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Moduli spaces of vector bundles on a curve and opers

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Abstract. Let X be a compact connected Riemann surface of genus g, with $g \geq 2$, and let ξ be a holomorphic line bundle on X with $\xi^{\otimes 2} = \mathcal{O}_X$. Fix a theta characteristic $\mathbb{L}$ on X. Let $\mathcal{M}_X(r, \xi)$ be the moduli space of stable vector bundles E on X of rank r such that $\bigwedge^r E = \xi$ and $H^0(X, E \otimes \mathbb{L}) = 0$. Consider the quotient of $\mathcal{M}_X(r, \xi)$ by the involution given by $E \longmapsto E^*$. We construct an algebraic morphism from this quotient to the moduli space of $\mathrm{SL}(r, \mathbb{C})$ opers on X. Since $\dim \mathcal{M}_X(r, \xi)$ coincides with the dimension of the moduli space of $\mathrm{SL}(r, \mathbb{C})$ opers, it is natural to ask about the injectivity and surjectivity of this map.

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1. Introduction

Opers were introduced by Beilinson and Drinfeld [2,3]. Our aim here is to construct $\mathrm{SL}(n, \mathbb{C})$ opers from stable vector bundles of degree zero. While a stable vector bundle of degree zero has a unique unitary flat connection, unitary connections are never an oper.

Take a compact connected Riemann surface of genus g, with $g \geq 2$, and fix a theta characteristic $\mathbb{L}$ on X. Let $\mathcal{M}_X(r)$ be the moduli space of stable vector bundles E of rank r and degree zero on X such that $H^0(X, E \otimes \mathbb{L}) = 0$. For $i = 1, 2$, the projection $X \times X \longrightarrow X$ to the i-th factor is denoted by p_i. The diagonal divisor in $X \times X$ is denoted by Δ; it is identified with X by p_i. For any $E \in \mathcal{M}_X(r)$, there is a unique section

$$\mathcal{A}_E \in H^0(X \times X, \ p_1^*(E \otimes \mathbb{L}) \otimes p_2^*(E^* \otimes \mathbb{L}) \otimes \mathcal{O}_{X \times X}(\Delta))$$

whose restriction to Δ is Id_E (using the identification of Δ with X).

Using $\mathcal{A}_E$, we construct an $\mathrm{SL}(n, \mathbb{C})$ oper on X for every $n \geq 2$; see Theorem 2 and Proposition 3. Related construction of opers from vector bundles were carried out in [5].

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Let $\mathrm{Op}_X(n)$ denote the moduli space of $\mathrm{SL}(n, \mathbb{C})$ opers on X. The above mentioned map

$$\mathcal{M}_X(r) \longrightarrow \mathrm{Op}_X(n)$$

factors through the quotient of $\mathcal{M}_X(r)$ by the involution $\mathcal{I}$ defined by $E \longmapsto E^*$.

Fix a holomorphic line bundle ξ on X such that $\xi^{\otimes 2} = \mathcal{O}_X$. Let

$$\mathcal{M}_X(r, \xi) \subset \mathcal{M}_X(r)$$

be the subvariety defined by the locus of all E such that $\bigwedge^r E = \xi$. We have

$$\dim \mathcal{M}_X(r, \xi)/\mathcal{I} = (r^2 - 1)(g - 1) = \dim \mathrm{Op}_X(r).$$

We end with a question (see Question (6)).

2. Vector bundles with trivial cohomology

Let X be a compact connected Riemann surface of genus g, with $g \geq 2$. The holomorphic cotangent bundle of X will be denoted by K_X. Fix a theta characteristic $\mathbb{L}$ on X. So, $\mathbb{L}$ is a holomorphic line bundle on X of degree $g - 1$, and $\mathbb{L} \otimes \mathbb{L}$ is holomorphically isomorphic to K_X.

For any $r \geq 1$, let $\tilde{\mathcal{M}}_X(r)$ denote the moduli space of stable vector bundles on X of rank r and degree zero. It is an irreducible smooth complex quasiprojective variety of dimension $r^2(g - 1) + 1$. Let

$$\mathcal{M}_X(r) \subset \tilde{\mathcal{M}}_X(r) \tag{1}$$

be the locus of all vector bundles $E \in \tilde{\mathcal{M}}_X(r)$ such that $H^0(X, E \otimes \mathbb{L}) = 0$. From the semicontinuity theorem, [10, p. 288, Theorem 12.8], we know that $\mathcal{M}_X(r)$ is a Zariski open subset of $\tilde{\mathcal{M}}_X(r)$. In fact, $\mathcal{M}_X(r)$ is known to be the complement of a theta divisor on $\tilde{\mathcal{M}}_X(r)$ [11]. For any $E \in \tilde{\mathcal{M}}_X(r)$, the Riemann–Roch theorem says

$$\dim H^0(X, E \otimes \mathbb{L}) - \dim H^1(X, E \otimes \mathbb{L}) = 0;$$

so $H^0(X, E \otimes \mathbb{L}) = 0$ if and only if we have $H^1(X, E \otimes \mathbb{L}) = 0$.

