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Topological differences at infinity for nonlinear problems related to the fractional Laplacian

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Abstract. In this paper, we are interested in the fractional Yamabe-type equation $A_s u = u^{\frac{n+2s}{n-2s}}$, $u > 0$ in Ω and $u = 0$ on $\partial\Omega$. Here Ω is a regular bounded domain of $\mathbb{R}^n$, $n \geq 2$ and A_s, $s \in (0, 1)$ represents the fractional Laplacian operator in Ω with zero Dirichlet boundary condition. Based on the theory of critical points at infinity of Bahri and the localization technique of Caffarelli and Silvestre, we compute the difference of topology induced by the critical points at infinity between the level sets of the variational functional associated to the problem. Our result can be seen as a nonlocal analog of the theorem of Bahri *et al*. (*Cal. Var. Partial. Differ. Equ.* **3** (1995) 67–94) on the classical Yamabe-type equation.

Keywords. Fractional partial differential equation; variational method; critical exponent; loss of compactness.

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1. Introduction

In this paper, we consider the nonlinear fractional Yamabe-type problem

$$\begin{cases} A_s u = u^p, \\ u > 0 \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n, n \geq 2$ is a regular bounded domain, $p = \frac{n+2s}{n-2s}$, $s \in (0, 1)$ and A_s represents the fractional Dirichlet Laplacian operator $(-\Delta)^s$ in Ω defined by using the spectrum of the Laplacian $-\Delta$ in Ω with zero Dirichlet boundary condition. It can be viewed as the nonlocal version of the Yamabe-type equation

$$\begin{cases} -\Delta u = u^{\frac{n+2}{n-2}}, \\ u > 0 \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega. \end{cases} \tag{1.2}$$

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Fractional equations involving $(-\Delta)^s$ has attracted the attention of a lot of researchers as it naturally appears in many fields in various scientific areas. The nonlocal character of the fractional Laplacian makes it difficult to handle. After the paper of Caffarelli and Silvestre [15] who provided a local interpretation to the fractional Laplacian in one more dimension, a large number of studies have been developed. In [14], Cabré and Tan studied the subcritical cases; that is, equation (1.1) with subcritical nonlinearities ($p < \frac{n+2s}{n-2s}$) in the particular case $s = 1/2$. They transform the equation in a local form as the Caffarelli-Silvestre extension and established the existence of positive solutions. For similar extensions, we refer to [13,16,28].

Motivated by the work of Pohozaev [22] on equation (1.2), Tan [27] proved that equation (1.1) has no solutions if Ω is a star-shaped domain and $s = 1/2$. The resemblance between (1.1) and (1.2) led the authors in [1] to investigate the effect of the topology of Ω on the existence of solutions of (1.1). Such a result can be seen as the fractional counterpart of the famous result of Bahri and Coron [9]. For more recent results on (1.1) and related problems, we refer to [2–10,18,23] and the references therein.

Problem (1.1) is delicate from the variational viewpoint because the failure of the Palais-Smale condition (PS). This leads to the possibility of existence of non-compact gradient-flow lines along which the associated variational functional J is bounded and its gradient tends to zero, the so-called critical points at infinity, see [7].

Trying to prove the existence of solutions of (1.1) by studying the topological differences between the level sets of J, it will be useful to compute the topological contributions of the critical points at infinity between these level sets. The main purpose of the present paper is to characterize the critical points at infinity of problem (1.1) and evaluate its topological contributions. We shall prove a fractional analog of the theorem of Bahri *et al.* [11] on the classical Yamabe-type equation.

2. General framework and statement of results

We start this section by recalling some preliminaries related to the fractional Laplacian. Let $(e_k)_{k\in\mathbb{N}}$ be the basis of $L^2(\Omega)$ such that for any $k \in \mathbb{N}$, $\|e_k\|_{L^2(\Omega)} = 1$, $\langle e_k, e_\ell\rangle = 0$, $\forall k \neq \ell$ and

$$\begin{cases} -\Delta e_k = \lambda_k e_k \text{ in } \Omega, \\ e_k \quad = 0 \qquad \text{on } \partial\Omega. \end{cases}$$

So for any $k \in \mathbb{N}$, $\lambda_k > 0$.

The fractional Laplacian A_s, $s \in (0, 1)$ is defined by

$$H_0^s(\Omega) \longrightarrow H_0^{-s}(\Omega) \simeq H_0^s(\Omega),$$
$$u = \sum_{k=1}^{\infty} b_k e_k \longmapsto A_s(u) = \sum_{k=1}^{\infty} b_k \lambda_k^s e_k,$$

where $H_0^s(\Omega) := \left\{u = \sum_{k=1}^{\infty} b_k e_k \in L^2(\Omega), \sum_{k=1}^{\infty} b_k^2 \lambda_k^s < \infty\right\}$ and $H_0^{-s}(\Omega)$ is the dual space of the Hilbert fractional Sobolev space $H_0^s(\Omega)$. Concerning the local equivalent problem to (1.1), we follow the results of [15] for $\Omega = \mathbb{R}^n$, and [14] for bounded domain Ω, see also [13,16,25,28]. Therefore, we consider the associated local problem on the half cylinder with base Ω. Define

$$C = \Omega \times [0, \infty) = \{(x, t), \text{ s.t. } x \in \Omega \text{ and } t \in [0, \infty)\}$$

and

$$C^{\infty}_{0L}(C) = \{v \in C^{\infty}(\bar{C}),\ \text{s.t. } v = 0 \text{ on } \partial_L C\},$$

where $\partial_L C$ denotes the lateral boundary of C, it is defined by $\partial\Omega \times [0, \infty)$. Let $H^s_{0L}(C)$ be the Hilbert Sobolev space defined by the closure of $C^{\infty}_{0L}(C)$ with respect to

$$|v| = \left(\int_C t^{1-2s}|\nabla v|^2\right)^{\frac{1}{2}},$$

and equipped by the following inner product:

$$\langle v, w\rangle_{H^s_{0L}(C)} = \int_C t^{1-2s}\nabla v \nabla w,\ \forall\, v, w \in H^s_{0L}(C).$$

Following [13,28], we associate to any $u \in H^s_0(\Omega)$ the unique s-harmonic function denoted $s - h(u)$ in $H^s_{0L}(C)$, the unique solution of the following problem:

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla v) = 0 & \text{in } C, \\ v = 0 & \text{on } \partial_L C, \\ v = u & \text{on } \Omega \times \{0\}. \end{cases}$$

See [13,28] for the explicit expression of $s - h(u)$. It follows that A_s is expressed by the following map:

$$u = \sum_{k=1}^{\infty} b_k e_k \longmapsto A_s(u) = \partial^s_{\nu}(s - h(u))/_{\Omega\times\{0\}},$$

where ν denotes the unit outward normal vector to C on $\Omega \times \{0\}$ and for any $v \in H^s_{0L}(C)$ and any $x \in \Omega$, we have

$$\partial^s_{\nu}(v)(x, 0) = -c_s \lim_{t\to 0^+} t^{1-2s}\frac{\partial v}{\partial t}(x, t), \quad \text{where } \ c_s := \frac{\Gamma(s)}{2^{1-2s}\Gamma(1-s)}.$$

In this way, problem (1.1) is equivalent to the following local problem

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla v) = 0 & \text{in } \ C, \\ v > 0 & \text{in } \ C, \\ v = 0 & \text{on } \ \partial_L C, \\ \partial^s_{\nu}(v) = v^{\frac{n+2s}{n-2s}} & \text{on } \ \Omega \times \{0\}. \end{cases} \tag{2.1}$$

Therefore, if v satisfies (2.1), then $u(x) = v(x, 0) := tr(v)(x), \forall\, x \in \Omega$ is a solution of (1.1). Notice that

$$H^s_0(\Omega) = \{u = tr(v), v \in H^s_{0L}(C),\ \text{with } \operatorname{div}(t^{1-2s}\nabla v) = 0 \text{ in } C\}.$$

In order to present the variational structure associated to (1.1), we introduce the following Hilbert space constructed by all s-harmonic functions in $H^s_{0L}(C)$. More precisely, let

$$\mathcal{H} = \{v \in H^s_{0L}(C),\ \text{s.t. } \operatorname{div}(t^{1-2s}\nabla v) = 0 \text{ in } C\}.$$

For all $v \in \mathcal{H}$, we denote

$$\|v\|^2 := |v|^2 = \int_C t^{1-2s}|\nabla v|^2 dxdt = c_s^{-1}\int_{\Omega\times\{0\}} \partial^s_{\nu} v(x, 0) \cdot v(x, 0)dx,$$

and for all $v, w \in \mathcal{H}$, we denote

$$\langle v, w\rangle = \langle v, w\rangle_{H^s_{0L}(C)} = c_s^{-1}\int_{\Omega\times\{0\}} \partial^s_{\nu} v(x, 0)w(x, 0)dx.$$

As mentioned above, problem (2.1) has a variational structure. The Euler–Lagrange functional is the following:

$$J(v) = c_s \frac{\|tr(v)\|^2}{\left(\int_\Omega |v(x,0|^{\frac{2n}{n-2s}} dx\right)^{\frac{n-2s}{n}}}$$

defined on

$$\Sigma = \{v \in \mathcal{H}, \|v\| = c_s^{-1/2}\}.$$

Problem (2.1) is equivalent to finding the critical points of J subjected to the constraint $u \in \Sigma^+ := \{u \in \Sigma, u > 0\}$.

Since $p+1$ is the critical Sobolev exponent of the Sobolev trace embedding $v \in \mathcal{H} \mapsto tr(v) \in L^{p+1}(\Omega)$ which is continuous but not compact for $p = \frac{n+2s}{n-2s}$, the functional J does not satisfy the Palais–Smale condition. This means that there exist sequences along which J is bounded, its gradient goes to zero and which do not converge.

