$, there exists $\nu_n \in \mathcal{M}_+$ such that

$$\nu = \underbrace{\nu_n \boxtimes \cdots \boxtimes \nu_n}_{n \text{ times}}.$$

The multiplicative free convolution power $\nu^{\boxtimes t}$ is defined at least for $t \geq 1$ by $S_{\nu^{\boxtimes t}}(z) = S_\nu(z)^t$. For more details about the S-transform, see [4].

The action of the free multiplicative convolution power on the pseudo-variance function is given in [16]. That is, for $\nu \in \mathcal{M}_+$ and $t > 0$ such that $\nu^{\boxtimes t}$ is well defined and for $m \in (m_-(\nu^{\boxtimes t}), m_0(\nu^{\boxtimes t})) = ((m_-(\nu))^t,\ (m_0(\nu))^t)$, we have

$$\mathbb{V}_{\nu^{\boxtimes t}}(m) = m^{2-2/t}\mathbb{V}_\nu(m^{1/t}). \tag{2.24}$$

3. Main result

Let $\nu \in \mathcal{M}_+$ be a non degenerate probability measure. We determine explicitly in terms of the pseudo-variance function $\mathbb{V}_\nu(.)$, the expressions of $\Phi(\nu^{\boxtimes t})$, $\Phi((D_{1/s}(\nu^{\boxplus s}))^{\boxtimes t})$, $\Phi((D_{1/r}(\nu^{\uplus r}))^{\boxtimes t})$ and $\Phi((\mathbb{B}_r(\nu))^{\boxtimes t})$ for $s > 1$, $t > 1$ and $r > 0$.

Theorem 3.1. *Let $\mathbb{V}_\nu(\cdot)$ be the pseudo-variance function of the CSK family $\mathcal{K}_-(\nu)$ generated by a non degenerate probability measure $\nu \in \mathcal{M}_+$. Set $\alpha = \nu(\{0\})$. With the notations introduced above, for all $s > 1$, $t > 1$ and $r > 0$ we have*

(i)
$$\Phi(\nu^{\boxtimes t})(dm) = \alpha\delta_0 + \left(\frac{m^{2/t}}{\mathbb{V}_\nu(m^{1/t})}\right)' \mathbf{1}_{((m_-(\nu))^t,\ (m_0(\nu))^t)}(m)dm. \tag{3.1}$$

(ii)
$$\begin{aligned}\Phi((D_{1/s}(\nu^{\boxplus s}))^{\boxtimes t})(dm) &= (s\alpha - (s-1))^+\delta_0\\ &\quad + \left(\frac{sm^{2/t}}{\mathbb{V}_\nu(m^{1/t})}\right)' \mathbf{1}_{\left(\left(-\frac{w_s(0)}{s-1}\right)^t,\ (m_0(\nu))^t\right)}(m)dm.\end{aligned} \tag{3.2}$$

(iii)
$$\begin{aligned}\Phi((D_{1/r}(\nu^{\uplus r}))^{\boxtimes t})(dm) &= \frac{\alpha}{r - \alpha(r-1)}\delta_0\\ &\quad + \left(\frac{rm^{2/t}}{\mathbb{V}_\nu(m^{1/t}) + (1-r)m^{2/t}}\right)' \mathbf{1}_{((m_-(\nu))^t,\ (m_0(\nu))^t)}(m)dm.\end{aligned} \tag{3.3}$$

(iv)
$$\begin{aligned}\Phi((\mathbb{B}_r(\nu))^{\boxtimes t})(dm) &= \frac{(1+r)((1+r)\alpha - r)^+}{1 + r((1+r)\alpha - r)^+}\delta_0\\ &\quad + \left(\frac{m^{2/t}}{\mathbb{V}_\nu(m^{1/t}) + rm^{2/t}}\right)' \mathbf{1}_{\left(\left(-\frac{w_{1+r}(0)}{r}\right)^t,\ (m_0(\nu))^t\right)}(m)dm.\end{aligned} \tag{3.4}$$

Proof.

(i) According to [14, Theorem 3.1], the free multiplicative law of large numbers $\Phi(\nu)$ is given in terms of the pseudo-variance function $\mathbb{V}_\nu(\cdot)$ by

$$\Phi(\nu)(dm) = \alpha\delta_0 + \left(\frac{m^2}{\mathbb{V}_\nu(m)}\right)' \mathbf{1}_{(m_-(\nu),\ m_0(\nu))}(m)dm. \tag{3.5}$$

Furthermore, according to [4, Lemma 6.9], if μ_1 and μ_2 are probability measures on $[0, +\infty)$ and $\mu = \mu_1 \boxtimes \mu_2$, we have $\mu(\{0\}) = \max\{\mu_1(\{0\}), \mu_2(\{0\})\}$. This implies that

$$\nu^{\boxtimes t}(\{0\}) = \nu(\{0\}) = \alpha. \tag{3.6}$$

Combining (2.24), (3.5) and (3.6), we get for all $t > 1$,

$$\Phi(\nu^{\boxtimes t})(dm) = \nu^{\boxtimes t}(\{0\})\delta_0 + \left(\frac{m^2}{\mathbb{V}_{\nu^{\boxtimes t}}(m)}\right)' \mathbf{1}_{(m_-(\nu^{\boxtimes t}),\ m_0(\nu^{\boxtimes t}))}(m)dm$$

$$= \alpha\delta_0 + \left(\frac{m^{2/t}}{\mathbb{V}_\nu(m^{1/t})}\right)' \mathbf{1}_{((m_-(\nu))^t,\ (m_0(\nu))^t)}(m)dm. \tag{3.7}$$