We will now recall a construction from [6,7].

For $i = 1, 2$, let $p_i : X \times X \longrightarrow X$ be the projection to the i-th factor. Let

$$\Delta := \{(x, x) \in X \times X \mid x \in X\} \subset X \times X$$

be the reduced diagonal divisor. We will identify Δ with X using the map $x \longmapsto (x, x)$. Using this identification, the restriction of the line bundle $\mathcal{O}_{X\times X}(\Delta)$ to $\Delta \subset X \times X$ gets identified with the holomorphic tangent bundle TX by the Poincaré adjunction formula [9, p. 146].

Take any $E \in \mathcal{M}_X(r)$. The restriction of the vector bundle

$$p_1^*(E \otimes \mathbb{L}) \otimes p_2^*(E^* \otimes \mathbb{L}) \otimes \mathcal{O}_{X\times X}(\Delta)$$

to Δ is identified with the vector bundle $\mathrm{End}(E)$ on X. Indeed, this follows immediately from the following facts:

- the restriction of $(p_1^* E) \otimes (p_2^* E^*)$ to Δ is identified with the vector bundle $\mathrm{End}(E)$ on X, and

- the above identification of $\mathcal{O}_{X\times X}(\Delta)|_\Delta$ with TX produces an identification of $(p_1^*\mathbb{L})\otimes(p_2^*\mathbb{L})\otimes\mathcal{O}_{X\times X}(\Delta)|_\Delta$ with $K_X\otimes TX = \mathcal{O}_X$.

Consequently, we have the following short exact sequence of sheaves on $X\times X$:

$$0 \longrightarrow p_1^*(E\otimes\mathbb{L})\otimes p_2^*(E^*\otimes\mathbb{L}) \longrightarrow p_1^*(E\otimes\mathbb{L}) \otimes p_2^*(E^*\otimes\mathbb{L})\otimes\mathcal{O}_{X\times X}(\Delta) \longrightarrow \mathrm{End}(E) \longrightarrow 0, \tag{2}$$

where $\mathrm{End}(E)$ is supported on Δ, using the identification of Δ with X. For $k = 0, 1$, since $H^k(X, E\otimes\mathbb{L}) = 0$, the Serre duality implies that $H^{1-k}(X, E^*\otimes\mathbb{L}) = 0$. Using Künneth formula,

$$H^j(X\times X, p_1^*(E\otimes\mathbb{L})\otimes p_2^*(E^*\otimes\mathbb{L})) = 0$$

for $j = 0, 1, 2$. Therefore, the long exact sequence of cohomologies for the short exact sequence of sheaves in (2) gives

$$\begin{aligned} 0 = {} & H^0(X\times X, p_1^*(E\otimes\mathbb{L})\otimes p_2^*(E^*\otimes\mathbb{L})) \longrightarrow \\ & H^0(X\times X, p_1^*(E\otimes\mathbb{L})\otimes p_2^*(E^*\otimes\mathbb{L})\otimes\mathcal{O}_{X\times X}(\Delta)) \\ & \xrightarrow{\gamma} H^0(X, \mathrm{End}(E)) \longrightarrow \\ & H^1(X\times X, p_1^*(E\otimes\mathbb{L})\otimes p_2^*(E^*\otimes\mathbb{L})) = 0. \end{aligned} \tag{3}$$

So the homomorphism γ in (3) is actually an isomorphism. For this isomorphism γ, let

$$\mathcal{A}_E := \gamma^{-1}(\mathrm{Id}_E) \in H^0(X\times X, p_1^*(E\otimes\mathbb{L})\otimes p_2^*(E^*\otimes\mathbb{L})\otimes\mathcal{O}_{X\times X}(\Delta)) \tag{4}$$

be the section corresponding to the identity automorphism of E.