For $x, y \in \Omega, t > 0$, let $\tilde{G}((x,t),y), x, y \in \Omega, t > 0$ be the s-harmonic extension of the Green's function of the fractional Dirichlet Laplacian A_s. It satisfies

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla \tilde{G}(\cdot,y)) = 0 & \text{in } C, \\ \tilde{G}(\cdot,y) = 0 & \text{on } \partial_L C, \\ \partial_\nu^s \tilde{G}(\cdot,y) = \delta_y & \text{on } \Omega \times \{0\}. \end{cases}$$

We have

$$\tilde{G}((x,t),y) = \frac{\hat{c}}{\|(x-y,t)\|^{n-2s}_{\mathbb{R}^{n+1}}} - \tilde{H}((x,t),y),$$

where $\hat{c}$ is a fixed constant and $\tilde{H}$ is the regular part of $\tilde{G}$ (see [19], page 6542). It satisfies

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla \tilde{H}(\cdot,y)) = 0 & \text{in } C, \\ \tilde{H}((x,t),y) = \dfrac{\hat{c}}{\|(x-y,t)\|^{n-2s}_{\mathbb{R}^{n+1}}} & \text{on } \partial_L C, \\ \partial_\nu^s \tilde{H}(\cdot,y) = 0 & \text{on } \Omega \times \{0\}. \end{cases}$$

For any $a \in \Omega$ and $\lambda > 0$, we set

$$\delta_{(a,\lambda)}(x) = \frac{\lambda^{\frac{n-2s}{2}}}{(1+\lambda^2|x-a|^2)^{\frac{n-2s}{2}}}, \quad x \in \mathbb{R}^n.$$

Following the classification results of [17,20,21], $\delta_{(a,\lambda)}(x), a \in \Omega, \lambda > 0$ are the only solutions of

$$\begin{cases} A_s u = c_0 u^{\frac{n+2s}{n-2s}} & \text{in } \mathbb{R}^n, \\ u > 0 & \text{in } \mathbb{R}^n, \\ \lim\limits_{|x|\to\infty} u(x) = 0, \end{cases}$$

where c_0 is a fixed positive constant which depends only on n and s. Notice that in the case where $\Omega = \mathbb{R}^n$, the Sobolev space $H^s(\mathbb{R}^n)$ is defined by

$$H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n), \int_{\mathbb{R}^n} (1+|2\pi x|^{2s})|\hat{u}(x)|^2 dx < \infty\}.$$

Here $\hat{u}$ denotes the Fourier transform of u. The fractional operator $A_s : H^s(\mathbb{R}^n) \to H^{-s}(\mathbb{R}^n)$ is defined by

$$\widehat{A_s(u)}(x) = |2\pi x|^{2s}\hat{u}(x).$$

Let $\hat{\delta}_{(a,\lambda)}$ be the s-harmonic extension of $\delta_{(a,\lambda)}$ in $\mathbb{R}^{n+1}_+$ and let

$$\hat{\gamma} = c_s^{\frac{-1}{2}}\|\hat{\delta}_{(a,\lambda)}\|^{-1}_{D^s(\mathbb{R}^{n+1}_+)} := \left(c_s\int_{\mathbb{R}^{n+1}} t^{1-2s}|\nabla\hat{\delta}_{(a,\lambda)}|^2 dxdt\right)^{\frac{-1}{2}}. \tag{2.2}$$

It is more convenient in the next to work with $\tilde{\delta}_{(a,\lambda)}$, $a \in \Omega$ and $\lambda > 0$ defined by

$$\tilde{\delta}_{(a,\lambda)} = \hat{\gamma}\hat{\delta}_{(a,\lambda)}.$$

We have

$$\|\tilde{\delta}_{(a,\lambda)}\|_{D^s(\mathbb{R}^{n+1}_+)} = c_s^{\frac{-1}{2}}, \tag{2.3}$$

$$tr(\tilde{\delta}_{(a,\lambda)}) = \hat{\gamma}\delta_{(a,\lambda)} \text{ on } \mathbb{R}^n \tag{2.4}$$

and

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla\tilde{\delta}_{(a,\lambda)}) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ \partial^s_\nu\tilde{\delta}_{(a,y)} = \gamma_0\tilde{\delta}^{\frac{n+2s}{n-2s}}_{(a,y)} & \text{on } \mathbb{R}^n \times \{0\}, \end{cases}$$

where $\gamma_0 = c_0\hat{\gamma}^{\frac{-4s}{n-2s}}$.

For any $a \in \Omega$ and $\lambda > 0$, we define the almost solutions $P\delta_{(a,\lambda)}$ of (2.1) as the unique solutions of the following problem:

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla P\delta_{(a,\lambda)}) = 0 & \text{in } C, \\ P\delta_{(a,\lambda)} = 0 & \text{on } \partial_L C, \\ \partial^s_\nu P\delta_{(a,y)} = \partial^s_\nu\tilde{\delta}_{(a,y)} = \gamma_0\tilde{\delta}^{\frac{n+2s}{n-2s}}_{(a,y)} & \text{on } \Omega \times \{0\}. \end{cases}$$

Next, we introduce the best constant of Sobolev. Let

$$\begin{aligned} \iota : H^s_{0L}(C) &\longrightarrow L^{\frac{2n}{n-2s}}(\Omega), \\ v &\longmapsto tr(v) \end{aligned}$$

be the Sobolev trace embedding. The best constant of Sobolev is given by

$$S = \frac{\|tr\ \tilde{\delta}_{(a,\lambda)}\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)}}{\|\tilde{\delta}_{(a,\lambda)}\|_{D^s(\mathbb{R}^{n+1}_+)}} = c_s^{\frac{1}{2}}\|tr\ \tilde{\delta}_{(a,\lambda)}\|_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)},$$

since $\|\tilde{\delta}_{(a,\lambda)}\|_{D^s(\mathbb{R}^{n+1}_+)} = c_s^{\frac{-1}{2}}$. Notice that S is independent of a and λ (see [29]). Observe that

$$\inf_{v\in\Sigma} J(v) = c_s^{\frac{n}{n-2s}} S^{\frac{-2n}{n-2s}} := \tilde{S} = \gamma_0.$$

Therefore,

$$\tilde{S} = \frac{1}{\|tr\ \tilde{\delta}_{(a,\lambda)}\|^{\frac{2n}{n-2s}}_{L^{\frac{2n}{n-2s}}(\mathbb{R}^n)}} = \frac{1}{\int_{\mathbb{R}^n}(tr\ \tilde{\delta}_{(a,\lambda)})^{\frac{2n}{n-2s}}dx}. \tag{2.5}$$

Arguing as [26], the following proposition describes the Palais–Smale sequences of J.

PROPOSITION 2.1

Assume that (2.1) *has no solution. Let* $(v_k)_k$ *be a sequence in* $\Sigma^+ := \{v \in \Sigma, v \geq 0\}$ *such that* $J(v_k) \to c$ *and* $\partial J(v_k) \to 0$. *There exists* $p \in \mathbb{N}^*$ *and a subsequence of* $(v_k)_k$ *denoted again* $(v_k)_k$ *such that* $v_k \in V(p, \varepsilon_k)$, *where* $\varepsilon_k \to 0$ *as* $k \to +\infty$ *and*

$$V(p,\varepsilon) = \Bigg\{ u \in \Sigma^+, \text{ s.t. } \exists (a_1,\dots,a_p) \in \Omega^p, \ \exists (\lambda_1,\dots,\lambda_p) \in \left[\frac{1}{\varepsilon}, \infty\right)^p \text{ and } (\alpha_1,\dots,\alpha_p) \in \mathbb{R}_+^p, \text{ s.t. } \left\| u - \frac{1}{\sqrt{c_s}} \frac{\sum_{i=1}^p \alpha_i P\delta_{(a_i,\lambda_i)}}{\| \sum_{i=1}^p \alpha_i P\delta_{(a_i,\lambda_i)} \|} \right\| < \varepsilon \text{ with } (\lambda_i d(a_i, \partial\Omega) > \frac{1}{\varepsilon} \text{ and } \varepsilon_{ij} < \varepsilon, \forall i \neq j \Bigg\}.$$

Here $\varepsilon_{ij} = \frac{1}{\left(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j |a_i - a_j|^2\right)^{\frac{n-2s}{2}}}$.

The following proposition gives suitable parameters for $V(p, \varepsilon)$. The proof is similar to ([9], Proposition 7).

PROPOSITION 2.2

Let $p \in \mathbb{N}^*$. *There exists* $\varepsilon_p > 0$ *such that for any* $0 < \varepsilon < \varepsilon_p$ *and* $u \in V(p, \varepsilon)$, *the following minimization problem*:

$$\inf_{\alpha_i, a_i, \lambda_i} \left\| u - \sum_{i=1}^p \alpha_i P\delta_{(a_i,\lambda_i)} \right\|$$

admits a unique solution $(\bar{\alpha}, \bar{a}, \bar{\lambda})$ *modulo a permutation on the indices set. Let* $v = u - \sum_{i=1}^p \bar{\alpha}_i P\delta_{(\bar{a}_i, \bar{\lambda}_i)}$. *It satisfies*

$$(V_0) : \langle v, \psi \rangle = 0 \ \textit{for} \ \psi \in \left\{ P\delta_{a_i,\lambda_i}, \frac{\partial P\delta_{a_i,\lambda_i}}{\partial \lambda_i}, \frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_i}, \ i = 1, \dots, p \right\}.$$

For $q \in \mathbb{N}^*$ and $x = (x_1, \dots, x_q) \in \Omega^q$, such that $x_i \neq x_j$ for $i \neq j$, we denote $M(x) = (m_{ij})_{1 \leq i,j \leq q}$ the matrix defined by

$$m_{ii} = H((x_i, 0), x_i), \qquad m_{ij} = -G((x_i, 0), x_j) \ \forall j \neq i. \tag{2.6}$$

Let $\rho(x)$ be the least eigenvalue and by $e(x)$ the eigenvector associated to $\rho(x)$ whose norm equals 1 and whose components are strictly positive.

Our main results are the following.

Theorem 2.3. *Assume that zero is a regular value of* ρ. *For* $\varepsilon > 0$ *sufficiently small, there exists a change of variables, such that for any* $u = \sum_{i=1}^p \alpha_i P\delta_{(a_i,\lambda_i)} \in V(p, \varepsilon)$, $(a_i, \lambda_i, v) \to (a'_i, \lambda'_i, V)$ *where* V *belongs to a neighborhood of zero in a fixed Hilbert* *space so that*

$$J\left(\sum_{i=1}^{p}\alpha_i P\delta_{(a_i,\lambda_i)}+v\right)=J\left(\sum_{i=1}^{p}\alpha_i P\delta_{(a_i',\lambda_i')}\right)+\|V\|^2.$$

Furthermore, if each a_i belongs to a neighborhood of x_i such that $\rho(x_1,\dots,x_p)>0$ and $\rho'(x_1,\dots,x_p)=0$, there exists another change of variables $(a_i,\lambda_i)\to(a_i',\lambda_i')$ such that

$$J\left(\sum_{i=1}^{p}\alpha_i P\delta_{(a_i,\lambda_i)}\right)=\frac{(\tilde{S})^{2/n}(\sum_{i=1}^{p}\alpha_i^2)}{(\sum_{i=1}^{p}\alpha_i^{\frac{2n}{n-2s}})^{\frac{n-2s}{n}}}\left(1+\eta\rho(a')\sum_{i=1}^{p}\frac{1}{\lambda_i'^{n-2s}}\right),$$

up to a multiplicative constant. Here $a'=(a_1',\dots,a_p')$ and η is a fixed positive constant.

The characterisation of the critical points at infinity is given in the following theorem.

Theorem 2.4. *Assume that zero is a regular value of ρ. Then we have*

(i) *For ε small enough, J does not have any critical point in $V(p,\varepsilon)$.*
(ii) *The only critical points at infinity of J in $V(p,\varepsilon)$ correspond to $\sum_{i=1}^{p}P\delta_{(x_i,+\infty)}$, where $p\in\mathbb{N}^*$ and the x_i's satisfy*

$$\rho(x_1,\dots,x_p)>0 \quad \textit{and} \quad \rho'(x_1,\dots,x_p)=0.$$

(iii) *There is $p_0\in\mathbb{N}^*$ such that J does not have any critical point at infinity in $V(p,\varepsilon)$ for each $p\geq p_0$.*

The following result illustrates the usefulness of the above theorems. It computes the difference of topology between the level sets of the functional J. More precisely, it evaluates the contribution of the critical points at infinity to the relative homology between the sets W_p and W_{p-1}, where

$$W_p=\{u\in\Sigma^+, s.t, J(u)<((p+1)\tilde{S})^{\frac{2}{n}}\}.$$

Theorem 2.5. *Assume that J has no critical point in Σ^+ and zero is a regular value of ρ. Then the relative homology $H_*(W_p,W_{p-1})$ between the sets W_p and W_{p-1} equals to*

$$H_*(\Omega^p\times_{\sigma_p}\Delta_{p-1},\Omega^p\times\partial\Delta_{p-1}\cup_{\sigma_p}I_p\times\Delta_{p-1}),$$

where $I_p=\{x\in\Omega^p, s.t. \rho(x)\leq 0\}$, $\Delta_{p-1}=\{(\alpha_1,\dots,\alpha_p)$, such that $\alpha_i\geq 0,\ \sum_{i=1}^{p}\alpha_i=1\}$ and σ_p is the permutation group.