(ii) According to [2, Theorem 3.1], $\nu^{\boxplus s}$ has an atom at 0 for $s > 1$ if and only if $\nu(\{0\}) = \alpha > 1 - 1/s$. In this case, we have

$$\nu^{\boxplus s}(\{0\}) = s\alpha - (s-1). \tag{3.8}$$

Then

$$(D_{1/s}(\nu^{\boxplus s}))^{\boxtimes t}(\{0\}) = D_{1/s}(\nu^{\boxplus s})(\{0\}) = \nu^{\boxplus s}(\{0\}) = (s\alpha - (s-1))^+. \tag{3.9}$$

Since ν is a non degenerate probability measure on $[0,\infty)$, this is the same for $\nu^{\boxplus s}$. Then $A = A(\nu^{\boxplus s}) = \min\{0, \inf \operatorname{supp}(\nu^{\boxplus t})\} = 0$. Thus,

$$m_-(\nu^{\boxplus s}) = A - 1/G_{\nu^{\boxplus s}}(A) = -1/G_{\nu^{\boxplus s}}(0). \tag{3.10}$$

Equation (3.10) together with (2.16) implies that for all $s > 1$,

$$m_-(\nu^{\boxplus s}) = -\frac{s w_s(0)}{s-1}. \tag{3.11}$$

Then

$$m_-((D_{1/s}(\nu^{\boxplus s}))^{\boxtimes t}) = (m_-(D_{1/s}(\nu^{\boxplus s})))^t = \left(-\frac{w_s(0)}{s-1}\right)^t. \tag{3.12}$$

Furthermore,

$$m_0((D_{1/s}(\nu^{\boxplus s}))^{\boxtimes t}) = (m_0(D_{1/s}(\nu^{\boxplus s})))^t = (m_0(\nu))^t. \tag{3.13}$$

From (2.24), (2.15) and (2.11), we have for all $s > 1$ and $t > 1$,

$$\mathbb{V}_{((D_{1/s}(\nu^{\boxplus s}))^{\boxtimes t})}(m) = \frac{m^{2-2/t}}{s}\mathbb{V}_\nu(m^{1/t}). \tag{3.14}$$

Combining (3.5), (3.9), (3.12), (3.13) and (3.14), we obtain for all $s > 1$ and $t > 1$,

$$\begin{aligned}\Phi((D_{1/s}(\nu^{\boxplus s}))^{\boxtimes t})(dm) &= (s\alpha - (s-1))^+\delta_0 \\ &+ \left(\frac{sm^{2/t}}{\mathbb{V}_\nu(m^{1/t})}\right)' \mathbf{1}_{\left(\left(-\frac{w_s(0)}{s-1}\right)^t,\ (m_0(\nu))^t\right)}(m)dm.\end{aligned} \tag{3.15}$$

(iii) From [22, Corollary 2.3], it is well known that $\nu^{\uplus r}$ has an atom at 0 if and only if ν has an atom at 0. In this case, we have

$$\nu^{\uplus r}(\{0\}) = \frac{\alpha}{r - \alpha(r-1)}, \qquad r > 0. \tag{3.16}$$

Then for all $r > 0$ and $t > 1$, we have

$$(D_{1/t}(\nu^{\uplus t}))^{\boxtimes t}(\{0\}) = D_{1/r}(\nu^{\uplus r})(\{0\}) = \nu^{\uplus r}(\{0\}) = \frac{\alpha}{r-\alpha(r-1)}. \tag{3.17}$$

Since ν is a non degenerate probability measure on $[0,\infty)$, this is the same for $\nu^{\uplus r}$. Then $A(\nu^{\uplus r}) = \min\{0, \inf \operatorname{supp}(\nu^{\uplus r})\} = 0$. Thus,

$$m_-(\nu^{\uplus r}) = -1/G_{\nu^{\uplus r}}(0) = -r/G_\nu(0) = rm_-(\nu). \tag{3.18}$$

Then, for all $r > 0$ and $t > 1$,

$$m_-((D_{1/r}(\nu^{\uplus r}))^{\boxtimes t}) = (m_-(D_{1/r}(\nu^{\uplus r})))^t = (m_-(\nu))^t. \tag{3.19}$$

Furthermore,

$$m_0((D_{1/r}(\nu^{\uplus r}))^{\boxtimes t}) = (m_0(D_{1/r}(\nu^{\uplus r})))^t = (m_0(\nu))^t. \tag{3.20}$$

From (2.24), (2.20) and (2.11), we have for all $r > 0$ and $t > 1$,

$$\mathbb{V}_{((D_{1/r}(\nu^{\uplus r}))^{\boxtimes t})}(m) = \frac{m^{2-2/t}}{r}\mathbb{V}_\nu(m^{1/t}) + m^2(1/r-1). \tag{3.21}$$

Combining (3.5), (3.17), (3.19), (3.20) and (3.21), we get for all $r > 0$ and $t > 1$,

$$\Phi((D_{1/r}(\nu^{\uplus r}))^{\boxtimes t})(dm) = \frac{\alpha}{r-\alpha(r-1)}\delta_0 + \left(\frac{rm^{2/t}}{\mathbb{V}_\nu(m^{1/t}) + (1-r)m^{2/t}}\right)' \mathbf{1}_{((m_-(\nu))^t,\ (m_0(\nu))^t)}(m)dm. \tag{3.22}$$