3. A section around the diagonal

Using the section $\mathcal{A}_E$ in (4), we will construct a section of $(p_1^*\mathbb{L})\otimes(p_2^*\mathbb{L})\otimes\mathcal{O}_{X\times X}(\Delta)$ on an analytic neighborhood of the diagonal Δ. For that, we first recall a description of the holomorphic differential operators on X.

3.1 *Differential operators*

Fix holomorphic vector bundles V, W on X, and also fix an integer $d \geq 1$. The ranks of V and W are denoted by r and r' respectively. Let $\mathrm{Diff}^d_X(V, W)$ denote the holomorphic vector bundle on X of rank $rr'(d+1)$ corresponding to the sheaf of differential operators of degree d from V to W. We recall that $\mathrm{Diff}^d_X(V, W) = W\otimes J^d(V)^*$, where

$$J^d(V) := p_{1*}\big((p_2^*V)/((p_2^*V)\otimes\mathcal{O}_{X\times X}(-(d+1)\Delta))\big) \longrightarrow X$$

is the d-th order jet bundle for V.

We have a short exact sequence of coherent analytic sheaves on $X\times X$ as follows:

$$\begin{aligned} 0 \longrightarrow (p_1^*W)\otimes p_2^*(V^*\otimes K_X) \longrightarrow (p_1^*W)\otimes p_2^*(V^*\otimes K_X)\otimes\mathcal{O}_{X\times X}((d+1)\Delta) \\ \longrightarrow \mathcal{Q}_d(V, W) := \frac{(p_1^*W)\otimes p_2^*(V^*\otimes K_X)\otimes\mathcal{O}_{X\times X}((d+1)\Delta)}{(p_1^*W)\otimes p_2^*(V^*\otimes K_X)} \longrightarrow 0; \end{aligned} \tag{5}$$

the support of the quotient sheaf $\mathcal{Q}_d(V, W)$ in (5) is $(d+1)\Delta$. The direct image

$$\mathcal{K}_d(V, W) := p_{1*}\mathcal{Q}_d(V, W) \longrightarrow X \tag{6}$$

is a holomorphic vector bundle on X of rank $rr'(d+1)$. It is known that

$$\mathcal{K}_d(V, W) = \mathrm{Hom}(J^d(V), W) = \mathrm{Diff}^d_X(V, W), \tag{7}$$

where $\mathcal{K}_d(V, W)$ is the vector bundle constructed in (6) (see [4, Section 2.1]).

Note that $R^1 p_{1*}\mathcal{Q}_d(V, W) = 0$, because $\mathcal{Q}_d(V, W)$ is supported on $(d+1)\Delta$. We have $H^0(X, p_{1*}\mathcal{Q}_d(V, W)) = H^0(X \times X, \mathcal{Q}_d(V, W))$. So from (6) and (7), it follows that

$$H^0(X, \mathrm{Diff}^d_X(V, W)) = H^0(X \times X, \mathcal{Q}_d(V, W)). \tag{8}$$

The restriction of the vector bundle $(p_1^* W) \otimes p_2^*(V^* \otimes K_X) \otimes \mathcal{O}_{X\times X}((d+1)\Delta)$ to $\Delta \subset X \times X$ is $\mathrm{Hom}(V, W) \otimes (TX)^{\otimes d}$, because the restriction of $\mathcal{O}_{X\times X}(\Delta)$ to Δ is TX. Therefore, we get a surjective homomorphism

$$\mathcal{K}_d(V, W) \longrightarrow p_{1*}\left(\frac{(p_1^* W) \otimes p_2^*(V^* \otimes K_X) \otimes \mathcal{O}_{X\times X}((d+1)\Delta)}{(p_1^* W) \otimes p_2^*(V^* \otimes K_X) \otimes \mathcal{O}_{X\times X}(d\Delta)}\right)$$
$$= \mathrm{Hom}(V, W) \otimes (TX)^{\otimes d},$$

where $\mathcal{K}_d(V, W)$ is constructed in (6). Using (7), this gives a surjective homomorphism

$$\mathrm{Diff}^d_X(V, W) \longrightarrow \mathrm{Hom}(V, W) \otimes (TX)^{\otimes d}. \tag{9}$$

The homomorphism in (9) is known as the *symbol map*.

3.2 *Construction of a connection*

Consider the de Rham differential operator $d : \mathcal{O}_X \longrightarrow K_X$. Using the isomorphism in (8), this d produces a section

$$d_1 \in H^0(X \times X, \mathcal{Q}_1(\mathcal{O}_X, K_X)).$$

From (5), we conclude that d_1 is a section of $(p_1^* K_X) \otimes (p_2^* K_X) \otimes \mathcal{O}_{X\times X}(2\Delta)$ over 2Δ. The restriction of d_1 to $\Delta \subset 2\Delta$ is the section of $\mathcal{O}_X$ given by the constant function 1 (see (9)); note that the symbol of the differential operator d is the constant function 1.

