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A note on stable equivalences of finite dimensional algebras

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Abstract. Here, we will show that, under mild conditions, stable equivalences preserve Gorenstein global dimension, CM-freeness, CM-finiteness, GP-conv-finiteness and τ-tilting-finiteness.

Keywords. Stable equivalence; Gorenstein projective module; τ-tilting module.

Mathematics Subject Classification. 16G10.

1. Introduction

To study which homological invariants of finite-dimensional algebras are preserved by stable equivalence is an interesting problem. However, even the simplest examples of stable equivalence (for instance, between radical zero and hereditary algebras) show that global dimension is not a stable invariant. In [20], Martínez Villa has proved that global dimensions and dominant dimensions are preserved by stable equivalences between algebras without nodes. Based on the results of this paper, Dugas [14] has extended this fact to relative homological dimensions which are defined with respect to contravariantly finite subcategories, and in particular, he has proved that the representation dimension is invariant under stable equivalence which was proved by Guo [16] independently. Recently, Xi and Zhang [22] proved that the de-looping levels and ϕ-dimensions are stable invariants. They also proved the Auslander–Reiten conjecture on stable equivalence holds true for principal centralizer algebras of matrices over algebraically closed field.

If one wants to prove that some properties are preserved under stable equivalences between algebras (maybe with nodes), one can often first check if these properties are preserved under stable equivalence between algebras without nodes. Then, by using Theorem 2.10 in [20], i.e., the separation of nodes, we can check if these properties are preserved under general stable equivalences.

In this paper, we further study homological invariants based on the results in [20]. We concentrate on Gorenstein homological properties and tilting theory between algebras which are stable equivalent without nodes. Note that, for a finite dimensional k-algebra A, the subcategory of Gorenstein projective A-modules is, in general, not a contravariant finite subcategory of A-mod, see [8,23]. So, our results are different from [14]. More precisely, we proved the following results.

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Theorem 1. *Let A and B be finite dimensional k-algebras, $\alpha : A\text{-}\underline{\text{mod}} \to B\text{-}\underline{\text{mod}}$ is a stable equivalence without nodes and semisimple summands. Then we have*

(1) $\text{Ggldim}(A) = \text{Ggldim}(B)$;
(2) *A is CM-free if and only if B is CM-free*;
(3) *A is CM-finite if and only if B is CM-finite*;
(4) *$\mathcal{GP}(A)$ is cokntravariant finite if and only if $\mathcal{GP}(B)$ is contravariant*;
(5) *A is τ-tilting finite if and only if B is τ-tilting finite*;
Furthermore, assume e and e' are the idempotents of A and B corresponding to maximal basic projective-injective modules in A-mod and B-mod.
(6) $Q(\text{s}\tau\text{-tilt } A/AeA) \simeq Q(\text{s}\tau\text{-tilt } B/Be'B)$.

Here, $\text{Ggldim}(A)$ denotes the Gorenstein global dimension of A and $\mathcal{GP}(A)$ the subcategory of finitely generated Gorenstein projective modules of A-mod. And $Q(\text{s}\tau\text{-tilt } A)$ denotes the Hasse quiver of support τ-tilting A-modules.

Note that the hereditary algebra $A = kA_2$ is stably equivalent to $B = k[x]/(x^2)$ [7] and the unique simple module in B-mod is a node. We know that $\text{Ggldim}(A) = \text{gldim}(A) = 1$ and $\text{Ggldim}(B) = 0$, so, they are different. And, A is CM-free but B is not. Therefore, the properties (1) and (2) are in general not true when the algebras have nodes. But, we do not know whether the properties (3), (4) and (5) are preserved under stable equivalences between finite dimensional k-algebras (maybe with nodes).

As a special case of τ-tilting modules, we also consider classic tilting modules under stable equivalences. In particular, we prove that from a tilting module T of A-mod we can obtain a tilting B-module. And, when T is separating and splitting, the obtained tilting B-module is also separating and splitting.

Throughout this paper, k is a field, and A and B denote a pair of stably equivalent, basic finite-dimensional k-algebras with no nodes and semisimple summands. We shall denote by A-mod the category of finitely generated left A-modules, by $A\text{-}\underline{\text{mod}}$ the stable module category, and by $A\text{-mod}_{\mathcal{P}}$ the full subcategory of A-mod consisting of modules with no projective direct summands. If $\alpha : A\text{-}\underline{\text{mod}} \to B\text{-}\underline{\text{mod}}$ is an equivalence, we shall also use α to denote the induced map $A\text{-mod} \to B\text{-mod}$ which takes projectives to 0. Furthermore, we shall write $\bar{\alpha} : \underline{\text{mod}}(A\text{-mod}) \to \underline{\text{mod}}(B\text{-mod})$ for the induced equivalence of functor categories as in [5] or [6], where $\underline{\text{mod}}(A\text{-mod})$ denotes the abelian category consisting of finitely presented, contravariant, additive functors from A-mod to the category of abelian groups, which vanish on A. We denote the morphism sets in $A\text{-}\underline{\text{mod}}$ by $\underline{\text{Hom}}_A(M, N)$, where $M, N \in A\text{-mod}$. If $f \in \text{Hom}_A(M, N)$, $\underline{f}$ will denote the image of f in $\underline{\text{Hom}}_A(M, N)$. Finally, we denote $\text{pd}_A(X)$ the projective dimension of A-module X and $\text{id}_A(X)$ the injective dimension of X. For any A-module M, we denote by $|M|$ the number of non-isomorphic indecomposable direct summands.