(iv) Using (3.8) and (3.16), we get for all $r > 0$ and $t > 1$,

$$\begin{aligned}(B_r(\nu))^{\boxtimes t}\{0\} &= (B_r(\nu))\{0\} = \left(\nu^{\boxplus 1+r}\right)^{\uplus \frac{1}{1+r}}\{0\} = \frac{\nu^{\boxplus 1+r}\{0\}}{\frac{1}{1+r} - \left(\frac{1}{1+r}-1\right)\nu^{\boxplus 1+r}\{0\}} \\ &= \frac{((1+r)\alpha - r)^+}{\frac{1}{1+r} - \left(\frac{1}{1+r}-1\right)((1+r)\alpha-r)^+} = \frac{(1+r)((1+r)\alpha-r)^+}{1+r((1+r)\alpha-r)^+}.\end{aligned} \tag{3.23}$$

Furthermore, from (3.11) and (3.18), one see that

$$m_-(B_r(\nu)) = m_-\left(\nu^{\boxplus 1+r}\right)^{\uplus\frac{1}{1+r}} = \frac{1}{1+r}m_-(\nu^{\boxplus 1+r}) = -\frac{w_{1+r}(0)}{r}. \tag{3.24}$$

Thus, for all $r > 0$ and $t > 1$, we have

$$m_-((B_r(\nu))^{\boxtimes t}) = (m_-(B_r(\nu)))^t = \left(-\frac{w_{1+r}(0)}{r}\right)^t. \tag{3.25}$$

We also have

$$m_0((B_r(\nu))^{\boxtimes t}) = (m_0(B_r(\nu)))^t = (m_0(\nu))^t\ . \tag{3.26}$$

In addition, one sees from (2.24) and (2.21) that for all $r > 0, t > 1$ and for $m < (m_0(\nu))^t$ close enough to $(m_0(\nu))^t$, we have

$$\mathbb{V}_{((B_r(\nu))^{\boxtimes t})}(m) = m^{2-2/t}\mathbb{V}_\nu(m^{1/t}) + rm^2. \tag{3.27}$$

Combining (3.5), (3.23), (3.25), (3.26) and (3.27), we get for all $r > 0$ and $t > 1$,

$$\Phi((\mathbb{B}_r(\nu))^{\boxtimes t})(dm) = \frac{(1+r)((1+r)\alpha - r)^+}{1 + r((1+r)\alpha - r)^+}\delta_0 + \left(\frac{m^{2/t}}{\mathbb{V}_\nu(m^{1/t}) + rm^{2/t}}\right)' \mathbf{1}_{\left(\left(-\frac{w_{1+r}(0)}{r}\right)^t,\ (m_0(\nu))^t\right)}(m)dm. \tag{3.28}$$

□

4. Examples

The following examples illustrate the usefulness of Theorem 3.1 and provide examples of the limiting distributions $\Phi(\nu^{\boxtimes t})$, $\Phi((D_{1/s}(\nu^{\boxplus s}))^{\boxtimes t})$, $\Phi((D_{1/r}(\nu^{\uplus r}))^{\boxtimes t})$ and $\Phi((\mathbb{B}_r(\nu))^{\boxtimes t})$ (with $s > 1$, $t > 1$ and $r > 0$) for probability measures ν of importance in non-commutative probability. However probability measures ν presented in the following examples generates CSK families having quadratic and cubic pseudo-variance functions. The quadratic CSK families are described in [5] and cubic CSK families are described in [6].

Example 4.1. Let $\gamma = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ be the symmetric Bernoulli distribution. It generates the CSK family with variance function $\mathcal{V}_\gamma(m) = 1 - m^2 = \mathbb{V}_\gamma(m)$ and $m_0(\gamma) = 0$. By the translation $f : x \longmapsto x + 1$, the probability measure $\nu = f(\gamma) = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_2$ generates the CSK family with $m_0(\nu) = 1$ and pseudo-variance function

$$\mathbb{V}_\nu(m) = \frac{m^2(2-m)}{m-1}.$$

The Cauchy-transform of ν is given by

$$G_\nu(z) = \frac{z-1}{z(z-2)}.$$

We have that $m_-(\nu) = -\frac{1}{G_\nu(0)} = 0$ and the one-sided domain of means of the family $\mathcal{K}_-(\nu)$ is $(m_-(\nu), m_0(\nu)) = (0,\ 1)$.

From (2.17), we get

$$H_t(z) = tz - (t-1)\frac{z(z-2)}{z-1} = \frac{z^2 + z(t-2)}{z-1}$$

and therefore,

$$w_t(y) = \frac{1}{2}\left(2 + y - t - \sqrt{y^2 - 2ty + (t-2)^2}\right),$$

for $y \in \mathbb{C}\backslash(t - 2\sqrt{t-1},\ t + 2\sqrt{t-1})$. Then

$$w_t(0) = \frac{2 - t - |t-2|}{2}.$$

In this case, for all $s > 1$, $t > 1$ and $r > 0$, we have

(i)
$$\Phi(\nu^{\boxtimes t})(dm) = \frac{1}{2}\delta_0 + \frac{m^{1/t-1}}{t(2 - m^{1/t})^2}\mathbf{1}_{(0,\ 1)}(m)dm.$$

(ii)
$$\Phi((D_{1/s}(\nu^{\boxplus s}))^{\boxtimes t})(dm) = (1 - s/2)^+\delta_0 + \frac{s}{t}\frac{m^{1/t-1}}{(2 - m^{1/t})^2}\mathbf{1}_{\left(\left(\frac{s-2+|s-2|}{2(s-1)}\right)^t,\ 1\right)}(m)dm.$$

(iii)
$$\Phi((D_{1/r}(\nu^{\uplus r}))^{\boxtimes t})(dm) = \frac{1}{1+r}\delta_0 + \frac{r}{t}\frac{m^{1/t-1}}{(1 + r - rm^{1/t})^2}\mathbf{1}_{(0,\ 1)}(m)dm.$$