As before, $\mathbb{L}$ is a theta characteristic on X. There is a unique section

$$\delta \in H^0(2\Delta, (p_1^*\mathbb{L}) \otimes (p_2^*\mathbb{L}) \otimes \mathcal{O}_{X\times X}(\Delta)) \tag{10}$$

such that

(1) $\delta \otimes \delta = d_1$, and
(2) the restriction of δ to $\Delta \subset 2\Delta$ is the constant function 1 (note that since the restriction of $\mathcal{O}_{X\times X}(\Delta)$ to Δ is TX, the restriction of $(p_1^*\mathbb{L}) \otimes (p_2^*\mathbb{L}) \otimes \mathcal{O}_{X\times X}(\Delta)$ to Δ is $K_X \otimes TX = \mathcal{O}_X$).

See [8, p. 754, Theorem 2.1(b)] for an alternative construction of δ.

There is a unique section

$$\Phi_E \in H^0(2\Delta, (p_1^* E) \otimes (p_2^* E^*)) \tag{11}$$

such that $(\mathcal{A}_E)|_{2\Delta} = \Phi_E \otimes \delta$, where $\mathcal{A}_E$ and δ are the sections in (4) and (10) respectively. Indeed, this follows immediately from the fact that the section δ is nowhere zero, so δ^{-1} is a holomorphic section of $((p_1^*\mathbb{L}) \otimes (p_2^*\mathbb{L}) \otimes \mathcal{O}_{X\times X}(\Delta))^*|_{2\Delta}$. Now set

$$\Phi_E = ((\mathcal{A}_E)\big|_{2\Delta}) \otimes \delta^{-1}$$

and consider it as a section of $((p_1^* E) \otimes (p_2^* E^*))\big|_{2\Delta}$ using the natural duality pairing

$$((p_1^*\mathbb{L}) \otimes (p_2^*\mathbb{L}) \otimes \mathcal{O}_{X\times X}(\Delta))|_{2\Delta} \otimes ((p_1^*\mathbb{L}) \otimes (p_2^*\mathbb{L}) \otimes \mathcal{O}_{X\times X}(\Delta))^*|_{2\Delta} \longrightarrow \mathcal{O}_{2\Delta}.$$

Since the restriction of $\mathcal{A}_E$ to Δ is Id_E (see (4)), and the restriction of δ to Δ is the constant function 1, it follows that the restriction of the section Φ_E in (11) to Δ is Id_E. Therefore, Φ_E defines a holomorphic connection on U, which will be denoted by D^E. To describe the connection D^E explicitly, take an open subset $U \subset X$ and a holomorphic section $s \in H^0(U, E\big|_U)$. Consider the section $\Phi_E \otimes p_2^* s$ over $\mathcal{U} := (2\Delta) \bigcap (U \times U)$. Using the natural pairing $E^* \otimes E \longrightarrow \mathcal{O}_X$, it produces a section of $((p_1^* E) \otimes (p_2^* \mathcal{O}_X))|_{\mathcal{U}} = (p_1^* E)|_{\mathcal{U}}$; denote this section of $(p_1^* E)|_{\mathcal{U}}$ by $\tilde{s}$. Since $\Phi_E|_\Delta = \mathrm{Id}_E$, we know that $\tilde{s}$ and $p_1^* s$ coincide on $\Delta \bigcap (U \times U)$. So

$$\tilde{s} - (p_1^* s)|_{2\Delta} \in H^0(U, E \otimes K_X);$$

the Poincaré adjunction formula identifies K_X with the restriction of the line bundle $\mathcal{O}_{X\times X}(-\Delta)$ to Δ. Then we have

$$D^E(s) = \tilde{s} - (p_1^* s)|_{2\Delta} \in H^0(U, E \otimes K_X). \tag{12}$$

It is straightforward to check that D^E satisfies the Leibniz identity thus making it a holomorphic connection on E.

4. Opers from vector bundles

We will construct an $\mathrm{SL}(n, \mathbb{C})$-oper on X, for every $n \geq 2$, from the section $\mathcal{A}_E$ in (4).

We will use that any holomorphic connection on a holomorphic bundle over X is integrable (same as flat) because $\Omega_X^2 = 0$.