2. Preliminaries

We recall some results in [21] by Martinez Villa about stable equivalence without nodes and semisimple summands. At first, we recall the definition of a node as given in [20].

DEFINITION 1

A simple non-projective, non-injective module S will be called a node if the almost split sequence for S: $0 \to S \to P \to \tau^{-1}(S) \to 0$ has P projective.

PROPOSITION 1 [21, Proposition 1.5]

Let P be an indecomposable, non-injective projective A-module, then we have $\bar{\alpha}(\mathrm{Ext}^1_A(-, P)) \simeq \mathrm{Ext}^1_B(-, P')$, *with P' an indecomposable projective non-injective B-module.*

Following [14,21], we shall denote the B-module P' by $\alpha'(P)$ and extend α' additively to all projective modules with no injective summands. And, in this case, α' gives a bijection between the isomorphism classes of indecomposable non-injective projective modules over A and B. Finally, recall that a short exact sequence is said to be minimal if it has no nonzero split exact sequence as a direct summand.

Theorem 2 [21, Theorem 1.7]. *Let $\alpha : A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ be a stable equivalence between two algebras without nodes and semisimple summands. Let*

$$0 \to X \oplus Q_1 \xrightarrow{f} Y \oplus P \oplus Q_2 \xrightarrow{g} Z \to 0$$

be a minimal short exact sequence in A-mod, where $X, Y, Z \in A\text{-}\mathrm{mod}_{\mathcal{P}}$, Q_1, Q_2 are projective modules with no injective summand, and P is projective and injective. Then there exists a minimal short exact sequence

$$0 \to \alpha(X) \oplus \alpha'(Q_1) \xrightarrow{u} \alpha(Y) \oplus P_1 \oplus \alpha'(Q_2) \xrightarrow{v} \alpha(Z) \to 0$$

in B-mod with P_1 projective and injective, and $\underline{v} = \alpha(\underline{g})$.

Remark 1. According to Corollary 2.3 in [14], the condition of minimality in the theorem is unnecessary.

PROPOSITION 2 [21, Proposition 2.2]

Let $X, Y \in A\text{-}\mathrm{mod}_{\mathcal{P}}$ and P be a projective non-injective module. Then for every $n \geq 1$, we have $\mathrm{Ext}^n_A(X, Y \oplus P) \simeq \mathrm{Ext}^n_B(\alpha(X), \alpha(Y) \oplus \alpha'(P))$.

Lemma 1 [14, *Corollary* 2.4]. *Suppose*

$$\cdots \to C_{i+2} \oplus P_{i+2} \oplus Q_{i+2} \to C_{i+1} \oplus P_{i+1} \oplus Q_{i+1} \to C_i \oplus P_i \oplus Q_i \to \cdots$$

is an exact sequence in A-mod such that for each i, C_i belongs to $A\text{-}\mathrm{mod}_{\mathcal{P}}$, P_i is projective with no injective summands and Q_i is projective-injective. Then there exists an exact sequence of B-modules

$$\begin{aligned} \cdots &\to \alpha(C_{i+2}) \oplus \alpha'(P_{i+2}) \oplus Q'_{i+2} \to \alpha(C_{i+1}) \oplus \alpha'(P_{i+1}) \oplus Q_{i+1} \\ &\to \alpha(C_i) \oplus \alpha'(P_i) \oplus Q_i \to \cdots \end{aligned}$$

such that for each i, Q'_i is projective-injective. Moreover, the i-th term of this sequence may be taken to be zero whenever the i-th term of the given sequence is zero.

3. Proof of the main result

In this section, we show that, under mild conditions, stable equivalences preserve CM-freeness, CM-finiteness, GP-conv-finiteness and τ-tilting-finiteness. At first, we recall the definitions of Gorenstein projective and injective modules in A-mod where A is a finite-dimensional k-algebra.

DEFINITION 2 [4,15]

A complete projective resolution is an exact sequence of projective A-modules, $\mathbf{P} = \cdots \to P^{-1} \to P^0 \to P^1 \to \cdots$, such that $\mathrm{Hom}_A(\mathbf{P}, A)$ is exact. An A-module G is called Gorenstein projective, if there exists a complete projective resolution $\mathbf{P}$ with $G \simeq \mathrm{Im}(P^0 \to P^1)$. The class of all Gorenstein projective A-modules is denoted by $\mathscr{GP}(A)$. The Gorenstein injective modules are defined dually, and the class of all such modules is denoted by $\mathscr{GI}(A)$.

DEFINITION 3

The Gorenstein projective dimension of $M \in A$-mod denoted by $\mathrm{Gpd}_A(M)$ is defined to be the smallest $n \in \mathbb{N}$ such that M has an exact sequence $0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$ with $G_i \in \mathscr{GP}(A)$. If M has no such exact sequence of finite length, define $\mathrm{Gpd}_A(M) = \infty$. The Gorenstein injective dimension is defined dually. The Gorenstein global dimension of A denoted by $\mathrm{Ggldim}(A)$ is defined to be $\sup\{\mathrm{Gpd}({}_AM) | M \in A\text{-mod}\}$.