(iv)
$$\Phi((\mathbb{B}_r(\nu))^{\boxtimes t})(dm) = \frac{(1+r)(1-r/2)^+}{1 + r(1-r/2)^+}\delta_0 + \frac{1}{t}\frac{m^{1/t-1}}{((r-1)m^{1/t} + 2 - r)^2}\mathbf{1}_{\left(\left(\frac{r-1+|r-1|}{2r}\right)^t,\ 1\right)}(m)dm.$$

Example 4.2. The Wigner's semicircle (free Gaussian) distribution

$$\gamma(dx) = \frac{\sqrt{4 - x^2}}{2\pi}\mathbf{1}_{(-2,2)}(x)dx,$$

generates the CSK family with variance function $\mathcal{V}_\gamma(m) = 1 = \mathbb{V}_\gamma(m)$. The Cauchy transform of γ is given by

$$G_\gamma(z) = \frac{1}{2}\left(z - \sqrt{z^2 - 4}\right).$$

The one- sided domain of means of the family $\mathcal{K}_-(\gamma)$ is $(m_-(\gamma), m_0(\gamma)) = (-1,\ 0)$.

By the translation $f : x \longmapsto x + 2$, the probability measure

$$\nu(dx) = f(\gamma)(dx) = \frac{\sqrt{x(4-x)}}{2\pi}\mathbf{1}_{(0,\ 4)}(x)dx,$$

generates the CSK family with $m_0(\nu) = 2$ and pseudo-variance function

$$\mathbb{V}_\nu(m) = \frac{m}{m-2}.$$

The Cauchy transform of ν is given by

$$G_\nu(z) = G_\gamma(z-2) = \frac{1}{2}\left(z - 2 - \sqrt{z(z-4)}\right).$$

We have that $G_\nu(0) = -1$ and the domain of means of the family $\mathcal{K}_-(\nu)$ is $(m_-(\nu), m_0(\nu)) = (1,\ 2)$. From (2.17), we have that

$$H_t(z) = tz - \frac{(t-1)}{G_\nu(z)} = tz - \frac{2(t-1)}{z - 2 - \sqrt{z(z-4)}}$$

and consequently

$$w_t(y) = \frac{2t - 2t^2 + y + ty - (t-1)\sqrt{4t(t-1) - 4ty + y^2}}{2t}$$

for all $y \in \mathbb{C}\backslash(2t - 2\sqrt{t},\ 2t + 2\sqrt{t})$. Then

$$w_t(0) = (t-1)\left(-1 - \sqrt{\frac{t-1}{t}}\right).$$

In this case, for all $s > 1$, $t > 1$ and $r > 0$, we have

(i) $$\Phi(\nu^{\boxtimes t})(dm) = \frac{2}{t}(m^{2/t-1} - m^{1/t-1})\mathbf{1}_{(1,\ 2^t)}(m)dm.$$

(ii) $$\Phi((D_{1/s}(\nu^{\boxplus s}))^{\boxtimes t})(dm) = \frac{2s}{t}(m^{2/t-1} - m^{1/t-1})\mathbf{1}_{\left(\left(1+\sqrt{\frac{s-1}{s}}\right)^t,\ 2^t\right)}(m)dm.$$

(iii) $$\Phi((D_{1/r}(\nu^{\uplus r}))^{\boxtimes t})(dm) = \frac{2r}{t}\frac{(m^{2/t-1} - m^{1/t-1})}{\left(1 + (1-r)\left(m^{2/t-1} - m^{1/t-1}\right)\right)^2}\mathbf{1}_{(1,\ 2^t)}(m)dm.$$

(iv) $$\Phi((\mathbb{B}_r(\nu))^{\boxtimes t})(dm) = \frac{2}{t}\frac{m^{2/t-1} - m^{1/t-1}}{(1 + r(m^{2/t} - 2m^{1/t}))^2}\mathbf{1}_{\left(\left(1+\sqrt{\frac{r}{1+r}}\right)^t,\ 2^t\right)}(m)dm.$$

Example 4.3. For $0 < a^2 < 1$, the (absolutely continuous) centered Marchenko–Pastur distribution

$$\gamma(dx) = \frac{\sqrt{4 - (x-a)^2}}{2\pi(1+ax)}\mathbf{1}_{(a-2,a+2)}(x)dx$$

generates the CSK family with variance function $\mathcal{V}(m) = 1 + am = \mathbb{V}(m)$. The Cauchy transform of γ is given by

$$G_\gamma(z) = \frac{\left(a + z - \sqrt{(a-z)^2 - 4}\right)}{2(1+az)}.$$

The one-sided domain of means of the family $\mathcal{K}_-(\gamma)$ is $(m_-(\gamma), m_0(\gamma)) = (-1,\ 0)$. This with the affine transformation $f : x \longmapsto ax + 1$ leads to the distribution given by

$$\nu(dx) = f(\gamma)(dx) = \frac{\sqrt{\left((a+1)^2 - x\right)\left(x - (a-1)^2\right)}}{2\pi a^2 x}\mathbf{1}_{((a-1)^2,(a+1)^2)}(x)dx.$$

It generates the CSK family with $m_0(\nu) = 1$, and pseudo-variance function of the form

$$\mathbb{V}_\nu(m) = \frac{a^2m^2}{m-1}.$$

The Cauchy transform of ν is given by

$$G_\nu(z) = \frac{1}{a}G_\gamma\left(\frac{z-1}{a}\right) = \frac{1}{2za}\left(a + \frac{(z-1)}{a} - \sqrt{\left(a - \frac{(z-1)}{a}\right)^2 - 4}\right). \tag{4.1}$$

Solving $z(m) = m + \mathbb{V}_\nu(m)/m = 0$ for $m > 0$, we obtain that $m = 1 - a^2$. Using the relation (2.10), we get $G_\nu(0) = -\frac{1}{1-a^2}$. The domain of means of the family $\mathcal{K}_-(\nu)$ is $(m_-(\nu), m_0(\nu)) = (1 - a^2,\ 1)$.