The remainder of the present paper is organized as follow. Section 3 will be devoted to the expansion of J and its gradient. In Section 4, we will study the v-part of u. In Section 5, we will construct a suitable pseudo-gradient to characterize the critical points associated to problem (1.1). The proofs of Theorems 2.3, 2.4 and 2.5 are given in Section 6.

3. Expansion of the functional and its gradient near potential critical points at infinity

First, we deal with the asymptotic expansion of the functional J.

PROPOSITION 3.1

For $\varepsilon > 0$ small enough and $u = \sum_{i=1}^{p} \alpha_i P\delta_{(a_i,\lambda_i)} + v \in V(p,\varepsilon)$, we have the following expansion:

$$
\begin{aligned}
J(u) = \frac{(c_s^{-1})^{2/n}(\sum_{i=1}^{p}\alpha_i^2)}{(\sum_{i=1}^{p}\alpha_i^{\frac{2n}{n-2s}})^{\frac{n-2s}{n}}} \Bigg[1 &+ \frac{c_1}{c_s^{-1}} \sum_{i=1}^{p} \frac{H((a_i,0),a_i)}{\lambda_i^{n-2s}} \left(\frac{2\alpha_i^{\frac{2n}{n-2s}}}{\sum_{j=1}^{p}\alpha_j^{\frac{2n}{n-2s}}} - \frac{\alpha_i^2}{\sum_{j=1}^{p}\alpha_j^2} \right) \\
&- \frac{c_1}{c_s^{-1}} \sum_{i\neq j\geq 1} \left(\varepsilon_{ij} - \frac{H((a_i,0),a_j)}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}} \right) \left(\frac{2\alpha_i^{\frac{n+2s}{n-2s}}\alpha_j}{\sum_{j=1}^{p}\alpha_j^{\frac{2n}{n-2s}}} - \frac{\alpha_i\alpha_j}{\sum_{j=1}^{p}\alpha_k^2} \right) \\
&+ \frac{Q(v,v)}{c_s^{-1}\sum_{j=1}^{p}\alpha_j^2} - f(v) \\
&+ O\left(\sum_{j\neq i} \varepsilon_{ij}^{\frac{n}{n-2s}} \log \varepsilon_{ij}^{-1} + \sum_{k=1}^{p} \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^{n+2-2s}} \right) + O(\|v\|^{\inf(3,\frac{2n}{n-2s})}) \Bigg],
\end{aligned}
$$

where c_1 is a positive constant, and

$$
Q(v,v) = \|v\|^2 - \frac{(n+2s)}{(n-2s)} \frac{\sum_j \alpha_j^2}{\sum_j \alpha_j^{\frac{2n}{n-2s}}} \int \left(\sum_i \alpha_i P\delta_i \right)^{\frac{4s}{n-2s}} v^2,
$$

$$
f(v) = \frac{2c_s}{\sum_j \alpha_j^{\frac{2n}{n-2s}}} \int \left(\sum_i \alpha_i P\delta_i \right)^{\frac{n+2s}{n-2s}} v.
$$

Proof. Let us recall that

$$
J(u) = \frac{\|u\|^2}{\left(\int_\Omega u^{\frac{2n}{n-2s}} \right)^{\frac{n-2s}{n}}}.
$$

We need to estimate

$$
N(u) = \|u\|^2 = \left\| \sum_{i=1}^{p} \alpha_i P\delta_i + v \right\|^2 \quad \text{and} \quad D^{\frac{n}{n-2s}} = \int_\Omega u^{\frac{2n}{n-2s}}.
$$

Using the fact that v satisfies (V_0), we have

$$
N(u) = \sum_{i=1}^{p} \alpha_i^2 \|P\delta_i\|^2 + \sum_{i\neq j} \alpha_i\alpha_j \langle P\delta_i, P\delta_j \rangle + \|v\|^2. \tag{3.1}
$$

A computation similar to the one performed in [1,7] shows that, for $\lambda_i d_i$ large enough, we have the following estimates:

$$\|P\delta_i\|^2 = c_s^{-1} - c_1 \frac{H((a_i,0),a_i)}{\lambda_i^{n-2s}} + O\left(\frac{\log(\lambda_i d_i)}{(\lambda_i d_i)^{n+2-2s}}\right), \tag{3.2}$$

$$\langle P\delta_i, P\delta_j\rangle = c_1\left(\varepsilon_{ij} - \frac{H((a_i,0),a_j)}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}}\right) + O\left(\varepsilon_{ij}^{\frac{n}{n-2s}}\log\varepsilon_{ij}^{-1} + \sum_{k=i,j}\frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^{n+2-2s}}\right). \tag{3.3}$$

Using (3.1), (3.2) and (3.3), we derive that

$$N(u) = \sum_{i=1}^{p}\alpha_i^2\left(c_s^{-1} - c_1\frac{H((a_i,0),a_i)}{\lambda_i^{n-2s}}\right) + \sum_{j\neq i}\alpha_i\alpha_j c_1\left(\varepsilon_{ij} - \frac{H((a_i,0),a_j)}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}}\right) + \|v\|^2 + R, \tag{3.4}$$

where

$$R = O\left(\sum_{j\neq i}\varepsilon_{ij}^{\frac{n}{n-2s}}\log\varepsilon_{ij}^{-1} + \sum_{k=1}^{p}\frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^{n+2-2s}}\right).$$

For the denominator, we have

$$D^{\frac{n}{n-2s}} = \int_\Omega\left(\sum_{i=1}^{p}\alpha_i P\delta_i + v\right)^{\frac{2n}{n-2s}} = \int_\Omega\left(\sum_{i=1}^{p}\alpha_i P\delta_i\right)^{\frac{2n}{n-2s}} + \frac{2n}{n-2s}\int_\Omega\left(\sum_{i=1}^{p}\alpha_i P\delta_i\right)^{\frac{n+2s}{n-2s}} v + \frac{n(n+2s)}{(n-2s)^2}\int_\Omega\left(\sum_{i=1}^{p}\alpha_i P\delta_i\right)^{\frac{4s}{n-2s}} v^2 + O(\|v\|^{\inf(3,\frac{2n}{n-2s})}). \tag{3.5}$$

Observe that

$$\int_\Omega\left(\sum_{i=1}^{p}\alpha_i P\delta_i\right)^{\frac{2n}{n-2s}} = \sum_{i=1}^{p}\alpha_i^{\frac{2n}{n-2s}}\int_\Omega P\delta_i^{\frac{2n}{n-2s}} + \frac{2n}{n-2s}\sum_{i\neq j}\alpha_i^{\frac{n+2s}{n-2s}}\alpha_j\int_\Omega P\delta_i^{\frac{n+2s}{n-2s}}P\delta_j + O\left(\sum_{i\neq j}\int P\delta_i^{\frac{4s}{n-2s}}\inf(P\delta_i,P\delta_j)^2\right). \tag{3.6}$$

A computation similar to the one performed in [1,7] shows that

$$\int P\delta_i^{\frac{4s}{n-2s}}\inf(P\delta_i,P\delta_j)^2 = O\left(\sum_{j\neq i}\varepsilon_{ij}^{\frac{n}{n-2s}}\log\varepsilon_{ij}^{-1} + \sum_{k=i,j}\frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^{n+2-2s}}\right), \tag{3.7}$$

$$\int P\delta_i^{\frac{n+2s}{n-2s}} P\delta_j = c_1 \left(\varepsilon_{ij} - \frac{H((a_i,0),a_j)}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}} \right) + O\left(\sum_{j\neq i} \varepsilon_{ij}^{\frac{n}{n-2s}} \log \varepsilon_{ij}^{-1} + \sum_{k=i,j} \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^{n+2-2s}} \right), \tag{3.8}$$

$$\int P\delta_i^{\frac{2n}{n-2s}} = c_s^{-1} - c_1 \frac{2n}{n-2s} \frac{H((a_i,0),a_i)}{\lambda_i^{n-2s}} + O\left(\frac{\log(\lambda_i d_i)}{(\lambda_i d_i)^{n+2-2s}} \right). \tag{3.9}$$

Using (3.7), (3.8) and (3.9), we get

$$\int \left(\sum_{i=1}^{p} \alpha_i P\delta_i \right)^{\frac{2n}{n-2s}} = \sum_{i=1}^{p} \alpha_i^{\frac{2n}{n-2s}} \left(c_s^{-1} - c_1 \frac{2n}{n-2s} \frac{H((a_i,0),a_i)}{\lambda_i^{n-2s}} \right) + \frac{2n}{n-2s} c_1 \sum_{i\neq j} \alpha_i^{\frac{n+2s}{n-2s}} \alpha_j \left(\varepsilon_{ij} - \frac{H((a_i,0),a_j)}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}} \right) + O\left(\sum_{j\neq i} \varepsilon_{ij}^{\frac{n}{n-2s}} \log \varepsilon_{ij}^{-1} + \sum_{k=i,j} \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^{n+2-2s}} \right). \tag{3.10}$$

Combining (3.4), (3.5) and (3.10) and the fact that $J(u)^{\frac{n}{n-2s}} \alpha_i^{\frac{4s}{n-2s}} = 1 + o(1)$, for each i, the result follows.

PROPOSITION 3.2

For $u = \sum_{i=1}^{p} \alpha_i P\delta_{(a_i,\lambda_i)} \in V(p,\varepsilon)$, *we have the following expansion*:

$$\left\langle \partial J(u), \lambda_i \frac{\partial P\delta_{a_i,\lambda_i}}{\partial \lambda_i} \right\rangle = 2c_1 J(u) \left[-\frac{n-2s}{2} \alpha_i \frac{H((a_i,0),a_i)}{\lambda_i^{n-2s}} (1+o(1)) - \sum_{j\neq i} \alpha_j \left(\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{n-2s}{2} \frac{H((a_i,0),a_j)}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}} \right) (1+o(1)) + R \right].$$

Proof. For any $h \in \mathcal{H}$, we have

$$\langle \partial J(u), h \rangle = 2J(u) \left(\langle u, h \rangle - J(u)^{\frac{n}{n-2s}} \int_\Omega u^{\frac{n+2s}{n-2s}} h \right). \tag{3.11}$$

Thus

$$\left\langle \partial J(u), \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \right\rangle = 2J(u) \left[\left\langle \sum_{j=1}^{p} \alpha_j P\delta_j, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \right\rangle - J(u)^{\frac{n}{n-2s}} \int_\Omega \left(\sum_{j=1}^{p} \alpha_j P\delta_j \right)^{\frac{n+2s}{n-2s}} \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \right].$$

Observe that

$$\int_\Omega \left(\sum_{j=1}^{p} \alpha_j P\delta_j\right)^{\frac{n+2s}{n-2s}} \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} = \sum_{j=1}^{p} \alpha_j^{\frac{n+2s}{n-2s}} \int_\Omega P\delta_j^{\frac{n+2s}{n-2s}} \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}$$
$$+\frac{n+2s}{n-2s} \sum_{i\neq j} \int_\Omega (\alpha_i P\delta_i)^{\frac{4s}{n-2s}} \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} (\alpha_j P\delta_j) + O\left(\sum_{j\neq i} \int_\Omega P\delta_j^{\frac{4s}{n-2s}} \inf(\delta_j, \delta_i)^2\right). \tag{3.12}$$