As before, take any $E \in \mathcal{M}_X(r)$. Consider the holomorphic connection D^E on E in (12). Let $U \subset X$ be a simply connected open subset, and let $x_0 \in U$ be a point. Since the connection D^E is integrable, using parallel translations, for D^E, along paths emanating from x_0, we get a holomorphic isomorphism of $E|_U$ with the trivial vector bundle $U \times E_{x_0} \longrightarrow U$. This isomorphism takes the connection $D^E|_U$ on $E|_U$ to the trivial connection on the trivial bundle. Let

$$\Delta \subset \mathcal{U} \subset X \times X$$

be an open neighborhood of Δ that admits a deformation retraction to Δ. For $i = 1, 2$, the restriction of the projection $p_i : X \times X \longrightarrow X$ to the open subset $\mathcal{U} \subset X \times X$ will be denoted by q_i. There is a unique holomorphic isomorphism over $\mathcal{U}$,

$$f : q_1^* E \longrightarrow q_2^* E \tag{13}$$

that satisfies the following two conditions:

(1) the restriction of f to Δ is the identity map of E, and
(2) f takes the connection $q_1^* D^E$ on $q_1^* E$ to the connection $q_2^* D^E$ on $q_2^* E$.

Since the inclusion map $\Delta \hookrightarrow \mathcal{U}$ is a homotopy equivalence, the flat vector bundle $(E,\ D^E)$ on $X = \Delta$ has a unique extension to a flat vector bundle on $\mathcal{U}$. On the other hand, both $(q_1^* E,\ q_1^* D^E)$ and $(q_2^* E,\ q_2^* D^E)$ are extensions of $(E,\ D^E)$. Therefore, there is a unique holomorphic isomorphism f as in (13) satisfying the above two conditions.

Using the isomorphism f in (13), the section $\mathcal{A}_E$ in (4) produces a holomorphic section

$$\mathcal{A}'_E \in H^0(\mathcal{U},\ p_1^*(E \otimes E^* \otimes \mathbb{L}) \otimes (p_2^* \mathbb{L}) \otimes \mathcal{O}_{X \times X}(\Delta)) . \tag{14}$$

Consider the trace pairing

$$tr\ :\ E \otimes E^* = \mathrm{End}(E) \longrightarrow \mathcal{O}_X,\quad B \longmapsto \frac{1}{r}\mathrm{trace}(B);$$

recall that $r = \mathrm{rank}(E)$. Note that $r \cdot tr$ is the natural pairing $E \otimes E^* \longrightarrow \mathcal{O}_X$. Using tr, the section $\mathcal{A}'_E$ in (14) produces a section

$$\widehat{\beta}_E \in H^0(\mathcal{U},\ (p_1^* \mathbb{L}) \otimes (p_2^* \mathbb{L}) \otimes \mathcal{O}_{X \times X}(\Delta)) . \tag{15}$$

The following lemma is straightforward to prove.

Lemma 1. *The restriction of the section $\hat{\beta}_E$ (in (15)) to $2\Delta \subset \mathcal{U}$ coincides with the section δ in (10).*

Proof. This follows from the constructions of Φ_E (in (11)) and $\hat{\beta}_E$. □

For any integer $k \geq 1$, the holomorphic line bundles $\mathbb{L}^{\otimes k}$ and $(\mathbb{L}^{\otimes k})^*$ will be denoted by $\mathbb{L}^k$ and $\mathbb{L}^{-k}$ respectively.

Theorem 2. *Take any integer $n \geq 2$. The section $\hat{\beta}_E$ in (15) produces a holomorphic connection $\mathcal{D}(E)$ on the holomorphic vector bundle $J^{n-1}(\mathbb{L}^{(1-n)})$.*

Proof. Consider the $(n+1)$-th tensor power of $\hat{\beta}_E$:

$$(\hat{\beta}_E)^{\otimes(n+1)} \in H^0(\mathcal{U},\ (p_1^* \mathbb{L}^{(n+1)}) \otimes (p_2^* \mathbb{L}^{(n+1)}) \otimes \mathcal{O}_{X \times X}((n+1)\Delta)),$$

and restrict it to $(n+1)\Delta \subset \mathcal{U}$. From (8), we have

$$\beta_E^n := (\hat{\beta}_E)^{\otimes(n+1)}|_{(n+1)\Delta} \in H^0(X,\ \mathrm{Diff}_X^n(\mathbb{L}^{(1-n)},\ \mathbb{L}^{(n+1)})) . \tag{16}$$

The symbol of the differential operator β_E^n in (16) is the section of $\mathcal{O}_X$ given by the constant function 1. Indeed, this follows immediately from the fact that the restriction of $\hat{\beta}_E$ to Δ is the constant function 1 (see Lemma 1).