We have that

$$H_t(z) = tz - \frac{2za(t-1)}{\left(a + \frac{(z-1)}{a} - \sqrt{\left(a - \frac{(z-1)}{a}\right)^2 - 4}\right)} \tag{4.2}$$

and

$$w_t(y) = \frac{-a^2 + t + a^2t - t^2 + y + ty - (t-1)\sqrt{(t-a^2)^2 - 2y(a^2+t) + y^2}}{2t}, \tag{4.3}$$

for all $y \in \mathbb{C}\backslash(a^2 + t - 2\sqrt{ta^2},\ a^2 + t + 2\sqrt{ta^2})$. Then

$$w_t(0) = -\frac{(t-1)(t-a^2)}{t}.$$

In this case, for all $s > 1$, $t > 1$ and $r > 0$, we have

(i) $$\Phi(\nu^{\boxtimes t})(dm) = \frac{1}{a^2t}m^{1/t-1}\mathbf{1}_{((1-a^2)^t,\ 1)}(m)dm.$$

(ii) $$\Phi((D_{1/s}(\nu^{\boxplus s}))^{\boxtimes t})(dm) = \frac{s}{a^2t}m^{1/t-1}\mathbf{1}_{\left(\left(\frac{s-a^2}{s}\right)^t,\ 2^t\right)}(m)dm.$$

(iii) $$\Phi((D_{1/r}(\nu^{\uplus r}))^{\boxtimes t})(dm) = \frac{ra^2}{t}\frac{m^{1/t-1}}{\left(a^2 + (1-r)\left(m^{1/t} - 1\right)\right)^2}\mathbf{1}_{((1-a^2)^t,\ 1)}(m)dm.$$

(iv)
$$\Phi((\mathbb{B}_r(\nu))^{\boxtimes t})(dm) = \frac{a^2}{t}\frac{m^{1/t-1}}{(a^2+r(m^{1/t}-1))^2}\mathbf{1}_{\left(\left(\frac{1+r-a^2}{1+r}\right)^t,\ 2^t\right)}(m)dm.$$

Example 4.4. For $a^2 > 1$, the Marchenko–Pastur distribution is given by

$$\gamma(dx) = \frac{\sqrt{4-(x-a)^2}}{2\pi(1+ax)}1_{(a-2,a+2)}(x)dx + (1-1/a^2)\delta_{-1/a}(dx).$$

It generates the CSK family with $m_0(\gamma) = 0$ and pseudo-variance function $\mathcal{V}_\gamma(m) = 1 + am = \mathbb{V}_\gamma(m)$. By the affine transformation $f : x \longmapsto ax + 1$, the probability distribution is given by

$$\begin{aligned}\nu(dx) &= f(\gamma)(dx)\\ &= \frac{\sqrt{\left((a+1)^2-x\right)\left(x-(a-1)^2\right)}}{2\pi a^2 x}\mathbf{1}_{((a-1)^2,(a+1)^2)}(x)dx + (1-1/a^2)\delta_0\end{aligned}$$

which generates the CSK family with $m_0(\nu) = 1$, and pseudo-variance function of the form

$$\mathbb{V}_\nu(m) = \frac{a^2m^2}{m-1}.$$

The Cauchy transform of ν is given by (4.1). The probability measure ν has a Dirac mass at 0. This implies that $\int_0^\infty x^{-1}\nu(dx) = +\infty$ and so $m_-(\nu) = -1/G_\nu(0) = 0$.

The functions $H_t(\cdot)$ and $w_t(\cdot)$ are given respectively by (4.2) and (4.3). We have that

$$w_t(0) = \frac{(t-1)(a^2-t-|t-a^2|)}{2t}.$$

In this case, for all $s > 1$, $t > 1$ and $r > 0$, we have

(i)
$$\Phi(\nu^{\boxtimes t})(dm) = (1-1/a^2)\delta_0 + \frac{1}{a^2t}m^{1/t-1}\mathbf{1}_{(0,\ 1)}(m)dm.$$

(ii)
$$\begin{aligned}\Phi((D_{1/s}(\nu^{\boxplus s}))^{\boxtimes t})(dm) &= (1-s/a^2)^+\delta_0\\ &\quad+\frac{s}{a^2t}m^{1/t-1}\mathbf{1}_{\left(\left(\frac{s-a^2+|s-a^2|}{2s}\right)^t,\ 1\right)}(m)dm.\end{aligned}$$

(iii)
$$\begin{aligned}\Phi((D_{1/r}(\nu^{\uplus r}))^{\boxtimes t})(dm) &= \left(\frac{a^2-1}{a^2+r-1}\right)\delta_0\\ &\quad+\frac{a^2r}{t}\frac{m^{1/t-1}}{(a^2+(1-r)(m^{1/t}-1))^2}\mathbf{1}_{(0,\ 1)}(m)dm.\end{aligned}$$