Lemma 2. *Let A be a finite-dimensional k-algebra, $G, E \in A\text{-mod}_{\mathcal{P}}$. Then $G \in \mathscr{GP}(A)$ if and only if there exists a complete projective resolution*

$$\cdots \to P^{-2} \xrightarrow{f^{-2}} P^{-1} \xrightarrow{f^{-1}} P^0 \xrightarrow{f^0} P^1 \xrightarrow{f^1} P^2 \cdots$$

with $G \simeq \mathrm{Im}(f^0)$ such that $\mathrm{Im}(f^i) \in A\text{-mod}_{\mathcal{P}}$ for any $i \in \mathbb{Z}$. $E \in \mathscr{GI}(A)$ if and only if there exists a complete injective co-resolution

$$\cdots \to I^{-2} \xrightarrow{g^{-2}} I^{-1} \xrightarrow{g^{-1}} I^0 \xrightarrow{g^0} I^1 \xrightarrow{g^1} I^2 \cdots$$

with $E \simeq \mathrm{Im}(g^0)$ such that $\mathrm{Im}(g^i) \in A\text{-mod}_{\mathcal{P}}$ for any $i \in \mathbb{Z}$.

Proof. We first prove the first statement. The sufficiency is by the definition of Gorenstein projective module, so we just prove the necessity. Take a minimal projective resolution of G: $\cdots \to P^{-2} \xrightarrow{f^{-2}} P^{-1} \xrightarrow{f^{-1}} P^0 \to G \to 0$. Since $\mathrm{Ext}^i_A(G, A) = 0$ for $i \geq 1$ and by the minimality of the resolution, we have $\mathrm{Im}(f^i) \in A\text{-mod}_{\mathcal{P}}$ for $i \leq -1$. Take the minimal left $\mathrm{add}(A)$-approximation of G: $0 \to G \to P^1 \to G^1 \to 0$, we have $G^1 \in A\text{-mod}_{\mathcal{P}} \cap \mathscr{GP}(A)$. Continuing this process, we can get an exact sequence: $0 \to G \to P^1 \xrightarrow{f^1} P^2 \to \cdots$ which is $\mathrm{Hom}_A(-, A)$-exact. Combining these two exact sequences, we obtain the resolution we want.

For the second statement, the sufficiency is by the definition of Gorenstein injective module, so we also just need to prove the necessity. Take a minimal injective co-resolution of E: $0 \to E \to I^0 \xrightarrow{g^0} I^1 \xrightarrow{g^1} I^2 \cdots$, similar to the proof above. By the minimality of the resolution, we have $\mathrm{Im}(g^i) \in A\text{-mod}_{\mathcal{P}}$ for $i \geq 0$. Take a minimal right add(DA)-approximation of E: $0 \to E^{-1} \to I^{-1} \xrightarrow{g} E \to 0$, then we have $E^{-1} \in \mathcal{GI}(A)$ and there exists epimorphism $I^{-2} \to E^{-1} \to 0$. If there exists a projective module P which is a direct summand of E^{-1}, then P is a projective-injective module which contradicts with the minimality of g. So we have $E^{-1} \in A\text{-mod}_{\mathcal{P}}$. Continuing this process, and similar to the proof of the first statement, the proof of the lemma is complete.

Lemma 3. *Assume* $0 \to P \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *is a short exact sequence in* A-mod *with* P *projective and* $\underline{g}$ *is an isomorphism in* A-$\underline{\mathrm{mod}}$. *Then the sequence is split.*

Proof. Since $\underline{g}$ is an isomorphism in A-$\underline{\mathrm{mod}}$, there exists $h : Z \to Y$ such that $\underline{gh} = \underline{1_Z}$. Then we have $gh - 1_Z$ factors through p, where $p : P_Z \to Z$ is the projective cover of Z. So there exists $a : Z \to P_Z$ such that $1_Z = gh + pa$. Since P_Z is projective, we have p factors through g, which means that there exists $b : P_Z \to Y$ such that $p = gb$. Then we have $1_Z = gh + pa = g(h + ba)$ which shows that g is a split epimorphism.

Lemma 4. *Let* $\alpha : A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ *be a stable equivalence between two algebras without nodes and semisimple summands. Assume* $M \in A\text{-mod}_{\mathcal{P}}$ *with* $\mathrm{Ext}^1_A(M, A) = 0$. *Then we have* $\mathrm{Ext}^1_B(\alpha(M), B) = 0$.

Proof. If $\mathrm{Ext}^1_B(\alpha(M), B) \neq 0$, there exists a non-split short exact sequence $0 \to Q \to U \xrightarrow{f} \alpha(M) \to 0$, where Q is projective. Without loss of generality, we assume it is minimal. By applying α^{-1} and Theorem 2, we get an exact sequence $0 \to P \to V \xrightarrow{g} M \to 0$ with $\underline{g} = \alpha^{-1}(\underline{f})$ which is split by the assumption. So we have $\underline{g}$ is an isomorphism. Since α^{-1} is an equivalence, we have $\underline{f}$ is also an isomorphism. By the above lemma, we have f is a split epimorphism, and consequently, we have $\mathrm{Ext}^1_B(\alpha(M), B) = 0$.

Lemma 5. *Let* $\alpha : A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ *be a stable equivalence between two algebras without nodes and semisimple summands. Assume* $G, E \in A\text{-mod}_{\mathcal{P}}$ *and* $G \in \mathcal{GP}(A)$, $E \in \mathcal{GI}(A)$, *we have* $\alpha(G) \in \mathcal{GP}(B)$ *and* $\alpha(E) \in \mathcal{GI}(B)$.