We recall that there is a natural injective homomorphism $J^{m+n}(V) \longrightarrow J^m(J^n(V))$ for all $m,\ n \geq 0$ and every holomorphic vector bundle V. We have following commutative diagram of vector bundles on X:

$$\begin{array}{ccccccccc} 0 \longrightarrow \mathbb{L}^{(1-n)} \otimes K_X^{\otimes n} = \mathbb{L}^{(n+1)} & \xrightarrow{\iota_1} & J^n(\mathbb{L}^{(1-n)}) & \xrightarrow{\psi_1} & J^{n-1}(\mathbb{L}^{(1-n)}) \longrightarrow 0 \\ \downarrow & & \downarrow \mathbf{h} & & \| \\ 0 \longrightarrow J^{n-1}(\mathbb{L}^{(1-n)}) \otimes K_X & \xrightarrow{\iota_2} & J^1(J^{n-1}(\mathbb{L}^{(1-n)})) & \xrightarrow{\psi_2} & J^{n-1}(\mathbb{L}^{(1-n)}) \longrightarrow 0 \end{array}, \tag{17}$$

where the rows are exact. From (7), we know that the differential operator β_E^n in (16) produces a homomorphism

$$\rho : J^n(\mathbb{L}^{(1-n)}) \longrightarrow \mathbb{L}^{(n+1)}.$$

Since the symbol of β_E^n is the constant function 1, we have

$$\rho \circ \iota_1 = \mathrm{Id}_{\mathbb{L}^{(n+1)}}, \tag{18}$$

where ι_1 is the homomorphism in (17). From (18), it follows immediately that ρ gives a holomorphic splitting of the top exact sequence in (17). Let

$$\mathcal{D}_1 : J^{n-1}(\mathbb{L}^{(1-n)}) \longrightarrow J^n(\mathbb{L}^{(1-n)})$$

be the unique holomorphic homomorphism such that

- $\rho \circ \mathcal{D}_1 = 0$, and
- $\psi_1 \circ \mathcal{D}_1 = \mathrm{Id}_{J^{n-1}(\mathbb{L}^{(1-n)})}$, where ψ_1 is the projection in (17).

Now consider the homomorphism

$$\mathcal{D}_2 := \mathbf{h} \circ \mathcal{D}_1 : J^{n-1}(\mathbb{L}^{(1-n)}) \longrightarrow J^1(J^{n-1}(\mathbb{L}^{(1-n)})), \tag{19}$$

where $\mathbf{h}$ is the homomorphism in (17). Since $\psi_1 \circ \mathcal{D}_1 = \mathrm{Id}_{J^{n-1}(\mathbb{L}^{(1-n)})}$, from the commutativity of (17), it follows that

$$\psi_2 \circ \mathcal{D}_2 = \psi_2 \circ \mathbf{h} \circ \mathcal{D}_1 = \mathrm{Id}_{J^{n-1}(\mathbb{L}^{(1-n)})} \circ \psi_1 \circ \mathcal{D}_1 = \mathrm{Id}_{J^{n-1}(\mathbb{L}^{(1-n)})},$$

where ψ_2 is the projection in (17). This implies that $\mathcal{D}_2$ in (19) gives a holomorphic splitting of the bottom exact sequence in (17). Let

$$\mathcal{D}(E) : J^1(J^{n-1}(\mathbb{L}^{(1-n)})) \longrightarrow J^{n-1}(\mathbb{L}^{(1-n)}) \otimes K_X$$

be the unique holomorphic homomorphism such that

- $\mathcal{D}(E) \circ \mathcal{D}_2 = 0$, and
- $\mathcal{D}(E) \circ \iota_2 = \mathrm{Id}_{J^{n-1}(\mathbb{L}^{(1-n)}) \otimes K_X}$, where ι_2 is the homomorphism in (17).