(iv)
$$\Phi((\mathbb{B}_r(\nu))^{\boxtimes t})(dm) = \left(\frac{(1+r)\left(1-\frac{1+r}{a^2}\right)^+}{1+r\left(1-\frac{1+r}{a^2}\right)^+}\right)\delta_0$$
$$+\frac{a^2}{t}\frac{m^{1/t-1}}{(a^2+r(m^{1/t}-1))^2}\mathbf{1}_{\left(\left(\frac{r+1-a^2+|r+1-a^2|}{2(r+1)}\right)^t,\ 1\right)}(m)dm.$$

Example 4.5. If ν is the standard free gamma distribution,

$$\gamma(dx) = \frac{\sqrt{4(1+a^2)-(x-2a)^2}}{2\pi(a^2x^2+2ax+1)}\mathbf{1}_{(2a-2\sqrt{1+a^2},2a+2\sqrt{1+a^2})}(x),$$

for $a \neq 0$, it generates the CSK family with $m_0(\gamma) = 0$, and pseudo-variance function will be equal to the variance function $\mathbb{V}_\gamma(m) = \mathcal{V}_\gamma(m) = (1+am)^2$. The Cauchy transform of γ is given by

$$G_\gamma(z) = \frac{2a+z+2a^2z-\sqrt{(2a-z)^2-4(1+a^2)}}{2(1+az)^2}.$$

By the affine transformation $f : x \longmapsto ax+1$, the probability distribution

$$\nu(dx) = f(\gamma)(dx) = \frac{\sqrt{((\sqrt{a^2+1}+a)^2-x)(x-(\sqrt{a^2+1}-a)^2)}}{2\pi a^2x^2}$$
$$\mathbf{1}_{((\sqrt{a^2+1}-|a|)^2,(\sqrt{a^2+1}+|a|)^2)}(x)dx$$

generates the CSK family with $m_0(\nu) = 1$, and pseudo-variance function of the form becomes

$$\mathbb{V}_\nu(m) = \frac{a^2m^3}{m-1}.$$

The Cauchy transform of ν is given by

$$G_\nu(z) = \frac{1}{a}G_\gamma\left(\frac{z-1}{a}\right) = \frac{\frac{z-1}{a}+2az-\sqrt{(2a-\frac{z-1}{a})^2-4(1+a^2)}}{2az^2}.$$

Solving $z(m) = m + \mathbb{V}_\nu(m)/m = 0$ for $m > 0$, we obtain that $m = \frac{1}{1+a^2}$. Using the relation (2.10), we get $G_\nu(0) = -(1+a^2)$. The one-sided domain of means of the family $\mathcal{K}_-(\nu)$ is $(m_-(\nu), m_0(\nu)) = \left(\frac{1}{1+a^2},\ 1\right)$. We have that

$$H_t(z) = tz - \frac{(t-1)}{G_\nu(z)} = tz - \frac{2(t-1)az^2}{\frac{z-1}{a}+2az-\sqrt{(2a-\frac{z-1}{a})^2-4(1+a^2)}}$$

and

$$w_t(y) = \frac{t-t^2+y(1+2a^2+t)-(t-1)\sqrt{t^2-(4a^2+2t)y+y^2}}{2(a^2+t)}$$

for all $y \in \mathbb{C}\backslash(2a^2+t-\sqrt{(2a^2+t)^2-1},\ 2a^2+t+\sqrt{(2a^2+t)^2-1})$. Then

$$w_t(0) = -\frac{t(t-1)}{a^2+t}.$$

In this case, for all $s > 1$, $t > 1$ and $r > 0$, we have

(i)
$$\Phi(\nu^{\boxtimes t})(dm) = \frac{1}{a^2 t}\frac{m^{1/t-1}}{m^{2/t}}\mathbf{1}_{\left(\left(\frac{1}{1+a^2}\right)^t,\,1\right)}(m)dm.$$

(ii)
$$\Phi((D_{1/s}(\nu^{\boxplus s}))^{\boxtimes t})(dm) = \frac{s}{a^2 t}\frac{m^{1/t-1}}{m^{2/t}}\mathbf{1}_{\left(\left(\frac{s}{s+a^2}\right)^t,\,1\right)}(m)dm.$$

(iii)
$$\Phi((D_{1/r}(\nu^{\uplus r}))^{\boxtimes t})(dm) = \frac{a^2 r}{t}\frac{m^{1/t-1}}{(a^2 m^{1/t} + (1-r)(m^{1/t}-1))^2}\mathbf{1}_{\left(\left(\frac{1}{1+a^2}\right)^t,\,1\right)}(m)dm.$$

(iv)
$$\Phi((\mathbb{B}_r(\nu))^{\boxtimes t})(dm) = \frac{a^2}{t}\frac{m^{1/t-1}}{((a^2+r)m^{1/t}-r)^2}\mathbf{1}_{\left(\left(\frac{r+1}{r+1+a^2}\right)^t,\,1\right)}(m)dm.$$

Example 4.6. The inverse semicircle distribution

$$\gamma(dx) = \frac{\sqrt{-1-4x}}{2\pi x^2}\mathbf{1}_{\left(-\infty,-\frac{1}{4}\right)}(x)dx,$$

generates the CSK family with pseudo-variance function $\mathbb{V}_\gamma(m) = m^3$, and with domain of means $(m_0(\gamma), m_+(\gamma)) = (-\infty, -1)$. By the transformation $f : x \longmapsto -x$, the probability distribution

$$\nu(dx) = f(\gamma)(dx) = \frac{\sqrt{-1+4x}}{2\pi x^2}\mathbf{1}_{\left(\frac{1}{4},+\infty\right)}(x)dx$$

generates the CSK family $\mathcal{K}_-(\nu)$ with pseudo-variance function $\mathbb{V}_\nu(m) = -m^3$ and the domain of means is $(m_-(\nu), m_0(\nu)) = (1, +\infty)$. According to [6, Section 4], the Cauchy transform of ν is given by