Proof. Since $G \in \mathcal{GP}(A)$, by Lemma 2, we have a complete projective resolution of G:

$$\cdots \to P^{-2} \xrightarrow{f^{-2}} P^{-1} \xrightarrow{f^{-1}} P^0 \xrightarrow{f^0} P^1 \xrightarrow{f^1} P^2 \cdots$$

such that $\mathrm{Im}(f^i) \in A\text{-mod}_{\mathcal{P}}$ for any $i \in \mathbb{Z}$. For any $i \in \mathbb{Z}$, there is a short exact sequence $0 \to \mathrm{Im}(f^i) \to P^{i+1} \to \mathrm{Im}(f^{i+1}) \to 0$. Then, by Theorem 2, we have a short exact sequence: $0 \to \alpha(\mathrm{Im}(f^i)) \to Q^{i+1} \to \alpha(\mathrm{Im}(f^{i+1})) \to 0$ in B-mod. Since $\mathrm{Ext}^1_A(\mathrm{Im}(f^i), A) = 0$ for $i \in \mathbb{Z}$, by Lemma 4, we have $\mathrm{Ext}^1_B(\alpha(\mathrm{Im}(f^i)), B) = 0$ for $i \in \mathbb{Z}$. Combining them, we obtain a complete projective resolution of $\alpha(G)$.

For the second statement, assume I is an injective non-projective module. By Proposition 2, we have $\alpha(I)$ is also injective non-projective. Combining the above proof and Proposition 2, one can easily prove $\alpha(E) \in \mathcal{GI}(B)$.

COROLLARY 1

Let $\alpha : A\text{-}\underline{\text{mod}} \to B\text{-}\underline{\text{mod}}$ be a stable equivalence between two algebras without nodes and semisimple summands. Suppose ${}_AM \in A\text{-mod}_{\mathcal{P}}$, then $\mathrm{Gpd}(M) = \mathrm{Gpd}(\alpha(M))$ *and* $\mathrm{Gid}(M) = \mathrm{Gid}(\alpha(M))$. *In particular*, $\mathrm{Ggldim}(A) = \mathrm{Ggldim}(B)$.

Proof. We first prove $\mathrm{Gpd}(\alpha(M)) \leq \mathrm{Gpd}(M)$. Suppose $\mathrm{Gpd}(M) = \infty$, then there is nothing to prove. Let $\mathrm{Gpd}(M) = n < \infty$. Then there exists a sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0.$$

Combining Lemma 1 and Lemma 5, we have $\mathrm{Gpd}(\alpha(M)) \leq \mathrm{Gpd}(M)$. Since ${}_AM \in A\text{-mod}_{\mathcal{P}}$, we have $\alpha^{-1}\alpha(M) \simeq M$. Then, by the same method, we have $\mathrm{Gpd}(M) \leq \mathrm{Gpd}(\alpha(M))$. Therefore, $\mathrm{Gpd}(\alpha(M)) = \mathrm{Gpd}(M)$. Thus $\mathrm{Gid}(M) = \mathrm{Gid}(\alpha(M))$ can be proved similarly.

We say the algebra A is CM-free if $\mathscr{G}\mathscr{P}(A) = \mathrm{add}(A)$. If there exists only finite number of isomorphism classes of indecomposable Gorenstein projective module, we say A is CM-finite. These algebras have being widely studied in [9,11,12,17,19,24]. And we have the following results.

Theorem 3. *Let $\alpha : A\text{-}\underline{\text{mod}} \to B\text{-}\underline{\text{mod}}$ be a stable equivalence without nodes and semisimple summands, then we have*

(1) *A is CM-free if and only if B is* CM-*free.*
(2) *A is CM-finite if and only if B is* CM-*finite.*

Proof. It is from Lemma 5.

Recall that a subcategory $\mathscr{X} \subset A$-mod is said to be contravariantly finite if for any $M \in A$-mod, there exists a morphism $f : X \to M$ with $X \in \mathscr{X}$ such that for any $g : X' \to M$, the term $X' \in \mathscr{X}$ factors through f and the morphism f is called a $\mathscr{X}$-approximation of M. In the following proposition, we prove contravariant finiteness of the category of finitely-generated Gorenstein projective modules denoted by GP-conv-finiteness is preserved under stable equivalence of algebras without nodes. Note that for a finite dimensional k-algebra A, the category of finitely-generated Gorenstein projective modules is not necessarily contravariantly finite in A-mod, see[8,23].

PROPOSITION 3

Let $\alpha : A\text{-}\underline{\text{mod}} \to B\text{-}\underline{\text{mod}}$ be a stable equivalence without nodes and semisimple summands, then $\mathscr{G}\mathscr{P}(A)$ is contravariantly finite if and only if $\mathscr{G}\mathscr{P}(B)$ is contravariantly finite.

Proof. We only prove the ‘only if’ part. The ‘if’ part can be proved by using the inverse of α. Let $X \in B\text{-mod}_{\mathcal{P}}$. Then there exists $M \in A\text{-mod}_{\mathcal{P}}$ such that $\alpha(M) \simeq X$ in B-mod. Since $\mathscr{G}\mathscr{P}(A)$ is contravariantly finite, there exists an exact sequence

$$0 \to K \to P \oplus G \xrightarrow{f} M \to 0,$$

where f is a $\mathcal{GP}(A)$-approximation of M, P is projective and $G \in \mathcal{GP}(A)$ without projective summands. By Theorem 2, we have the following exact sequence:

$$0 \to K' \to P' \oplus \alpha(G) \xrightarrow{g} X \to 0$$

in B-mod with P' projective, and $\underline{g} = \alpha(\underline{f})$. We claim g is a $\mathcal{GP}(B)$-approximation of X.