Using (7), we know that

$$\mathcal{D}(E) \in H^0(X, \mathrm{Diff}_X^1(J^{n-1}(\mathbb{L}^{(1-n)}), J^{n-1}(\mathbb{L}^{(1-n)}) \otimes K_X)). \tag{20}$$

From the homomorphism in (9) and the above equality $\mathcal{D}(E) \circ \iota_2 = \mathrm{Id}_{J^{n-1}(\mathbb{L}^{(1-n)}) \otimes K_X}$, it follows that the symbol of the differential operator $\mathcal{D}(E)$ in (20) is $\mathrm{Id}_{J^{n-1}(\mathbb{L}^{(1-n)})}$. This implies that $\mathcal{D}(E)$ satisfies the Leibniz rule. Consequently, $\mathcal{D}(E)$ is a holomorphic connection on the holomorphic vector bundle $J^{n-1}(\mathbb{L}^{(1-n)})$. □

For $1 \le i \le n-1$, consider the short exact sequence

$$0 \longrightarrow \mathbb{L}^{(1-n)} \otimes K_X^{\otimes i} \longrightarrow J^i(\mathbb{L}^{(1-n)}) \longrightarrow J^{i-1}(\mathbb{L}^{(1-n)}) \longrightarrow 0.$$

Using these together with the fact that $\mathbb{L} \otimes \mathbb{L} = K_X$, it is deduced that

$$\det J^i(\mathbb{L}^{(1-n)}) := \bigwedge^{i+1} J^i(\mathbb{L}^{(1-n)}) = \mathbb{L}^{(i+1)(i+1-n)}.$$

In particular, we have

$$\det J^{n-1}(\mathbb{L}^{(1-n)}) := \bigwedge^{n} J^{n-1}(\mathbb{L}^{(1-n)}) = \mathcal{O}_X.$$

So $\det J^{n-1}(\mathbb{L}^{(1-n)})$ has a unique holomorphic connection whose monodromy is trivial; it will be called the trivial connection on $\det J^{n-1}(\mathbb{L}^{(1-n)})$.

PROPOSITION 3

The holomorphic connection on $\det J^{n-1}(\mathbb{L}^{(1-n)})$ *induced by the connection* $\mathcal{D}(E)$ *on* $J^{n-1}(\mathbb{L}^{(1-n)})$ (*see Theorem* 2) *is the trivial connection.*

Proof. Any holomorphic connection D on $\det J^{n-1}(\mathbb{L}^{(1-n)}) = \mathcal{O}_X$ can be uniquely expressed as

$$D = D_0 + \omega,$$

where D_0 is the trivial connection on $\det J^{n-1}(\mathbb{L}^{(1-n)})$ and $\omega \in H^0(X, K_X)$. Let D^1 be the holomorphic connection on $\det J^{n-1}(\mathbb{L}^{(1-n)})$ induced by the connection $\mathcal{D}(E)$ on $J^{n-1}(\mathbb{L}^{(1-n)})$. Decompose it as

$$D^1 = D_0 + \omega^1,$$

where $\omega^1 \in H^0(X, K_X)$. Then

$$\omega^1 = (n-1) \cdot ((\hat{\beta}_E)|_{2\Delta} - \delta),$$

where $\hat{\beta}_E$ and δ are the sections in (15) and (10) respectively. Note that two sections of $((p_1^*\mathbb{L}) \otimes (p_2^*\mathbb{L}) \otimes \mathcal{O}_{X\times X}(\Delta))|_{2\Delta}$ that coincide on $\Delta \subset 2\Delta$ differ by an element of $H^0(\Delta, ((p_1^*\mathbb{L})\otimes(p_2^*\mathbb{L}))|_\Delta) = H^0(X, K_X)$. Now from Lemma 1, it follows that $\omega^1 = 0$. □

Let $\mathrm{Op}_X(n)$ denote the moduli space of $\mathrm{SL}(n, \mathbb{C})$ opers on X [3]. It is a complex affine space of dimension $(n^2-1)(g-1)$. We recall a description of $\mathrm{Op}_X(n)$. Let $C_n(X)$ denote the space of all holomorphic connections D' on $J^{n-1}(\mathbb{L}^{(1-n)})$ such that the holomorphic connection on $\det J^{n-1}(\mathbb{L}^{(1-n)})$ induced by D' is the trivial connection on $\det J^{n-1}(\mathbb{L}^{(1-n)}) = \mathcal{O}_X$. Then

$$\mathrm{Op}_X(n) = C_n(X)/\mathrm{Aut}(J^{n-1}(\mathbb{L}^{(1-n)})),$$

where $\mathrm{Aut}(J^{n-1}(\mathbb{L}^{(1-n)}))$ denotes the group of all holomorphic automorphisms of the vector bundle $J^{n-1}(\mathbb{L}^{(1-n)})$; note that $\mathrm{Aut}(J^{n-1}(\mathbb{L}^{(1-n)}))$ has a natural action on $C_n(X)$.