A computation similar to the one performed in [7], shows that

$$\left\langle P\delta_i, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\right\rangle = \frac{n-2s}{2} c_1 \frac{H((a_i,0),a_i)}{\lambda_i^{n-2s}} + O\left(\frac{\log(\lambda_i d_i)}{(\lambda_i d_i)^{n+2-2s}}\right), \tag{3.13}$$

$$\left\langle P\delta_j, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\right\rangle = c_1 \left(\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + \frac{n-2s}{n} \frac{H((a_i,0),a_j)}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}}\right)$$
$$+O\left(\sum_{j\neq i} \varepsilon_{ij}^{\frac{n}{n-2s}} \log \varepsilon_{ij}^{-1} + \sum_{k=i,j} \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^{n+2-2s}}\right), \tag{3.14}$$

$$\int_\Omega P\delta_i^{\frac{n+2s}{n-2s}} \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} = (n-2s) c_1 \frac{H((a_i,0),a_i)}{\lambda_i^{n-2s}} + O\left(\frac{\log(\lambda_i d_i)}{(\lambda_i d_i)^{n+2-2s}}\right), \tag{3.15}$$

$$\int_\Omega P\delta_j^{\frac{n+2s}{n-2s}} \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} = \left\langle P\delta_j, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\right\rangle + O\left(\sum_{j\neq i} \varepsilon_{ij}^{\frac{n}{n-2s}} \log \varepsilon_{ij}^{-1} + \sum_{k=i,j} \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^{n+2-2s}}\right), \tag{3.16}$$

$$\frac{n+2s}{n-2s} \int_\Omega P\delta_j P\delta_i^{\frac{4s}{n-2s}} \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} = \left\langle P\delta_j, \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}\right\rangle$$
$$+O\left(\sum_{j\neq i} \varepsilon_{ij}^{\frac{n}{n-2s}} \log \varepsilon_{ij}^{-1} + \sum_{k=i,j} \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^{n+2-2s}}\right). \tag{3.17}$$

Using (3.13)–(3.17) and the fact that $J(u)^{\frac{n}{n-2s}} \alpha_i^{\frac{4s}{n-2s}} = 1+o(1)$, for each i, Proposition 3.2 follows.

PROPOSITION 3.3

For $u = \sum_{i=1}^{p} \alpha_i P\delta_{(a_i,\lambda_i)} \in V(p,\varepsilon)$, *we have the following expansion*:

$$\left\langle \partial J(u), \frac{1}{\lambda_i} \frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_i}\right\rangle = J(u) c_1 \left[\frac{\alpha_i}{\lambda_i^{n+1-2s}} \frac{\partial H((a_i,0),a_i)}{\partial a_i}(1+o(1))\right.$$
$$+2\sum_{j\neq i} \alpha_j \left(\frac{1}{\lambda_i} \frac{\partial \varepsilon_{ij}}{\partial a_i} - \frac{\frac{\partial H((a_i,0),a_j)}{\partial a_i}}{\lambda_i(\lambda_i\lambda_j)^{\frac{n-2s}{2}}}\right)$$

$$\times(1 - J(u)^{\frac{n}{n-2s}}(\alpha_j^{\frac{4s}{n-2s}} + \alpha_j^{\frac{4s}{n-2s}}))\Bigg]$$
$$+R + O\left(\sum_{i\neq j}\lambda_i|a_i - a_j|\varepsilon_{ij}^{\frac{n+3-2s}{n-2s}}\right).$$

Proof. Using (3.11), we have

$$\left\langle \partial J(u), \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\right\rangle = 2J(u)\left[\left\langle \sum_{j=1}^{p}\alpha_j P\delta_j, \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\right\rangle - J(u)^{\frac{n}{n-2s}}\int_\Omega\left(\sum_{j=1}^{p}\alpha_j P\delta_j\right)^{\frac{n+2s}{n-2s}}\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\right].$$

Observe that

$$\int_\Omega\Big(\sum_{j=1}^{p}\alpha_j P\delta_j\Big)^{\frac{n+2s}{n-2s}}\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i} = \sum_{j=1}^{p}\alpha_j^{\frac{n+2s}{n-2s}}\int_\Omega P\delta_j^{\frac{n+2s}{n-2s}}\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i} + \frac{n+2s}{n-2s}\sum_{i\neq j}\int_\Omega(\alpha_i P\delta_i)^{\frac{4s}{n-2s}}\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}(\alpha_j P\delta_j) + O\Big(\sum_{j\neq i}\int_\Omega P\delta_j^{\frac{4s}{n-2s}}\inf(\delta_j,\delta_i)^2\Big). \tag{3.18}$$

A computation similar to the one performed in [7], shows that

$$\left\langle P\delta_i, \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\right\rangle = -\frac{1}{2}\frac{c_1}{\lambda_i^{n+1-2s}}\frac{\partial H((a_i,0),a_i)}{\partial a_i} + O\left(\frac{1}{(\lambda_i d_i)^{n+2-2s}}\right), \tag{3.19}$$

$$\left\langle P\delta_j, \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\right\rangle = \frac{c_1}{\lambda_i}\left(\frac{\partial\varepsilon_{ij}}{\partial a_i} - \frac{1}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}}\frac{\partial H((a_i,0),a_j)}{\partial a_i}\right) + O\left(\sum_{k=i,j}\frac{1}{(\lambda_k d_k)^{n+2-2s}} + \sum_{i\neq j}\lambda_i|a_i - a_j|\varepsilon_{ij}^{\frac{n+3-2s}{n-2s}}\right), \tag{3.20}$$

$$\int_\Omega P\delta_i^{\frac{n+2s}{n-2s}}\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i} = -\frac{c_1}{\lambda_i^{n+1-2s}}\frac{\partial H((a_i,0),a_i)}{\partial a_i} + O\left(\frac{\log(\lambda_i d_i)}{(\lambda_i d_i)^{n+2-2s}}\right), \tag{3.21}$$

$$\int_\Omega P\delta_j^{\frac{n+2s}{n-2s}}\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i} = \left\langle P\delta_j, \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\right\rangle + O\left(\sum_{k=i,j}\frac{1}{(\lambda_k d_k)^{n+2-2s}} + \sum_{j\neq i}\varepsilon_{ij}^{\frac{n}{n-2s}}\log\varepsilon_{ij}^{-1}\right), \tag{3.22}$$

$$\frac{n+2s}{n-2s}\int_{\Omega} P\delta_j P\delta_i^{\frac{4s}{n-2s}}\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i} = \left\langle P\delta_j, \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\right\rangle + O\left(\sum_{k=i,j}\frac{1}{(\lambda_k d_k)^{n+2-2s}} + \sum_{j\neq i}\varepsilon_{ij}^{\frac{n}{n-2s}}\log\varepsilon_{ij}^{-1}\right). \tag{3.23}$$

Using (3.18)–(3.23) and the fact that $J(u)^{\frac{n}{n-2s}}\alpha_i^{\frac{4s}{n-2s}} = 1+o(1)$, for each i, Proposition 3.3 follows.

4. The v-part of u

In this section, we deal with the v-part of u, in order to show that it is negligible with respect to the concentration phenomenon.

PROPOSITION 4.1

There is a C^1-map which to each $(\alpha_i, a_i, \lambda_i)$ such that $\sum_{i=1}^{p}\alpha_i P\delta_{(a_i,\lambda_i)}$ belongs to $V(p,\varepsilon)$ associates $\bar{v} = \bar{v}(\alpha, a, \lambda)$ such that $\bar{v}$ is unique and satisfies

$$J\left(\sum_{i=1}^{p}\alpha_i P\delta_{(a_i,\lambda_i)} + \bar{v}\right) = \min_{v\in(V_0)}\left\{J\left(\sum_{i=1}^{p}\alpha_i P\delta_{(a_i,\lambda_i)} + v\right)\right\}.$$

Furthermore, we have the following estimate:

$$\|\bar{v}\| \leq \begin{cases}\displaystyle\sum_{i=1}^{p}\frac{\log(\lambda_i d_i)}{(\lambda_i d_i)^{\frac{n+2s}{2}}} + \sum_{j\neq i}\varepsilon_{ij}^{\frac{n+2s}{2(n-2s)}}(\log\varepsilon_{ij}^{-1})^{\frac{n+2s}{2n}}, & \text{if } n\geq 6\\ \displaystyle\sum_{i=1}^{p}\frac{1}{(\lambda_i d_i)^{n-2s}} + \sum_{j\neq i}\varepsilon_{ij}(\log\varepsilon_{ij}^{-1})^{\frac{n-2s}{n}}, & \text{if } n\leq 5\end{cases}$$

Proof. Since $\frac{\alpha_i}{\alpha_j} = 1+o(1)$, then the quadratic form $Q(v,v)$ defined in Proposition 3.1 is close to

$$\|v\|^2 - \frac{(n+2s)}{(n-2s)}\sum_i\int P\delta_i^{\frac{4s}{n-2s}}v^2. \tag{4.1}$$

Arguing as in [7], the existence of $\bar{v}$ follows, since $Q(v,v)$ is definitive and positive. Thus

$$\exists\,\alpha > 0,\ \text{s.t. } \alpha\|\bar{v}\|^2 \leq |(f,\bar{v})| \leq \alpha|f|\|\bar{v}\|, \tag{4.2}$$

where f is the linear form defined in Proposition 3.1. Thus, it is sufficient to estimate $|f|$. We have

$$f(v) = \frac{2c_s}{\sum_j\alpha_j^{\frac{2n}{n-2s}}}\sum_{i=1}^{p}\alpha_i^{\frac{n+2s}{n-2s}}\int P\delta_i^{\frac{n+2s}{n-2s}}v + O\Big(\sum_{j\neq i}\int_{P\delta_j\leq P\delta_i}P\delta_i^{\frac{4s}{n-2s}}P\delta_j|v|\Big). \tag{4.3}$$

Observe that

$$\int P\delta_i^{\frac{n+2s}{n-2s}} v = \int \delta_i^{\frac{n+2s}{n-2s}} v + O\left(\int_{B_i \cup B_i^c} \delta_i^{\frac{4s}{n-2s}} \theta_i |v|\right), \tag{4.4}$$

where $B_i = \{x, |x - a_i| < d_i\}$ and $\theta_i = \delta_i - P\delta_i$. Then, using the Holder's inequality, we need to estimate

$$\int_{B_i^c} \left(\delta_i^{\frac{4s}{n-2s}} \theta_i\right)^{\frac{2n}{n-2s}} \leq \int_{B_i^c} \delta_i^{\frac{2n}{n-2s}} = O\left(\frac{1}{(\lambda_i d_i)^{n+2-2s}}\right), \tag{4.5}$$

$$|\theta_i|_\infty \int_{B_i} \left(\delta_i^{\frac{8n}{(n+2s)(n-2s)}}\right)^{\frac{n+2s}{2n}}$$

$$= O\left(\frac{1}{(\lambda_i d_i)^{\frac{n+2s}{2}}} + (\text{if } n = 6)\frac{\log(\lambda_i d_i)}{(\lambda_i d_i)^4} + (\text{if } n \leq 5)\frac{1}{(\lambda_i d_i)^{n-2s}}\right). \tag{4.6}$$

Also, we have

$$\int_{P\delta_j \leq P\delta_i} P\delta_i^{\frac{4s}{n-2s}} P\delta_j |v| \leq |v| \left[\int_{P\delta_j \leq P\delta_i} \left(P\delta_i^{\frac{4s}{n-2s}} P\delta_j\right)^{\frac{2n}{n+2s}}\right]^{\frac{n+2s}{2n}}. \tag{4.7}$$

If $n \geq 6$, then $\frac{2n}{n+2s} \geq \frac{n}{n-2s}$. Therefore,

$$\int_{P\delta_j \leq P\delta_i} \left(P\delta_i^{\frac{4s}{n-2s}} P\delta_j\right)^{\frac{2n}{n+2s}} \leq \int (\delta_i \delta_j)^{\frac{n}{n-2s}} = O\left(\varepsilon_{ij}^{\frac{n}{n-2s}} \log \varepsilon_{ij}^{-1}\right). \tag{4.8}$$

If $n \leq 5$, then $1 < \frac{4s}{n-2s}$. In this case,

$$\int_{P\delta_j \leq P\delta_i} \left(P\delta_i^{\frac{4s}{n-2s}} P\delta_j\right)^{\frac{2n}{n+2s}} \leq \left[\int (\delta_i \delta_j)^{\frac{n}{n-2s}}\right]^{\frac{2(n-2s)}{n+2s}} = O\left(\varepsilon_{ij}^{\frac{2n}{n+2s}} (\log \varepsilon_{ij}^{-1})^{\frac{2(n-2s)}{n+2s}}\right). \tag{4.9}$$

This concludes the proof.