By Lemma 5, we have $\alpha(G) \in \mathcal{GP}(B)$. Let $h : G' \to X$ be a morphism in B-mod, where $G' \in \mathcal{GP}(B) \cap B\text{-mod}_{\mathcal{P}}$. If h factors through a projective module, then it also factors through g. Assume $\underline{h} \neq 0$, there exists $h' : \alpha^{-1}(G') \to M$ such that $\alpha(\underline{h'}) = \underline{h}$. Since $\alpha^{-1}(G') \in \mathcal{GP}(A)$, we have h' factors through f, i.e., $h' = fa$ for some $a : \alpha^{-1}(G') \to P \oplus G$. Therefore, we have $\underline{h} - \underline{g}\alpha(\underline{a}) = 0$ which shows $h - g\alpha(a)$ factors through g and g is a $\mathcal{GP}(B)$-approximation. So, $\mathcal{GP}(B)$ is contravariantly finite.

The τ-tilting theory introduced in [1] plays an important role in the representation theory of finite-dimensional algebras. In particular, support τ-tilting modules are in bijection with the two-term silting complexes, functorially-finite torsion classes, left-finite semibricks and two-term simple-minded collections [1,3,10].

DEFINITION 4 [1]

Let ${}_AM$ be an A-module, τ be the Auslander–Reiten translation in A-mod. Then we call

(1) M is τ-rigid if $\mathrm{Hom}_A(M, \tau M) = 0$.
(2) M is τ-tilting if M is τ-rigid and $|M| = |{}_AA|$.
(3) M is support τ-tilting if there exists an idempotent e of A such that M is a τ-tilting A/AeA-module.

We will denote by τ-tilt A (respectively, $s\tau$-tilt A) the set of isomorphism classes of basic τ-tilting (respectively, support τ-tilting) A-modules. Given two support τ-tilting A-modules M and N, we say $M \geq N (M > N)$ if $\mathrm{Fac}(M) \supset \mathrm{Fac}(N) (\mathrm{Fac}(M) \supsetneq \mathrm{Fac}(N))$. And $\geq$ gives a partial order on $s\tau$-tilt A. The associated Hasse quiver $Q(s\tau\text{-tilt } A)$ is as follows:

- The set of vertices is isomorphism classes of basic support τ-tilting A-modules.
- Draw an arrow from M to N if $M > N$ and there is no support τ-tilting A-module L such that $M > L > N$.

By Proposition 1.2 in [1], an A-module M is τ-rigid if and only if $\mathrm{Ext}^1_A(T, \mathrm{Fac}\, T) = 0$, where $\mathrm{Fac}\, T$=$\{X \in A\text{-mod} | \exists\, T^{(n)} \to X$ is an epimorphism for some $n \in \mathbb{N}\}$. A is said to be τ-tilting finite if there are only finitely many isomorphism classes of basic τ-tilting A-modules. By Corollary 2.9 in [13], it is equivalent to the condition that there are only finitely many isomorphism classes of indecomposable τ-rigid A-modules.

PROPOSITION 4

Let $\alpha : A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ be a stable equivalence between two algebras without nodes and semisimple summands, and ${}_AT \in A\text{-mod}_{\mathcal{P}}$ be an A-module. Then ${}_AT$ is τ-rigid if and only if $\alpha({}_AT)$ is also τ-rigid.

Proof. Assume ${}_AT \in A\text{-mod}_{\mathcal{P}}$ is τ-rigid. Let $X \in \mathrm{Fac}(\alpha(T))$, then we have $\alpha^{-1}(X) \in \mathrm{Fac}\, T$, by Theorem 2. We have $\mathrm{Ext}^1_B(\alpha(T), X) \simeq \mathrm{Ext}^1_A(T, \alpha^{-1}(X)) = 0$, by Proposition 2 which shows that $\alpha({}_AT)$ is τ-rigid. One can prove the converse is also true similarly.

COROLLARY 2

Let $\alpha : A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ be a stable equivalence between two algebras without nodes and semisimple summands. Then A is τ-tilting finite if and only if B is τ-tilting finite.

COROLLARY 3

Let $\alpha : A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ be a stable equivalence between two algebras without nodes and semisimple summands. Suppose $M \oplus P$ is a basic support τ-tilting A-module with $M \in A\text{-mod}_{\mathcal{P}}$ and P a projective module with no injective direct summands, then $\alpha(M) \oplus \alpha'(P)$ is a basic support τ-tilting module.

Proof. We first show $\alpha(M) \oplus \alpha'(P)$ is τ-rigid. Since $\alpha(M)$ is τ-rigid, by Proposition 4, we just need to prove $\mathrm{Hom}_B(\alpha'(P), \tau_B\alpha(M)) \simeq 0$ which is equivalent to prove $\mathrm{Ext}^1_B(\alpha(M), \mathrm{Fac}(\alpha'(P))) \simeq 0$. By Proposition 2, we have $\mathrm{Ext}^1_B(\alpha(M), \alpha'(P)) \simeq \mathrm{Ext}\, A^1(M, P) \simeq 0$. And, for any $Y \in B\text{-mod}_{\mathcal{P}} \cap \mathrm{Fac}(\alpha'(P))$, we have $Y \simeq \alpha(X)$ for some $X \in A\text{-mod}_{\mathcal{P}} \cap \mathrm{Fac}(P)$, by Theorem 2. Therefore, by Proposition 2 again, we have $\mathrm{Ext}^1_B(\alpha(M), Y) \simeq \mathrm{Ext}^1_A(M, X) \simeq 0$.