$$G_\nu(z) = \frac{2z-1+\sqrt{1-4z}}{2z^2}.$$

We have that

$$H_t(z) = tz - \frac{(t-1)}{G_\nu(z)} = tz - \frac{2(t-1)z^2}{2z-1+\sqrt{1-4z}}$$

and

$$w_t(y) = \frac{1}{2}\left((t-1)(-t-\sqrt{t^2-4y}) + 2y\right)$$

for all $y \in \mathbb{C}\backslash(t^2/4,\ +\infty)$. Then

$$w_t(0) = -t(t-1).$$

In this case, for all $s > 1$, $t > 1$ and $r > 0$, we have

(i)

$$\Phi(\nu^{\boxtimes t})(dm) = \frac{1}{tm^{1/t+1}}\mathbf{1}_{(1,\ +\infty)}(m)dm.$$

(ii)

$$\Phi((D_{1/s}(\nu^{\boxplus s}))^{\boxtimes t})(dm) = \frac{s}{tm^{1/t+1}}\mathbf{1}_{(s^t,\ +\infty)}(m)dm.$$

(iii)

$$\Phi((D_{1/r}(\nu^{\uplus r}))^{\boxtimes t})(dm) = \frac{r}{t}\frac{m^{1/t-1}}{((1-r)-m^{1/t})^2}\mathbf{1}_{(1,\ +\infty)}(m)dm.$$

(iv)

$$\Phi((\mathbb{B}_r(\nu))^{\boxtimes t})(dm) = \frac{1}{t}\frac{m^{1/t-1}}{(r-m^{1/t})^2}\mathbf{1}_{((r+1)^t,\ +\infty)}(m)dm.$$

Example 4.7. The free Ressel (or free Kendall) distribution

$$\gamma(dx) = \frac{-1}{\pi x\sqrt{-1-x}}\mathbf{1}_{(-\infty,-1)}(x)dx$$

generates the CSK family with domain of means $(m_0(\gamma), m_+(\gamma)) = (-\infty, -2)$ and pseudo-variance function $\mathbb{V}_\gamma(m) = m^2(m+1)$. With the transformation $f : x \longmapsto -x$, the probability distribution

$$\nu(dx) = f(\gamma)(dx) = \frac{1}{\pi x\sqrt{x-1}}\mathbf{1}_{(1,+\infty)}(x)dx.$$

generates the CSK family $\mathcal{K}_-(\nu)$ with pseudo-variance function $\mathbb{V}_\nu(m) = m^2(1-m)$, and domain of means $(m_-(\nu), m_0(\nu)) = (2, +\infty)$. According to [6, Section 4], the Cauchy transform of ν is given by

$$G_\nu(z) = \frac{1-z-\sqrt{1-z}}{z(1-z)}.$$

We have that

$$H_t(z) = tz - \frac{(t-1)}{G_\nu(z)} = tz - \frac{(t-1)z(1-z)}{1-z-\sqrt{1-z}}$$

and

$$w_t(y) = \frac{1}{2}\big(1-t^2-(t-1)\sqrt{(1+t)^2-4y}+2y\big)$$

for all $y \in \mathbb{C}\backslash((1+t^2)/4,\ +\infty)$. Then

$$w_t(0) = -(t-1)(1+t).$$

In this case, for all $s > 1$, $t > 1$ and $r > 0$, we have

(i)

$$\Phi(\nu^{\boxtimes t})(dm) = \frac{1}{t}\frac{m^{1/t-1}}{(1-m^{1/t})^2}\mathbf{1}_{(2^t,\ +\infty)}(m)dm.$$

(ii)

$$\Phi((D_{1/s}(\nu^{\boxplus s}))^{\boxtimes t})(dm) = \frac{s}{t}\frac{m^{1/t-1}}{(1-m^{1/t})^2}\mathbf{1}_{((1+s)^t,\ +\infty)}(m)dm.$$

(iii)

$$\Phi((D_{1/r}(\nu^{\uplus r}))^{\boxtimes t})(dm) = \frac{r}{t}\frac{m^{1/t-1}}{(2-r-m^{1/t})^2}\mathbf{1}_{(2^t,\ +\infty)}(m)dm.$$

(iv)

$$\Phi((\mathbb{B}_r(\nu))^{\boxtimes t})(dm) = \frac{1}{t}\frac{m^{1/t-1}}{(1+r-m^{1/t})^2}\mathbf{1}_{((r+2)^t,\ +\infty)}(m)dm.$$

Example 4.8. The free Abel (or free Borel–Tanner) distribution

$$\gamma(dx) = \frac{1}{\pi(1-x)\sqrt{-x}}\mathbf{1}_{(-\infty,0)}(x)dx$$

generates the CSK family with domain of means $(m_0(\gamma), m_+(\gamma)) = (-\infty, 0)$ and pseudo-variance function $\mathbb{V}_\gamma(m) = m^2(m-1)$. By the transformation $f : x \longmapsto -x$, the probability distribution

$$\nu(dx) = f(\gamma)(dx) = \frac{1}{\pi(1+x)\sqrt{x}}\mathbf{1}_{(0,+\infty)}(x)dx$$

generates the CSK family $\mathcal{K}_-(\nu)$ with pseudo-variance function $\mathbb{V}_\nu(m) = -m^2(1+m)$ and domain of means $(m_-(\nu), m_0(\nu)) = (0, +\infty)$. The Cauchy transform of ν is given by

$$G_\nu(z) = \frac{z+\sqrt{-z}}{z(1+z)}.$$

We have that

$$H_t(z) = tz - \frac{(t-1)}{G_\nu(z)} = tz - \frac{(t-1)z(1+z)}{z+\sqrt{-z}}$$

and

$$w_t(y) = \frac{1}{2}\Big(-(t-1)^2 - (t-1)\sqrt{(t-1)^2-4y} + 2y\Big)$$

for all $y \in \mathbb{C}\backslash((t-1)^2/4,\ +\infty)$. Then

$$w_t(0) = -(t-1)^2.$$

In this case, for all $s > 1$, $t > 1$ and $r > 0$, we have

(i)

$$\Phi(\nu^{\boxtimes t})(dm) = \frac{1}{t}\frac{m^{1/t-1}}{(1+m^{1/t})^2}\mathbf{1}_{(0,\ +\infty)}(m)dm.$$