The moduli space $\mathrm{Op}_X(n)$ also coincides with the space of all holomorphic differential operators

$$\mathcal{B} \in H^0(X, \mathrm{Diff}_X^n(\mathbb{L}^{(1-n)}, \mathbb{L}^{(n+1)}))$$

such that

(1) the symbol of $\mathcal{B}$ is the section of $\mathcal{O}_X$ given by the constant function 1, and
(2) the sub-leading term of $\mathcal{B}$ vanishes.

From Theorem 2 and and Proposition 3, we get an algebraic morphism

$$\tilde{\Psi} : \mathcal{M}_X(r) \longrightarrow \mathrm{Op}_X(n) \tag{21}$$

that sends any $E \in \mathcal{M}_X(r)$ to the image in $\mathrm{Op}_X(n)$ of the holomorphic connection $\mathcal{D}(E)$ (see Theorem 2).

Since $H^i(X, E \otimes \mathbb{L}) = H^{1-i}(X, E^* \otimes \mathbb{L})$ (Serre duality), we have an involution

$$\mathcal{I} : \mathcal{M}_X(r) \longrightarrow \mathcal{M}_X(r), \quad F \longmapsto F^*. \tag{22}$$

Let

$$\tau : X \times X \longrightarrow X \times X, \quad (x_1, x_2) \longmapsto (x_2, x_1) \tag{23}$$

be the involution.

PROPOSITION 4

For any $E \in \mathcal{M}_X(r)$, the sections

$$\mathcal{A}_E \in H^0(X \times X, p_1^*(E \otimes \mathbb{L}) \otimes p_2^*(E^* \otimes \mathbb{L}) \otimes \mathcal{O}_{X\times X}(\Delta))$$

and

$$\mathcal{A}_{\mathcal{I}(E)} \in H^0(X \times X, p_1^*(E^* \otimes \mathbb{L}) \otimes p_2^*(E \otimes \mathbb{L}) \otimes \mathcal{O}_{X\times X}(\Delta))$$

(see (4) for $\mathcal{A}_E$ and (22) for $\mathcal{I}$) satisfy the equation

$$\tau^*\mathcal{A}_E = \mathcal{A}_{\mathcal{I}(E)},$$

where τ is the involution in (23).

Proof. We recall that $\mathcal{A}_E$ is the unique section of $p_1^*(E^* \otimes \mathbb{L}) \otimes p_2^*(E \otimes \mathbb{L}) \otimes \mathcal{O}_{X\times X}(\Delta)$ over $X \times X$ whose restriction to Δ coincides with Id_E using the identification of Δ with X. Now the restriction of $\tau^*\mathcal{A}_{\mathcal{I}(E)}$ to Δ is also Id_E. So, $\tau^*\mathcal{A}_{\mathcal{I}(E)} = \mathcal{A}_E$, which implies that $\tau^*\mathcal{A}_E = \mathcal{A}_{\mathcal{I}(E)}$. □

COROLLARY 5

The map $\tilde{\Psi}$ *in* (21) *descends to a map*

$$\tilde{\Psi}^0 : \mathcal{M}_X(r)/\mathcal{I} \longrightarrow \mathrm{Op}_X(n),$$

where $\mathcal{I}$ *is the involution in* (22).

Let ξ be a holomorphic line bundle on X such that $\xi \otimes \xi = \mathcal{O}_X$; for example, ξ can be $\mathcal{O}_X$. Let

$$\mathcal{M}_X(r, \xi) \subset \mathcal{M}_X(r)$$

be the sub-variety consisting of all $E \in \mathcal{M}_X(r)$ such that $\bigwedge^r E = \xi$. Since $\xi^{\otimes 2} = \mathcal{O}_X$, it follows that

$$\mathcal{I}(\mathcal{M}_X(r, \xi)) = \mathcal{M}_X(r, \xi),$$

where $\mathcal{I}$ is defined in (22). So restricting the map $\tilde{\Psi}^0$ in Corollary 5 to $\mathcal{M}_X(r, \xi)/\mathcal{I}$, we get morphism

$$\Psi : \mathcal{M}_X(r, \xi)/\mathcal{I} \longrightarrow \mathrm{Op}_X(n)\,. \tag{24}$$

We note that when $n = r$,

$$\dim \mathcal{M}_X(r, \xi)/\mathcal{I} = (r^2 - 1)(g - 1) = \dim \mathrm{Op}_X(r).$$

So it is natural to ask the following question.

Question 6. When $n = r$, how close is the map Ψ (constructed in (24)) to being injective or surjective?

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