Since $\alpha(M) \oplus \alpha'(P)$ is τ-rigid, we know $\mathrm{Fac}(\alpha(M) \oplus \alpha'(P))$ is a functorially finite torsion class. By the correspondence of support τ-tilting modules and functorially finite torsion classes, we have $\mathcal{P}(\mathrm{Fac}(\alpha(M) \oplus \alpha'(P))) = V \oplus \alpha'(P)$ is a support τ-tilting module, where $V \in B\text{-mod}_{\mathcal{P}}$ and $|M| = |\alpha(M)| \le |V|$. We consider $\alpha^{-1}(V) \oplus P$ which is also a τ_A-rigid module. Since $\mathrm{Fac}(\alpha(M) \oplus \alpha'(P)) = \mathrm{Fac}(V \oplus \alpha'(P))$, we have $\mathrm{Fac}(\alpha^{-1}(V) \oplus P) = \mathrm{Fac}(M \oplus P)$, by Theorem 2. Then, because $M \oplus P$ is a basic support τ-tilting A-module, we have $M \oplus P = \mathcal{P}(\mathrm{Fac}(\alpha^{-1}(V) \oplus P))$. So, $|V| = |\alpha^{-1}(V)| \le |M|$ and therefore $\alpha(M) \simeq V$ and $\alpha(M) \oplus \alpha'(P)$ is a support τ-tilting module.

Let e and e' be the idempotents of A and B corresponding to maximal basic projective-injective modules in A-mod and B-mod. The following proposition is similar to Corollary 3.

PROPOSITION 5

Let $\alpha : A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ be a stable equivalence between two algebras without nodes and semisimple summands. Suppose $M \oplus P \oplus Ae$ is a basic support τ-tilting A-module with $M \in A\text{-mod}_{\mathcal{P}}$ and P projective module with no injective direct summands, then $\alpha(M) \oplus \alpha'(P) \oplus Be'$ is a basic support τ-tilting module.

Proof. The proof is similar to Corollary 3. We only need to check $\alpha(M) \oplus \alpha'(P) \oplus Be'$ is τ-rigid, i.e., $\mathrm{Hom}_B(Be', \tau\alpha(M)) \simeq 0$.

It is equivalent to prove $\mathrm{Ext}^1_B(\alpha(M), \mathrm{Fac}(Be')) \simeq 0$. Let $N \in \mathrm{Fac}(Be')$. If it is projective, then $N \in \mathrm{add}(Be')$ which is also injective. So, $\mathrm{Ext}^1_B(\alpha(M), N) = 0$. Now, assume N has no projective direct summands. We have an exact sequence

$$0 \to K \to Q \to N \to 0.$$

By Theorem 2, we have the following exact sequence:

$$0 \to \alpha^{-1}(K) \to Q' \to \alpha^{-1}(N) \to 0$$

where Q' is a projective-injective A-module which shows $\alpha^{-1}(N) \in \mathrm{Fac}(Ae)$. By the assumption that $M \oplus P \oplus Ae$ is a basic support τ-tilting A-module, we have $\mathrm{Ext}^1_A(M, \alpha^{-1}(N)) = 0$. So we have $\mathrm{Ext}^1_B(\alpha(M), N) \simeq \mathrm{Ext}^1_A(M, \alpha^{-1}(N)) = 0$. Therefore, $\alpha(M) \oplus \alpha'(P) \oplus Be'$ is τ-rigid.

In [18], Jasso introduced the technique of reduction of τ-tilting modules. Let ${}_AU$ be a τ-rigid module. In Theorem 3.16 of [18], he proved that there exists an order-preserving bijection between $\mathrm{s}\tau\text{-tilt}_U\, A$ and $\mathrm{s}\tau\text{-tilt}\, C$ for some algebra C. Here, we denote by $\mathrm{s}\tau\text{-tilt}_U\, A$ the set of support τ-tilting A-modules which have U as a direct summand. Moreover, we have that C-mod is equivalent to the wide subcategory $U^{\perp} \cap {}^{\perp}(\tau U)$.

Theorem 4. *Let $\alpha : A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ be a stable equivalence between two algebras without nodes and semisimple summands. Then we have the associated Hasse quivers of $\mathrm{s}\tau\text{-tilt}\, A/AeA$ and $\mathrm{s}\tau\text{-tilt}\, B/Be'B$ are isomorphic.*

Proof. By Jasso's reduction of τ-tilting modules, we have order-preserving bijections between $\mathrm{s}\tau\text{-tilt}_{Ae} A$ and $\mathrm{s}\tau\text{-tilt}\, A/AeA$ and $\mathrm{s}\tau\text{-tilt}_{Be'} B$ and $\mathrm{s}\tau\text{-tilt}\, B/Be'B$. So, we only need to prove $Q(\mathrm{s}\tau\text{-tilt}_{Ae} A) \simeq Q(\mathrm{s}\tau\text{-tilt}_{Be'} B)$. By Proposition 5, we have an isomorphism of the set of vertices of $Q(\mathrm{s}\tau\text{-tilt}_{Ae} A)$ and $Q(\mathrm{s}\tau\text{-tilt}_{Be'} B)$. On the other hand, if there is an arrow from $M \oplus P \oplus Ae$ to $N \oplus Q \oplus Ae$, where $M \oplus P \oplus Ae$ and $N \oplus Q \oplus Ae$ are support τ-tilting A-modules, we claim there is also an arrow between the corresponding support τ-tilting B-modules. If this is not true, there exists another support τ-tilting B-modules $X \oplus P' \oplus Be'$ such that $\mathrm{Fac}(\alpha(M) \oplus \alpha'(P) \oplus Be') \supsetneq \mathrm{Fac}(X \oplus P' \oplus Be') \supsetneq \mathrm{Fac}(\alpha(N) \oplus \alpha'(Q) \oplus Be')$. Then, by Theorem 2, we have $\mathrm{Fac}(M \oplus P \oplus Ae) \supsetneq \mathrm{Fac}(\alpha^{-1}(X) \oplus \alpha'^{-1}(P') \oplus Ae) \supsetneq \mathrm{Fac}(N \oplus Q \oplus Ae)$ which is a contradiction.