5. Construction of the pseudo-gradient

This section is devoted to the construction of a suitable pseudo-gradient of J for which the Palais–Smale condition is satisfied along the decreasing flow lines as long as these flow lines do not enter in some neighborhood of $\sum_{i=1}^{p} P\delta_{(x_i,+\infty)}$, $p \in \mathbb{N}^*$ such that

$$\rho(x_1, \ldots, x_p) > 0 \quad \text{and} \quad \rho'(x_1, \ldots, x_p) = 0.$$

Such a construction allows us to identify the critical points at infinity of the variational structure associated to (1.1).

Theorem 5.1. *Assume that zero is a regular value of ρ. For any $p \geq 1$ and $\varepsilon > 0$ small enough, there exists a pseudo-gradient W in $V(p, \varepsilon)$ satisfying the following:*

There exists a constant $c > 0$ such that for any $u = \sum_{i=1}^{p} \alpha_i P\delta_{a_i,\lambda_i} \in V(p, \varepsilon)$, we have

$$(i)\ \langle \partial J(u), W(u)\rangle \leq -c\Bigg(\sum_{i=1}^{p}\Big(\frac{1}{(\lambda_i d_i)^{n+1-2s}}\Big) + \sum_{j\neq i}\varepsilon_{ij}^{\frac{n+1-2s}{n-2s}}\Bigg),$$

$$(ii)\ \Big\langle \partial J(u+\bar{v}), W(u)+\frac{\partial \bar{v}}{\partial(\alpha, a, \lambda)}(W(u))\Big\rangle \leq -c\Bigg(\sum_{i=1}^{p}\Big(\frac{1}{(\lambda_i d_i)^{n+1-2s}}\Big)+\sum_{j\neq i}\varepsilon_{ij}^{\frac{n+1-2s}{n-2s}}\Bigg).$$

$|W|$ is bounded, the minimal distance to the boundary only increases if it is small enough and the only case where $\lambda_i(s), i = 1, \ldots, p, s \geq 0$, tend to ∞ is when $\rho(X) > 0$ and $\rho'(X) = 0$, where $X = (a_1, \ldots, a_p)$.

In order to construct the required pseudo-gradient, we need to introduce the following notations: For each $i \in \{1, \ldots, p\}$, let

$$I_1 = \Bigg\{i,\ \text{s.t.}\ \frac{1}{2^{p+1}}\sum_{k\neq i}\varepsilon_{ki} \leq \sum_{j=1}^{p}\frac{H(a_i, 0), a_j)}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}}\Bigg\},$$

$$I_2 = \Bigg\{i,\ \text{s.t.}\ \frac{1}{2^{p+1}}\sum_{k\neq i}\varepsilon_{ki} > \sum_{j=1}^{p}\frac{H(a_i, 0), a_j)}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}}\Bigg\}.$$

Without loss of generality, we can assume that $\lambda_i d_i$: $\lambda_1 d_1 \leq \lambda_2 d_2 \leq \cdots \leq \lambda_p d_p$. Let us define

$$I = \{1\} \cup \{i,\ \text{s.t.}\ \forall\, k \leq i, c_2\lambda_k d_k \leq \lambda_{k-1}d_{k-1} \leq \lambda_k d_k\},$$

where c_2 is a constant chosen small enough.

Case 1: $I \cap I_2 \neq \emptyset$ and $I \neq \{1, \ldots, p\}$. We order all the concentrations λ_i, $i \in I_2$. Assume that

$$\lambda_{i_1} \leq \lambda_{i_2} \leq \cdots \leq \lambda_{i_r}.$$

Let

$$W_1 = -\sum_{k=1}^{r} 2^k \alpha_{i_k}\lambda_{i_k}\frac{\partial P\delta_{i_k}}{\partial \lambda_{i_k}}.$$

We claim that

$$\langle \partial J(u), W_1(u)\rangle \leq -c\sum_{i\in I_2}\Bigg(\frac{1}{(\lambda_i d_i)^{n-2s}} + \sum_{j\neq i}\varepsilon_{ij}\Bigg) + R. \tag{5.1}$$

Indeed, using Proposition (3.2), we derive that

$$\begin{aligned}\langle \partial J(u), W_1(u)\rangle = 2c_1 J(u)\sum_{k=1}^{r}\Bigg[&-\sum_{j\neq i_k} 2^k\alpha_j\alpha_{i_k}\lambda_{i_k}\frac{\partial \varepsilon_{j i_k}}{\partial \lambda_{i_k}}(1+o(1)) \\ &-\frac{n-2s}{2}\sum_{j=1}^{p}2^k\alpha_j\alpha_{i_k}\frac{H((a_j, 0), a_{i_k})}{(\lambda_j\lambda_{i_k})^{\frac{n-2s}{2}}}(1+o(1)) + R\Bigg].\end{aligned} \tag{5.2}$$

Observe that

$$-\lambda_i\frac{\partial \varepsilon_{ij}}{\partial \lambda_i} = \frac{n-2s}{2}\varepsilon_{ij}\Bigg(1-2\frac{\lambda_j}{\lambda_i}\varepsilon_{ij}^{\frac{2}{n-2s}}\Bigg). \tag{5.3}$$

Thus for $\lambda_i \geq \lambda_j$,

$$-2\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} - \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \geq -\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} = \frac{n-2s}{2} \varepsilon_{ij} + O\Big(\varepsilon_{ij}^{\frac{n+2-2s}{n-2s}}\Big). \tag{5.4}$$

Furthermore, arguing as in [7,24] and using the maximum principle, the regular part of the Green's function satisfies

$$H((a_i,0),a_j) \leq \max(d_i,d_j)^{2s-n}. \tag{5.5}$$

For $j \in I_1$ and $i \neq j$, if $d_j/2 \leq d_i \leq 2d_j$, using (5.5), we obtain

$$\frac{\lambda_j}{\lambda_i} \varepsilon_{ij}^{\frac{2}{n-2s}} = o(1). \tag{5.6}$$

In the other case (i.e. $d_i \leq d_j/2$ or $d_i \geq 2d_j$), we use the inequality $|a_i - a_j| \geq \frac{1}{2}\max(d_i,d_j)$ to obtain (5.6). Thus

$$\langle \partial J(u), W_1(u)\rangle \leq -(n-2s)c_1 J(u) \sum_{i\in I_2}\Big[\sum_{i\neq j}\varepsilon_{ij}(1+o(1)) - 2^p \sum_{j=1}^{p} \frac{H((a_i,0),a_j)}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}}\Big] + R. \tag{5.7}$$

Since $i \in I_2$, we obtain

$$\langle \partial J(u), W_1(u)\rangle \leq -c\sum_{i\in I_2}\Big[\sum_{i\neq j}\varepsilon_{ij} + 2^p \sum_{j=1}^{p} \frac{H((a_i,0),a_j)}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}}\Big] + R. \tag{5.8}$$

A similar computation as in the proof of (2.8) of [24] shows that

$$H((a_i,0),a_i) = (2d_i)^{2s-n} + o(d_i^{2s-n}), \tag{5.9}$$

for each point a_i near the boundary. From another part, for each a_i in a compact set K of Ω, we have $H((a_i,0)a_i) \geq c$. Thus

$$H((a_i,0),a_i) \geq c(d_i)^{2s-n}, \tag{5.10}$$

for each $a_i \in \Omega$. Using (5.10) and (5.8), claim (5.1) follows.

Since $\lambda_1 d_1 \leq \lambda_2 d_2 \leq \cdots \leq \lambda_p d_p$, we can make appear the term $(\lambda_i d_i)^{-(n-2s)}$ in the upper bound of (5.1). Therefore,

$$\langle \partial J(u), W_1(u)\rangle \leq -c\Big(\sum_{i=1}^{p}\frac{1}{(\lambda_i d_i)^{n-2s}} + \sum_{i\in I_2, j\neq i}\varepsilon_{ij}\Big) + R. \tag{5.11}$$

For $i \in I_1$, we have

$$\frac{1}{2^{p+1}}\sum_{k\neq i}\varepsilon_{ki} \leq \sum_{j=1}^{p}\frac{H(a_i,0),a_j)}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}} \leq \frac{p}{(\lambda_1 d_1)^{n-2s}}. \tag{5.12}$$

Therefore, from $(\lambda_1 d_1)^{-(n-2s)}$ we can make appear the term $\sum_{i\in I_1, j\neq i}\varepsilon_{ij}$ in the upper bound of (5.11). Hence the estimates (i) of Theorem 5.1 follows in this case.