(ii)

$$\Phi((D_{1/s}(\nu^{\boxplus s}))^{\boxtimes t})(dm) = \frac{s}{t}\frac{m^{1/t-1}}{(1+m^{1/t})^2}\mathbf{1}_{((s-1)^t,\ +\infty)}(m)dm.$$

(iii)

$$\Phi((D_{1/r}(\nu^{\uplus r}))^{\boxtimes t})(dm) = \frac{r}{t}\frac{m^{1/t-1}}{(r+m^{1/t})^2}\mathbf{1}_{(0,\ +\infty)}(m)dm.$$

(iv)

$$\Phi((\mathbb{B}_r(\nu))^{\boxtimes t})(dm) = \frac{1}{t}\frac{m^{1/t-1}}{(r-1-m^{1/t})^2}\mathbf{1}_{(r^t,\ +\infty)}(m)dm.$$

Example 4.9. The free strict arcsine distribution

$$\gamma(dx) = \frac{\sqrt{3-4x}}{2\pi(1+x^2)}\mathbf{1}_{(-\infty,3/4)}(x)dx$$

generates the CSK family with domain of means $(m_0(\gamma), m_+(\gamma)) = (-\infty, -1/2)$ and pseudo-variance function $\mathbb{V}_\gamma(m) = m(1+m^2)$. By the affine transformation $f : x \longmapsto -x + 3/4$, the probability distribution

$$\nu(dx) = f(\gamma)(dx) = \frac{\sqrt{x}}{\pi(1+(3/4-x)^2)}\mathbf{1}_{(0,+\infty)}(x)dx$$

generates the CSK family $\mathcal{K}_-(\nu)$ with pseudo-variance function $\mathbb{V}_\nu(m) = -m(m^2 - \frac{3}{2}m + \frac{25}{16})$. The Cauchy transform of ν is given by

$$G_\nu(z) = \frac{\frac{5}{4} - z - \sqrt{-z}}{-\frac{25}{16} + \frac{3}{2}z - z^2}.$$

We have that $G_\nu(0) = -\frac{4}{5}$. The domain of means of $\mathcal{K}_-(\nu)$ is $(m_-(\nu), m_0(\nu)) = (5/4, +\infty)$. We have

$$H_t(z) = tz - \frac{(t-1)}{G_\nu(z)} = tz - \frac{(t-1)(-\frac{25}{16} + \frac{3}{2}z - z^2)}{\frac{5}{4} - z - \sqrt{-z}}$$

and

$$w_t(y) = \frac{1}{4}\Big(-(t-1)(2t+3) - 2(t-1)\sqrt{(t-1)(t+4) - 4y} + 4y\Big)$$

for all $y \in \mathbb{C}\backslash((t-1)(t+4)/4,\ +\infty)$. Then

$$w_t(0) = \frac{1}{4}(t-1)\big[-(2t+3) - 2\sqrt{(t-1)(t+4)}\big].$$

In this case, for all $s > 1$, $t > 1$ and $r > 0$, we have

(i)
$$\Phi(\nu^{\boxtimes t})(dm) = \frac{1}{t}\frac{m^{3/t-1} - \frac{25}{16}m^{1/t-1}}{(m^{2/t} - \frac{3}{2}m^{1/t} + \frac{25}{16})^2}\mathbf{1}_{((5/4)^t,\ +\infty)}(m)dm.$$

(ii)
$$\Phi((D_{1/s}(\nu^{\boxplus s}))^{\boxtimes t})(dm) = \frac{s}{t}\frac{m^{3/t-1} - \frac{25}{16}m^{1/t-1}}{(m^{2/t} - \frac{3}{2}m^{1/t} + \frac{25}{16})^2}\mathbf{1}_{\left(\left(\frac{2s+3+2\sqrt{(s-1)(s+4)}}{4}\right)^t,\ +\infty\right)}(m)dm.$$

(iii)
$$\Phi((D_{1/r}(\nu^{\uplus r}))^{\boxtimes t})(dm) = \frac{r}{t}\frac{m^{3/t-1} - \frac{25}{16}m^{1/t-1}}{((1-r)m^{1/t} - (m^{2/t} - \frac{3}{2}m^{1/t} + \frac{25}{16}))^2}\mathbf{1}_{((5/4)^t,\ +\infty)}(m)dm.$$

(iv)
$$\Phi((\mathbb{B}_r(\nu))^{\boxtimes t})(dm) = \frac{1}{t}\frac{m^{3/t-1} - \frac{25}{16}m^{1/t-1}}{(rm^{1/t} - (m^{2/t} - \frac{3}{2}m^{1/t} + \frac{25}{16}))^2}\mathbf{1}_{\left(\left(\frac{2r+5+2\sqrt{r(r+5)}}{4}\right)^t,\ +\infty\right)}(m)dm.$$

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