As a special case of τ-tilting modules, in the following section, we will also consider tilting modules under stable equivalences.

DEFINITION 5

A basic module ${}_AT$ is called tilting module, if it satisfies the following conditions:

(1) $\mathrm{pd}({}_AT) \leq 1$;
(2) $\mathrm{Ext}^1_A(T, T) = 0$;
(3) There is an exact sequence $0 \to A \to T_0 \to T_1 \to 0$.

It is well-known that there is torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ associated to a tilting module ${}_AT$, where $\mathcal{T}(T) = \mathrm{Gen}(T) = \{X \in A\text{-mod} \mid \mathrm{Ext}^1(T, X) = 0\}$ and $\mathcal{F}(T) = \{Y \in A\text{-mod} \mid \mathrm{Hom}_A(T, Y) = 0\}$. Denoted by A' the endomorphism algebra of ${}_AT$. The famous Brenner–Butler theorem says there are equivalences

$$\mathcal{T}(T) \underset{{}_AT_{A'}\otimes -}{\overset{\mathrm{Hom}_A({}_AT_{A'},-)}{\rightleftarrows}} \mathcal{Y}(T), \qquad \mathcal{F}(T) \underset{\mathrm{Tor}_1^{A'}({}_AT_{A'},-)}{\overset{\mathrm{Ext}^1_A({}_AT_{A'},-)}{\rightleftarrows}} \mathcal{X}(T),$$

where $\mathcal{X}(T) = \{M \in A'\text{-mod} | T \otimes_{A'} M = 0\}$, $\mathcal{Y}(T) = \{N \in A'\text{-mod} | \mathrm{Tor}_1^{A'}(T, N) = 0\}$.

We recall the definition of a special kind of tilting modules, say separating and splitting tilting modules. We say a torsion pair $(\mathcal{T}, \mathcal{F})$ in A-mod is splitting if each indecomposable A-module lies either in $\mathcal{T}$ or in $\mathcal{F}$.

DEFINITION 6

Let A be an algebra, ${}_AT$ be a tilting module, and $A' = \mathrm{End}^{\mathrm{op}}({}_AT)$. Then

(1) ${}_AT$ is said to be separating if the torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in A-mod is splitting, and
(2) ${}_AT$ is said to be splitting if the torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in A'-mod is splitting.

PROPOSITION 6 [2, Proposition 1.7, Theorem 5.6, Chapter 6]

Let A be an algebra, ${}_AT$ be a tilting module, and $A' = \mathrm{End}^{\mathrm{op}}({}_AT)$. Then

(1) *${}_AT$ is separating if and only if for any $M \in \mathcal{T}(T)$, we have $\tau^{-1}M \in \mathcal{T}(T)$, where τ is the Auslander–Reiten translation;*
(2) *${}_AT$ is splitting if and only if $\mathrm{id}(N) = 1$ for every ${}_AN \in \mathcal{F}(T)$.*

A is a finite dimensional k-algebra, and we say a module ${}_AE$ is a projective–injective generator if $\mathrm{add}(E) = \{{}_AX | {}_AX \text{ is projective and injective}\}$.

PROPOSITION 7

Let A and B be finite dimensional k-algebras, $\alpha : A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ be a stable equivalence without nodes and semisimple summands. Assume E_A, E_B are basic projective–injective generators. If ${}_AT = T_0 \oplus P_0 \oplus E_A$ is a basic tilting module, where P_0 is projective non-injective, then we have ${}_BT' = \alpha(T_0) \oplus \alpha'(P_0) \oplus E_B$ is a basic tilting B-module.

Proof. By Proposition 2, we have $\mathrm{pd}({}_BT') \leq 1$. And, by Proposition 2, we obtain that $\mathrm{Ext}_B^1({}_BT', {}_BT') = 0$. Let $A = P \oplus E_A$, then we have an exact sequence $0 \to P \to \hat{T}_0 \to \hat{T}_1 \to 0$ with $\hat{T}_0, \hat{T}_1 \in \mathrm{add}(T)$ and $\hat{T}_1$ has no projective summands. By Remark 1, we have the exact sequence: $0 \to \alpha'(P) \to \hat{T}_0' \to \hat{T}_1' \to 0$ with $\hat{T}_0', \hat{T}_1' \in \mathrm{add}(T')$. Since α' gives a bijection between the isomorphism classes of indecomposable non-injective projective modules over A and B, we have $\alpha'(P) \oplus E_B \simeq {}_BB$. Therefore ${}_BT'$ is a basic tilting B-module.

Lemma 6. *Let $\alpha : A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ be a stable equivalence between two algebras, and ${}_AX$ be an indecomposable non-projective module. Assume $f : X \to X$, $g : \alpha(X) \to \alpha(X)$ such that $\underline{g} = \alpha(\underline{f})$, if f is an isomorphism, then g is also an isomorphism.*

Proof. If g is not an isomorphism, then by Chapter 1, Corollary 4.8(b) in [2], we have that g is nilpotent. So there exists $n > 0$ such that $g^n = 0$ and then $\alpha(\underline{f^n}) = 0$. Therefore f^n factors through projective modules. Since f is an isomorphism, X is isomorphic to a projective module which contradicts the assumption.