Case 2: $I \cap I_2 \neq \emptyset$ and $I = \{1,\ldots,p\}$. Let $i_1 = \min\{i,\ \text{s.t. } i \in I_2\}$ and $I_{i_1} = \{j \notin I_2,\ \text{s.t. } \lambda_{i_1} d_{i_1} \leq 2c_2\lambda_j d_j\}$. Let

$$W_2 = -\sum_{i\in I_2\cup I_{i_1}} \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}.$$

Using Proposition 3.2, we have

$$\langle \partial J(u), W_2(u)\rangle = 2c_1 J(u) \sum_{i\in I_2\cup I_{i_1}} \Big[-\sum_{j\neq i} 2^i \alpha_j \alpha_i \lambda_i \frac{\partial \varepsilon_{ji}}{\partial \lambda_i}(1+o(1)) - \frac{n-2s}{2}\sum_{j=1}^{p} 2^i \alpha_j\alpha_i \frac{H((a_j,0),a_i)}{(\lambda_j\lambda_i)^{\frac{n-2s}{2}}}(1+o(1))\Big] + R. \tag{5.13}$$

Since $I=\{1,\dots,p\}$, we have for each $i\neq j$,

$$\varepsilon_{ij} = \left(\frac{1}{\lambda_i\lambda_j|a_i-a_j|^2}\right)^{\frac{n-2s}{2}}\left(1+O\left(\frac{1}{\lambda_i^2|a_i-a_j|^2}+\frac{1}{\lambda_j^2|a_j-a_j|^2}\right)\right) = \left(\frac{1}{\lambda_i\lambda_j|a_i-a_j|^2}\right)^{\frac{n-2s}{2}} + O\left(\frac{1}{(\lambda_1 d_1)^{n+2-2s}}+\varepsilon_{ij}^{\frac{n}{n-2s}}\right). \tag{5.14}$$

Indeed, if $d_i\le d_j/2$ or $d_i\ge 2d_j$, we have $|a_i-a_j|\ge \frac{1}{2}\max(d_i,d_j)$ and the result follows. In the other case, if $d_j/2\le d_i\le 2d_j$, using that $i,j\in I$, we derive that $\frac{\lambda_i}{\lambda_j}$ and $\frac{\lambda_j}{\lambda_i}$ are bounded. Therefore, $(\lambda_k|a_i-a_j|)^{-2}=O(\varepsilon_{ij}^{\frac{2}{n-2s}})$ for $k=i,j$. Thus, for $i\in I_{i_1}$, using (5.3) and (5.14), we get

$$\sum_{j\neq i}\left(-\lambda_i\frac{\partial\varepsilon_{ji}}{\partial\lambda_i} - \frac{n-2s}{2}\frac{H((a_i,0),a_j)}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}}\right) = \frac{n-2s}{2}\sum_{j\neq i}\frac{G((a_i,0),a_j)}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}} + O\left(\frac{1}{(\lambda_1 d_1)^{n+2-2s}}+\varepsilon_{ij}^{\frac{n}{n-2s}}\right). \tag{5.15}$$

Furthermore, for $i\in I_{i_1}$,

$$\frac{H((a_i,0),a_i)}{(\lambda_i)^{n-2s}} = O\left(\frac{1}{(\lambda_i d_i)^{n-2s}}\right) = O\left(\frac{2c_2}{(\lambda_{i_1}d_{i_1})^{n-2s}}\right) = o\left(\frac{1}{(\lambda_{i_1}d_{i_1})^{n-2s}}\right).$$

For $i\in I_2$, using (5.3) and (5.14), we get

$$\sum_{j\neq i}\lambda_i\frac{\partial\varepsilon_{ji}}{\partial\lambda_i} + \frac{n-2s}{2}\sum_{j=1}^{p}\frac{H((a_i,0),a_j)}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}} \le -c\left(\sum_{j\neq i}\varepsilon_{ij}+\frac{1}{(\lambda_i d_i)^{\frac{n}{n-2s}}}\right). \tag{5.16}$$

Therefore, using the fact that the Green's function is positive, we derive

$$\langle\partial J(u), W_2(u)\rangle \le -c\sum_{i\in I_2}\left(\sum_{j\neq i}\varepsilon_{ji}+\frac{1}{(\lambda_i d_i)^{n-2s}}\right)+R. \tag{5.17}$$

Using the fact that $I\cap I_2\neq\emptyset$ and arguing as in the Case1, estimate (5.11) is valid. Therefore, we can make appear the term $\sum_{i\in I_1, j\neq i}\varepsilon_{ij}$ in the upper bound of (5.11). Thus estimate (i) of Theorem 5.1 follows in this case.

For c_3 a fixed small constant, let us define

$$L=\{j\in I_1,\ \text{s.t. } \exists\, i\in I_1,\ \text{s.t. } c_3\max(d_i,d_j)\ge|a_i-a_j|\}.$$

For $i\in L$, let i_0 the index such that

$$c_3\max(d_i,d_{i_0})\ge|a_i-a_{i_0}|. \tag{5.18}$$

Case 3: $I \cap I_2 = \emptyset$ and there exists $i, i_0 \in I$ satisfying (5.18). Let

$$W_3 = -\alpha_i \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}.$$

We have

$$\langle \partial J(u), W_3(u)\rangle = 2c_1 J(u)\Big[-\sum_{j\neq i} \alpha_j \alpha_i \lambda_i \frac{\partial \varepsilon_{ji}}{\partial \lambda_i}(1+o(1)) \\ - \frac{n-2s}{2}\sum_{j=1}^{p} \alpha_j \alpha_i \frac{H((a_j,0),a_i)}{(\lambda_j\lambda_i)^{\frac{n-2s}{2}}}(1+o(1))\Big] + R. \tag{5.19}$$

Arguing as in Case 1, the terms with $j \in I_2$ can be seen like $O(\varepsilon_{ij})$. Next, we interest with the indices $j \in I_1$. Observe that for $i, k \in I_1$, we have (5.14). Indeed, if $d_i \le d_k/2$ or $d_i \ge 2d_k$), we have $|a_i - a_k| \ge \frac{1}{2}\max(d_i, d_k)$ and the result follows. In the other case, if $d_k/2 \le d_i \le 2d_k$, using that $i, k \in I_1$, we have as in (5.6)

$$\varepsilon_{ij} = o\Big(\Big(\frac{\lambda_i}{\lambda_k}\Big)^{\frac{n-2s}{2}} + \Big(\frac{\lambda_k}{\lambda_i}\Big)^{\frac{n-2s}{2}}\Big). \tag{5.20}$$

Therefore,

$$\frac{1}{\lambda_r^2|a_i-a_k|^2} = \frac{\lambda_i}{\lambda_k}\frac{1}{\lambda_i\lambda_k|a_i-a_k|^2} \le c\frac{\lambda_i}{\lambda_k}\varepsilon_{ij}^{\frac{2}{n-2s}} \le c\frac{\lambda_i}{\lambda_k}\frac{1}{(\lambda_i d_i)(\lambda_1 d_1)} \\ = O\Big(\frac{1}{(\lambda_1 d_1)^2}\Big), \forall r = i,k.$$

Thus we obtain (5.15) with the indices $j \in I_1$. Using (5.10), and the fact that the Green's function is positive, we derive that

$$\langle \partial J(u), W_3(u)\rangle \le -c\Big(\frac{1}{(\lambda_i d_i)^{n-2s}} + \frac{G((a_i,0),a_{i_0})}{(\lambda_i\lambda_{i_0})^{\frac{n-2s}{2}}}\Big) + R + O\Big(\sum_{j\neq i, j\in I_2}\varepsilon_{ij}\Big). \tag{5.21}$$

Since $i, i_0 \in I$ satisfying (5.18), we can assume that $\lambda_i \ge \lambda_{i_0}$ and thus

$$\frac{1}{(\lambda_i d_i)^{n-2s}} + \frac{H((a_i,0),a_{i_0})}{(\lambda_i\lambda_{i_0})^{\frac{n-2s}{2}}} \le \Big(\frac{c_3^2}{\lambda_i\lambda_{i_0}|a_i-a_{i_0}|^2}\Big)^{\frac{n-2s}{2}}. \tag{5.22}$$

Using (5.14), we derive that

$$\langle \partial J(u), W_3(u)\rangle \le -c\Big(\varepsilon_{ii_0} + \frac{1}{(\lambda_i d_i)^{n-2s}}\Big) + R + O\Big(\sum_{j\neq i, j\in I_2}\varepsilon_{ij}\Big). \tag{5.23}$$

Since $i \in I$ and the term $(\lambda_i d_i)^{2s-n}$ appears in the upper-bound of the above estimate, we argue as in the Case 1, we can make appear all the $(\lambda_k d_k)^{2s-n}$ and $\sum_{k\neq j, k\in I_1}\varepsilon_{kj}$ in this upper-bound. For m_1, a fixed large constant, the pseudo-gradient $W_3 + m_1 W_1$ satisfies estimate (i) of Theorem 5.1.

Case 4: $I \cap I_2 = \emptyset$ and $\forall i, i_0 \in I$, $c_3 \max(d_i, d_{i_0}) < |a_i - a_{i_0}|$. Let d_0 be a fixed small positive constant. We introduce the following sets:

$$I' = \{i \in I, d_i < d_0\}$$

and

$$L_i = \{j \in L,\ \text{s.t. } i \text{ and } j \text{ satisfy (5.18)}\}.$$

Case 4.1: If $I' \neq \emptyset$, let

$$W_4 = \sum_{i \in I'} \alpha_i \frac{\partial P\delta_i}{\partial a_i}\Big(-\frac{\eta_i}{\lambda_{j_0}}\Big),$$

where $\lambda_{j_0} = \max\{\lambda_i, i \in I'\}$ and η_i is the outward normal to $\partial\Omega_{d_i} = \{x \in \Omega,\ \text{s.t. } d(x, \partial\Omega) = d_i\}$ at a_i.

Using Proposition 3.3, we have

$$\begin{aligned}
&\langle \partial J(u), W_4\rangle \\
&= \frac{J(u)c_1}{\lambda_{j_0}} \sum_{i \in I'} \Bigg[\frac{\alpha_i^2}{\lambda_i^{n-2s}} \frac{\partial H((a_i,0),a_i)}{\partial \eta_i}(1+o(1)) + O\Big(\lambda_i \sum_{i \neq j, j \in I_2 \cup L_i} \varepsilon_{ij}\Big) \\
&\quad + 2\sum_{j \neq i} \alpha_i \alpha_j \Bigg(\frac{\partial \varepsilon_{ij}}{\partial \eta_i} - \frac{\frac{\partial H((a_i,0),a_j)}{\partial \eta_i}}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}}\Bigg)\Big(1 - J(u)^{\frac{n}{n-2s}}(\alpha_j^{\frac{4s}{n-2s}} + \alpha_j^{\frac{4s}{n-2s}})\Big)\Bigg] \\
&\quad + O\Bigg(\lambda_i\Bigg(\sum_{i \neq j} \varepsilon_{ij}^{\frac{n}{n-2s}} \log \varepsilon_{ij}^{-1} + \sum_k \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^{n+2-2s}}\Bigg)\Bigg) \\
&\quad + O\Bigg(\sum_{i \neq j} \lambda_i^2 |a_i - a_j| \varepsilon_{ij}^{\frac{n+3-2s}{n-2s}}\Bigg).
\end{aligned}$$

Observe that for $i \in I'$ and $j \in I_1 \setminus (I \cup L_i)$, using (5.5), we have

$$\frac{\frac{\partial H((a_i,0),a_j)}{\partial \eta_i}}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}} \le \frac{cH((a_i,0),a_j)}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}} d_i} \le \frac{c}{d_i(\lambda_i d_i \lambda_j d_j)^{\frac{n-2s}{2}}} = o\Big(\frac{1}{d_i(\lambda_i d_i)^{n-2s}}\Big).$$

For i and $j \in I'$, if $\frac{d_i}{d_j}$, $\frac{d_j}{d_i}$ and $\frac{|a_i - a_j|}{d_i}$ are bounded and arguing as in the Appendix of [11], we derive that $\frac{\partial H}{\partial \eta_i}((a_i,0),a_j) > 0$. In the other case, we have

$$\begin{aligned}
\frac{\partial H}{\partial \eta_i}((a_i,0),a_j) &\le \frac{H((a_i,0),a_j)}{d_i} \le \frac{1}{d_i \max(d_i, d_j, |a_i - a_j|)^{n-2s}} \\
&= o\Bigg(\frac{1}{(d_i d_j)^{\frac{n+1-2s}{2}}}\Bigg).
\end{aligned} \tag{5.24}$$

Thus

$$\frac{1}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}} \frac{\partial H}{\partial \eta_i}((a_i,0),a_j) = o\Big(\frac{1}{d_i(\lambda_i d_i)^{n-2s}} + \frac{1}{d_j(\lambda_j d_j)^{n-2s}}\Big). \tag{5.25}$$