For convenience, we introduce a map $\tilde{\alpha}$ which combines α and α'. Let ${}_AM = M_0 \oplus M_1$, where $M_0 \in A\text{-mod}_{\mathcal{P}}$ and M_1 is projective non-injective. Let $\tilde{\alpha}({}_AM) = \alpha(M_0) \oplus \alpha'(M_1)$.

Lemma 7. *Let* $\alpha : A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ *be a stable equivalence between two algebras without nodes and semisimple summands. Assume* ${}_AX \in A\text{-mod}_{\mathcal{P}}$ *is indecomposable, then we have* $\tau_B(\alpha(X)) \simeq \tilde{\alpha}(\tau_A X)$. *Similarly, if* ${}_AX \in A$-mod *is indecomposable non-injective, then* $\tau_B^{-1}(\tilde{\alpha}(X)) \simeq \tilde{\alpha}(\tau_A^{-1}X)$.

Proof. We only prove $\tau_B(\alpha(X)) \simeq \tilde{\alpha}(\tau_A X)$, the other one can be proved similarly. Since ${}_AX$ is indecomposable non-projective, we have an almost split sequence

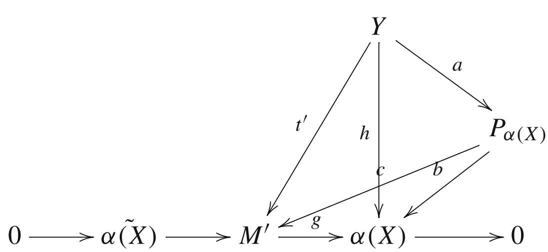
$$0 \to \tau(X) \to M \xrightarrow{f} X \to 0 \tag{$*$}$$

and this sequence is minimal. By Theorem 2, we have an exact sequence

$$0 \to \tilde{\alpha}(\tau(X)) \to M' \xrightarrow{g} \alpha(X) \to 0 \tag{$**$}$$

with $\alpha(\underline{f}) = \underline{g}$ and this sequence is minimal, and we show that it is almost split. Since $\tilde{\alpha}(\tau(X))$ is indecomposable, by Chapter 5, Proposition 1.14 in [7], we just need to prove g is right almost split. Since $(**)$ is minimal, it is not split, and by Chapter 5, Proposition 1.8 in [7], we just need to prove every non-isomorphism $h : Y \to \alpha(X)$ with Y indecomposable factors through g.

Assume that ${}_BY$ is non-projective, then there exists $s : \alpha^{-1}(Y) \to X$ such that $\alpha(\underline{s}) = \underline{h}$ and s is not an isomorphism by the above lemma. Since $(*)$ is almost split, there exists $t : \alpha^{-1}(Y) \to M$ such that $s = ft$. Therefore, we have $\underline{h} = \alpha(\underline{f})\alpha(\underline{t})$ which implies that $h - gt'$ factors through projective modules, hence $P_{\alpha(X)}$ the projective cover of $\alpha(X)$, where t' denotes a lift of $\alpha(\underline{t})$ which means $\underline{t'} = \alpha(\underline{t})$. We have the following commutative diagram:



where $h - gt' = ba = gca$. The existence of c is due to $P_{\alpha(X)}$ which is projective. Therefore, we have $h = g(t' + ca)$ which completes our proof.

COROLLARY 4

Let $\alpha : A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ *be a stable equivalence between two algebras without nodes and semisimple summands. Assume that* ${}_AT$ *is a separating and splitting tilting module, then* T' *constructed in Proposition 7 is also separating and splitting.*

Proof. We have T' is a tilting module by Proposition 7. To show ${}_BT'$ is separating, by Proposition 6, we just need to prove, for any $M \in \mathcal{T}(T')$, we have $\tau^{-1}M \in \mathcal{T}(T')$. We can assume that M is an indecomposable non-injective, then $M \in B\text{-mod}_{\mathcal{P}}$ or M is projective non-injective. We can assume $M \simeq \alpha(N)$ or $M \simeq \alpha'(Q)$ respectively, where

$N \in B\text{-mod}_{\mathcal{P}}$ and Q is a projective non-injective A-module. By Proposition 2 and the definition of $\mathcal{T}(T)$, we have $N \in \mathcal{T}(T)$ and $Q \in \mathcal{T}(T)$. Since ${}_AT$ is separating, we have $\tau^{-1}(N) \in \mathcal{T}(T)$ and $\tau^{-1}(Q) \in \mathcal{T}(T)$. By Lemma 7 and Proposition 2, we have $\alpha(\tau_A^{-1}(N)) \simeq \tau_B^{-1}(M) \in \mathcal{T}(T')$ and $\alpha(\tau_A^{-1}(Q)) \simeq \tau_B^{-1}(P) \in \mathcal{T}(T')$.

To prove ${}_BT'$ is splitting, by Proposition 6, we just need to prove $\mathrm{id}(N') = 1$ for every indecomposable $N' \in \mathcal{F}(T')$. We have $N' \simeq \tilde{\alpha}(N)$ for some indecomposable $N \in A$-mod, since ${}_AT$ is separating, we have $N \in \mathcal{T}(T)$ or $N \in \mathcal{F}(T)$. If $N \in \mathcal{T}(T)$, by Proposition 2, we have $\tilde{\alpha}(N) \in \mathcal{T}(T')$ which is a contradiction. Therefore, $N \in \mathcal{F}(T)$ and $\mathrm{id}(N) = 1$ since ${}_AT$ is splitting. So, we have $\mathrm{id}(N') = 1$, by using Proposition 2.

Question. Are the properties CM-finite, GP-conv-finiteness, and τ-tilting finiteness preserved under stable equivalences between finite dimensional k-algebras?

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