Observe that for each $i \in I'$, using (5.9) and arguing as [7,24], we have

$$\frac{\partial H}{\partial \eta_i}\big((a_i,0),a_i\big) = \frac{n-2s}{2^{n-2s}} \frac{1}{d_i^{n+1-2s}}\big(1 + o(1)\big). \tag{5.26}$$

Moreover, for $i, j \in I'$, we have $\eta_i - \eta_j = O(|a_i - a_j|)$. Therefore,

$$\frac{\partial \varepsilon_{ij}}{\partial a_i}\eta_i + \frac{\partial \varepsilon_{ij}}{\partial a_j}\eta_j = \frac{n-2s}{2}\lambda_i\lambda_j(a_i - a_j)\varepsilon_{ij}^{\frac{n+2-2s}{n-2s}}(\eta_j - \eta_i) = O(\varepsilon_{ij}). \tag{5.27}$$

Using the fact that $c_3 \max(d_i, d_j) \le |a_i - a_j|$, for $i \in I'$ and $j \in I_1 \setminus (I \cup L_i)$, we get

$$\left|\frac{\partial \varepsilon_{ij}}{\partial a_i}\right| = (n-2s)\lambda_i\lambda_j|a_i - a_j|\varepsilon_{ij}^{\frac{n+2-2s}{n-2s}} \le \frac{c}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}}\frac{1}{|a_i - a_j|^{n+1-2s}}$$
$$= O\Big(\frac{1}{c_3^{n+1-2s}(\lambda_i d_i \lambda_j d_j)^{\frac{n-2s}{2}} d_i}\Big)$$
$$= O\Big(\frac{c_2^{\frac{n-2s}{2}}}{c_3^{n+1-2s}(\lambda_i d_i)^{n-2s} d_i}\Big)$$
$$= o\Big(\frac{1}{(\lambda_i d_i)^{n-2s} d_i}\Big), \tag{5.28}$$

for c_2 and c_3 chosen such that $c_2^{\frac{n-2s}{2}} = o(c_3^{n+1-2s})$. For $i \in I'$ and $j \in I \setminus I'$, we claim that

$$\frac{\partial H}{\partial \eta_i}((a_i,0),a_j)\frac{1}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}} - \frac{\partial \varepsilon_{ij}}{\partial a_i}\eta_i = -\frac{\partial G((a_i,0),a_j)}{\partial \eta_i}\frac{1}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}}$$
$$+ o\Big(\frac{\lambda_i}{(\lambda_1 d_1)^{n+3-2s}}\Big). \tag{5.29}$$

Indeed, since $I \cap I_2 = \emptyset$ then i and j belong to I_1. Using (5.14) and the fact that (5.18) is not satisfied, we derive

$$\frac{\partial \varepsilon_{ij}}{\partial a_i} = (n-2s)\lambda_i\lambda_j(a_i - a_j)\varepsilon_{ij}^{\frac{n+2-2s}{n-2s}}$$
$$= \frac{(n-2s)(a_i - a_j)}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}|a_i - a_j|^{n+2-2s}}\Big(1 + O\Big(\frac{1}{(\lambda_1 d_1)^2}\Big)\Big).$$

Therefore,

$$\frac{\partial \varepsilon_{ij}}{\partial a_i} = \frac{1}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}}\frac{\partial}{\partial a_i}\Big(\frac{1}{|a_i - a_j|^{n-2s}}\Big) + O\Big(\frac{\lambda_i}{c_3^{n+1-2s}(\lambda_1 d_1)^{n+3-2s}}\Big).$$

and our claim follows. Thus

$$\langle \partial J(u), W_4(u)\rangle \le -\frac{c}{\lambda_{j_0}}\sum_{i\in I'}\Big[\frac{1}{(\lambda_i d_i)^{n-2s} d_i} - \sum_{j\in I\setminus I'}\frac{1}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}}\frac{\partial G((a_i,0),a_j)}{\partial \eta_i}$$
$$+ O\Big(\sum_{i\ne j, j\in I'}\varepsilon_{ij}\Big) + O\Big(\sum_{i\ne j, j\in I_2\cup L_i}\varepsilon_{ij}\Big)$$
$$+ O\Big(\sum_{i\ne j}\varepsilon_{ij}^{\frac{n}{n-2s}}\log\varepsilon_{ij}^{-1} + \sum_k \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^{n+2-2s}}\Big)$$
$$+ O\Big(\sum_{i\ne j}\lambda_i|a_i - a_j|\varepsilon_{ij}^{\frac{n+3-2s}{n-2s}}\Big)\Big].$$

Observe that $d_i \le d_j$ for $i \in I'$ and $j \in I \setminus I'$. Therefore, $-\frac{\partial G((a_i,0),a_j)}{\partial \eta_i} > 0$, see [7] and [24]. Now for $i, j \in I'$, we have

$$\varepsilon_{ij} = O\Big(\frac{1}{(\lambda_i d_i)^{n-2s}} + \frac{1}{(\lambda_j d_j)^{n-2s}}\Big).$$

Using the fact that d_i and d_j are small enough, we get

$$\varepsilon_{ij} = o\Big(\frac{1}{d_i(\lambda_i d_i)^{n-2s}} + \frac{1}{d_j(\lambda_j d_j)^{n-2s}}\Big).$$

Observe that $j_0 \in I'$ and $\lambda_{j_0} d_{j_0}$ and $\lambda_1 d_1$ are of the same order. Thus, we can make all the $(\lambda_i d_i)^{n+1-2s}$ for $i \in I_1$ appear in the upper bound of the last inequality. It follows that,

$$\langle \partial J(u), W_4(u)\rangle \le -c\sum_{i=1}^{p} \frac{1}{(\lambda_i d_i)^{n+1-2s}} + O\Big(\sum_{i\neq j, j\in I_2\cup L_i} \varepsilon_{ij}\Big) \\ +O\Big(\sum_{i\neq j} \varepsilon_{ij}^{\frac{n}{n-2s}} \log \varepsilon_{ij}^{-1} + \sum_{k} \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^{n+2-2s}}\Big) + O\Big(\sum_{i\neq j} \lambda_i |a_i - a_j| \varepsilon_{ij}^{\frac{n+3-2s}{n-2s}}\Big). \tag{5.30}$$

Let

$$W_5(u) = -\sum_{i\in I'} \alpha_i \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} - \sum_{j\in L_i} \alpha_j \lambda_j \frac{\partial P\delta_j}{\partial \lambda_j}.$$

Using Case 3, we get

$$\langle \partial J(u), W_5(u)\rangle \le -c\Big(\sum_{i\in I', j\in L_i} \varepsilon_{ij}\Big) + O\Big(\sum_{j\neq i, i\in I', j\in I_2} \varepsilon_{ij}\Big) + R. \tag{5.31}$$

For m_1 and m_2 two fixed large constants, using (5.11) (5.30) and (5.31), we get

$$\langle \partial J(u), W_4 + m_1 W_1 + m_2 W_5\rangle \le -c\Big(\sum_{i=1}^{p} \frac{1}{(\lambda_i d_i)^{n+1-2s}} + \sum_{i\neq j, j\in I_2} \varepsilon_{ij}\Big) + R \\ +O\Big(\sum_{i\neq j} \lambda_i |a_i - a_j| \varepsilon_{ij}^{\frac{n+3-2s}{n-2s}}\Big). \tag{5.32}$$

As in the Case 1, we can make appear the term $\sum_{i\in I_1, j\neq i} \varepsilon_{ij}$ in the upper bound of (5.11). Hence the estimates (i) of Theorem 5.1 follows in this case.

Case 4.2: If $I' = \emptyset$, then $d_i \ge d_0$ for any $i \in I$. Let $M = (m_{ij})_{i,j\in I}$ be the matrix defined in (2.6). Let ρ its least eigenvalue and e is the eigenvector associated to ρ. Fix $\eta > 0$. We set

$$C(e,\eta) = \Big\{y \in (\mathbb{R}_+^*)^r, r = card I, \text{ s.t. } \Big|\frac{y}{|y|} - e\Big| < \eta\Big\}.$$

For any $x \in C(e,\eta)$, η small, we have

$$t_x M x - \rho|x|^2 \le \frac{1}{2}|\rho||x|^2 \tag{5.33}$$

and

$$t_x \frac{\partial M}{\partial a_i} x = \Big(\frac{\partial \rho}{\partial a_i} + o(1)\Big)|x|^2. \tag{5.34}$$

For any $x \in C(e,\eta)^c$, we have

$$t_x M x - \rho|x|^2 > c|x|^2.$$

Denote $I = \{j_1, \ldots, j_r\}$ and $\Lambda = t\left(\frac{1}{\lambda_{j_1}^{\frac{n-2s}{2}}}, \ldots, \frac{1}{\lambda_{j_r}^{\frac{n-2s}{2}}}\right)$.

If Λ belongs to the set $C(e,\eta)^c$, then we move the vector Λ to $C(e,\eta)$ as in [12] along

$$\Lambda(t) = |\Lambda| \frac{(1-t)\Lambda + t|\Lambda|e}{|(1-t)\Lambda + t|\Lambda|e|}.$$

Using Proposition (3.2), we derive that there exists a pseudo-gradient W_6 such that

$$\langle \partial J(u), W_6 \rangle = c\left[\frac{d}{dt}(t_\Lambda(t) M \Lambda(t)) + o\left(\sum_{j\neq i, i, j \in I} \varepsilon_{ij} + \sum_{i\in I} \frac{1}{\lambda_i^{n-2s}}\right) + O\left(\sum_{i\neq j, i\in I, j\in I_2 \cup I_1 \setminus I} \varepsilon_{ij}\right) + R\right]. \tag{5.35}$$

As in [12], we have

$$\frac{d}{dt}(t_\Lambda(t) M \Lambda(t)) < -c|\Lambda|^2 = -c\sum_{i\in I} \frac{1}{\lambda_i^{n-2s}}$$

and

$$\varepsilon_{ij} = O\left(\left(\frac{1}{(\lambda_1 d_1)(\lambda_j d_j)}\right)^{\frac{n-2s}{2}}\right) = O\left(\frac{c_2^{\frac{n-2s}{2}}}{d_0^{n-2s}\lambda_1^{n-2s}}\right) = o\left(\frac{1}{\lambda_1^{n-2s}}\right), \tag{5.36}$$

for $i \in I$, $j \in I_1 \setminus I$. Thus

$$\langle \partial J(u), W_6(u) \rangle \leq -c\sum_{i\in I} \frac{1}{\lambda_i^{n-2s}} + o\left(\sum_{j\neq i, i, j\in I} \varepsilon_{ij} + \sum_{i\in I} \frac{1}{\lambda_i^{n-2s}}\right) + R. \tag{5.37}$$

If Λ belongs to $C(e,\eta)$, the construction of the vector-field W_6 depends on the value of ρ and $|\rho'|$. Since zero is a regular value of ρ then there exists a constant $\rho_0 > 0$ such that either $|\rho| > \rho_0$ or $|\rho'| > \rho_0$.

If $\rho < -\rho_0$, we decrease all the λ_i's for $i \in I$. If we assume that $c_2^{\frac{n-2s}{2}} = o(\rho_0 d_0^{n-2s})$ then using Proposition 3.2, (5.3) (5.5) and (5.36), we obtain (5.37) in this case.

If $|\rho'| > \rho_0$ and $\rho > -\rho_0$, then we move the points a_i's along $\lambda_{j_0}\dot{a}_i = -\frac{\partial\rho}{\partial a_i}$ for each $a_i \in I$ and $\lambda_{j_0} = \max\{\lambda_i, i \in I\}$. Using Proposition 3.3, we derive

$$\begin{aligned}
\langle \partial J(u), W_6 \rangle &= \frac{1}{\lambda_{j_0}} \sum_{i\in I} \left\langle \partial J(u), \frac{\partial P\delta_{a_i,\lambda_i}}{\partial a_i} \right\rangle \left(-\frac{\partial \rho}{\partial a_i}\right) \\
&= -\frac{1}{\lambda_{j_0}} \sum_{i\in I} \left[c\left(\frac{\partial\rho}{\partial a_i}\right)\left(t_\Lambda \frac{\partial M}{\partial a_i}\Lambda\right) + O\left(\sum_{j\in I_2 \cup L_i} \lambda_i \varepsilon_{ij}\right)\right. \\
&\quad + O\left(\sum_{j\in I_1 \setminus (I\cup L_i)} \frac{\partial H((a_i,0),a_j)}{\partial a_i} \frac{1}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}} + \frac{\partial \varepsilon_{ij}}{\partial a_i}\right) \\
&\quad + O\left(\sum_{i\neq j} \varepsilon_{ij}^{\frac{n}{n-2s}} \log \varepsilon_{ij}^{-1} + \sum_k \frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^{n+2-2s}}\right) \\
&\quad \left. + O\left(\sum_{i\neq j} \lambda_i |a_i - a_j| \varepsilon_{ij}^{\frac{n+3-2s}{n-2s}}\right)\right].
\end{aligned} \tag{5.38}$$

Observe that for $i \in I$ and $j \in I_1 \backslash (I \cup L_i)$, (5.18) is not satisfied. Thus

$$\frac{\partial H((a_i,0),a_j)}{\partial a_i}\frac{1}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}} \le \frac{1}{d_i^{n+1-2s}}\frac{1}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}} \le \frac{(c_2D)^{\frac{n-2s}{2}}}{\lambda_1^{n-2s}d_0^{\frac{3n+2-6s}{2}}}, \quad (5.39)$$

where D is the diameter of Ω. Arguing as in (5.28), we obtain

$$\begin{aligned}\left|\frac{\partial \varepsilon_{ij}}{\partial a_i}\right| &\le \frac{c}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}}\frac{1}{|a_i-a_j|^{n+1-2s}} \\ &\le \frac{(c_2D)^{\frac{n-2s}{2}}}{c_3^{n+1-2s}\lambda_1^{n-2s}d_0^{\frac{3n+2-6s}{2}}}. \end{aligned} \quad (5.40)$$

We can chose c_2 and c_3 such that

$$\frac{\partial H((a_i,0),a_j)}{\partial a_i}\frac{1}{(\lambda_i\lambda_j)^{\frac{n-2s}{2}}} + \left|\frac{\partial \varepsilon_{ij}}{\partial a_i}\right| = o\Big(\frac{1}{\lambda_1^{n-2s}}\Big). \quad (5.41)$$

Since $\Lambda \in C(e,\eta)$, then using (5.33), (5.34), (5.38) and (5.41), we obtain

$$\begin{aligned}\langle \partial J(u), W_6\rangle &\le -\frac{c}{2\lambda_{j_0}}|\rho'|^2|\Lambda|^2 + O\Big(\sum_{i\in I, j\in I_2\cup L_i}\lambda_i\varepsilon_{ij}\Big) \\ &\quad + R + O\Big(\sum_{i\neq j}\lambda_i|a_i-a_j|\varepsilon_{ij}^{\frac{n+3-2s}{n-2s}}\Big) \\ &\le -\frac{c}{2\lambda_{j_0}}\rho_0^2|\Lambda|^2 + O\Big(\sum_{i\in I, j\in I_2\cup L_i}\lambda_i\varepsilon_{ij}\Big) \\ &\quad + R + O\Big(\sum_{i\neq j}\lambda_i|a_i-a_j|\varepsilon_{ij}^{\frac{n+3-2s}{n-2s}}\Big). \end{aligned} \quad (5.42)$$

Thus in both cases, the pseudo-gradient $W_6 + m_1W_1 + m_2W_5$ where m_1 and m_2 are two positive constants large enough, satisfies (5.32) and then the estimate (i) of Theorem 5.1 follows in this case.

The required vector field W required in Theorem 5.1 will be defined by a convex combination of $W_i, i = 1, \dots, 6$. It satisfies Claim (i) of Theorem 5.1. Moreover, by the argument of Corollary B.3 of [12], it satisfies Claim (ii). This completes the proof.

COROLLARY 5.2

Under the assumption of Theorem 5.1, the only critical points at infinity for J are $\sum_{i=1}^p P\delta_{(a_i,\infty)}$, $p \ge 1$, where $\rho(a_1,\dots,a_p) > 0$ and $\rho'(a_1,\dots,a_p) = 0$. Moreover, all concentration points of any critical point at infinity lie in a compact set K of Ω.

6. Proof of the results

Proof of Theorem 2.3. It follows from the result of Theorem 5.1 and similar arguments of Appendix 2 of [8] (see also [12]).

Proof of Theorem 2.4. Claim (i) follows from the inequalities of Theorem 5.1. Claim (ii) corresponds to the result of Corollary 5.2. Concerning Claim (iii), we know that for a large integer p_0, there exists at least two points a_i and a_j in K, (where K is defined in Corollary 5.2), such that $|a_i - a_j|$ is very small. Therefore, any related $p_0 \times p_0$-matrix, $M(a_1, \ldots, a_{p_0})$ is not positive definite. Claim (iii) follows from Corollary 5.2.

Proof of Theorem 2.5. Using the result of Corollary 5.2, the only critical points at infinity for J are $\sum_{i=1}^{p} P\delta_{(a_i,\infty)}$ with $\rho(a_1, \ldots, a_p) > 0$ and $\rho'(a_1, \ldots, a_p) = 0$. Near each critical point at infinity, the normal form of the expansion of J presented by Theorem 2.3, shows that the relative homology between W_p and W_{p-1} is given by the product of the homologies defined by each variables. This conclude the proof.

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References

[1] Abdelhedi W, Chtioui H and Hajaiej H, The Bahri–Coron theorem for fractional Yamabe-type problems, *Adv. Nonlinear Stud.* **18**(2) (2018) 393–407

[2] Abdullah Sharaf K and Chtioui H, Conformal metrics with prescribed fractional *Q*-curvatures on the standard *n*-dimensional sphere, *Differential Geom. Appl.* **68** (2020) Article 101562 on the standard *n-dimensional sphere, Differential Geom. Appl.* **68** (2020) Article 101562

[3] Abdullah Sharaf K and Chtioui H, The topological contribution of the critical points at infinity for critical fractional Yamabe-type equations, *SN Partial Differ. Equ. Appl.* **3** (2020) https://doi.org/10.1007/s42985-020-00011-5

[4] Ahmedou M and Chtioui H, Conformal metrics of prescribed scalar curvature on 4-manifolds: the degree zero case, *Arab. J. Math.* **6(3)** (2017) 127–136

[5] Alghanemi A and Chtioui H, Prescribing scalar curvatures on *n*-dimensional manifolds, *C. R. Acad. Bulg. Sci.* **73(2)** (2020) 163–169

[6] Alghanemi A and Chtioui H, Perturbation theorems for fractional critical equations on bounded domains, *J. Aust. Math. Soc.* published online (2020) 1–20, https://doi.org/10.1017/S144678871900048X

[7] Bahri A, Critical point at infinity in some variational problems, Pitman Res. Notes Math. Ser., vol. 182 (1989) (Harlow: Longman Sci. Tech.)

[8] Bahri A, An invariant for yamabe-type flows with applications to scalar curvature problems in high dimensions, a celebration of J. F. Nash Jr., *Duke Math. J.* **81** (1996) 323–466

[9] Bahri A and Coron J M, On a nonlinear elliptic equation involving the critical Sobolev exponent: The effect of topology of the domain, *Comm. Pure Appl. Math.* **41** (1988) 255–294

[10] Bahri A and Brezis H, Elliptic differential equations involving the Sobolev critical exponent on manifolds, PNLDE, vol. 20 (1996) (Birkhauser, Basel)

[11] Bahri A, Li Y and Rey O, On a variational problem with lack of compactness: The topological effect of the critical points at infinity, *Calc. Var. Partial Differential Equations* **3** (1995) 67–94

[12] Ben Ayed M, Chen Y, Chtioui H and Hammami M, On the prescribed scalar curvature problem on 4-manifolds, *Duke Math. J.* **84** (1996) 633–677

[13] Brandle C, Colorado E, dePablo A and Sánchez U, A concave-convex elliptic problem involving the fractional Laplacian, *Proc. Roy. Soc. Edinburgh Sect. A* **143** (2013) 39–71

[14] Cabré X and Tan J, Positive solutions of nonlinear problems involving the square root of the Laplacian, *Adv. Math.* **224** (2010) 2052–2093

[15] Caffarelli L and Silvestre L, An extension problem related to the fractional Laplacian, *Comm. Partial. Diff. Equ.* **32** (2007) 1245–1260

[16] Capella A, Dávila J, Dupaigne L and Sire Y, Regularity of radial extremal solutions for some non-local semi linear equations, *Comm. Partial Diff. Equ.* **36** (2011) 1353–1384

[17] Chen W, Li C and Ou B, Classification of solutions for an integral equation, *Comm. Pure Appl. Math.* **59** (2006) 330–343

[18] Chtioui H, Hajaiej H and Soula M, The scalar curvature problem on four-dimensional manifolds, *Comm. Pure Appl. Anal.* **19(2)** (2020) 723–746

[19] Choi W, Kim S and Lee K, Asymptotic behavior of solutions for nonlinear elliptic problems with the fractional Laplacian, *J. Funct. Anal.* **266(11)** (2014) 6531–6598

[20] Li Y, Remark on some conformally invariant integral equations: the method of moving spheres, *J. Eur. Math. Soc. (JEMS)* **6** (2004) 153–180

[21] Li Y Y and Zhu M, Uniqueness theorems through the method of moving spheres, *Duke Math. J.* **80** (1995) 383–417

[22] Pohozaev S, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, *Soviet Math. Dokl.* **6** (1965) 1408–1411

[23] Sharaf K, On a critical fourth order PDE with Navier boundary condition, *Acta Math. Sinica* **35** (2019) 1906–1916

[24] Rey O, The role of the Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent, *J. Funct. Anal.* **89** (1990) 1–52

[25] Stinga P and Torrea J, Extension problem and Harnack's inequality for some fractional operators, *Comm. Partial Diff. Equ.* **35** (2010) 2092–2122

[26] Struwe M, A global compactness result for elliptic boundary value problem involving limiting nonlinearities, *Math. Z.* **187** (1984) 511–517

[27] Tan J, The Brezis Nirenberg type problem involving the square root of the Laplacian, *Cal. Var. Partial Diff. Equ.* **42** (2011) 21–41

[28] Tan J, Positive solutions for non local elliptic problems, *Discrete Contin. Dyn. Syst.* **33** (2013) 837–859

[29] Xiao J, A sharp Sobolev trace inequality for the fractional-order derivatives *Bull. Sci. Math.* **130** (2006) 87–96

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