Proc. Indian Acad. Sci. (Math. Sci.) (2021) 131:28
https://doi.org/10.1007/s12044-021-00625-8

A family of tetravalent half-arc-transitive graphs

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MS received 19 August 2020; revised 27 October 2020; accepted 19 November 2020

Abstract. Alspach *et al*. (*J. Austral. Math. Soc*. **56(3)** (1994) 391–402) constructed an infinite family of tetravalent graphs $M(a; m, n)$ and proved that if $n \geq 9$ be odd and $a^3 \equiv 1 (\mathrm{mod}\ n)$, then $M(a; 3, n)$ is half-arc-transitive. In this paper, we show that if $a^3 \equiv 1 (\mathrm{mod}\ n)$, then $M(a; 3, n)$ is an infinite family of tetravalent half-arc-transitive Cayley graphs for all integers n except 7 and 14.

Keywords. Half-arc-transitive graph; graph automorphism; cycles.

2008 Mathematics Subject Classification. 05C25, 20B25, 05E18.

1. Introduction

A graph $G = (V, E)$ is said to be vertex-transitive, edge-transitive and arc-transitive if the automorphism group of G, $\mathsf{Aut}(G)$, acts transitively on the vertices, on the edges and on the arcs of G respectively. It is known that an arc-transitive graph is both vertex-transitive and edge-transitive. However, a graph which is both vertex-transitive and edge-transitive may not be arc-transitive, the smallest example being the Holt graph [10] on 27 vertices. Such graphs are called *half-arc-transitive* graphs. For other definitions related to algebraic graph theory, one is referred to [9].

The study of half-arc-transitive graphs was initiated by Tutte [13], who proved that any half-arc-transitive graph is of even degree. Since any connected 2-regular is a cycle and a cycle is arc-transitive, the first possibility of finding a half-arc-transitive graph is a 4-regular or tetravalent graph. The first examples of tetravalent half-arc-transitive graphs were given by Bouwer [3] and the smallest example was given by Holt [10]. Though numerous papers have been published in the last 50 years, the classification of tetravalent half-arc-transitive graphs is not yet complete. In the absence of a complete classification, two major approaches have been fruitful so far: the first is to characterize half-arc-transitive graphs of some particular orders like p^3, p^4, p^5, pq, $2pq$, etc. [5–8], and the second is to come up with infinite families of half-arc-transitive graphs [4, 14].

In [1], Alspach *et al.* constructed an infinite family of tetravalent graphs $M(a; m, n)$ and proved as follows.

Published online: 25 September 2021

Theorem 1.1 [1, Theorem 3.3]. *Let $n \geq 9$ be odd and $a^3 \equiv 1 (\text{mod } n)$. Then $M(a; 3, n)$ is half-arc-transitive.*

In this paper, we prove that $M(a; 3, n)$ is half-arc-transitive for all integers n except 7 and 14. For this, we redefine $M(a; 3, n)$ (as $\Gamma(n, a)$ in Definition 1.1) in a different way which helps us to prove the half-arc-transitivity of the entire family (not only when n is odd). It turns out that $\Gamma(n, a)$ is a family of Cayley graphs of order $3n$ and for $(n, a) = (9, 4)$, we get the Holt graph. In fact, our definition (Definition 1.1) is a generalization of an alternative construction of the Holt graph (see the last paragraph of [10]).

DEFINITION 1.1

Let n be a positive integer such that $3 \mid \varphi(n)$, where φ denotes the Euler totient function. Then $\mathbb{Z}_n^*$, the group of units of $\mathbb{Z}_n$, is a group of order a multiple of 3. Let a be an element of order 3 in $\mathbb{Z}_n^*$ and $b \equiv a^2 (\text{mod } n)$. Define $\Gamma(n, a)$ to be the graph with vertex-set $\mathbb{Z}_n \times \mathbb{Z}_3$ and the edge-set composed of edges of the form $(i, j) \sim (ai \pm 1, j - 1)$ and $(i, j) \sim (bi \pm b, j + 1)$, where the operations in the first and second coordinates are done modulo n and modulo 3, respectively.

It is obvious that $\Gamma(n, a)$ is tetravalent. One can check that $\Gamma(n, a)$ is a suitable redefinition of $M(a; 3, n)$ and $\Gamma(9, 4)$ is the Holt graph. It is also to be noted that for a particular n, we can have two graphs, $\Gamma(n, a)$ and $\Gamma(n, b)$. However, these two graphs are isomorphic via the automorphism $\tau : \Gamma(n, a) \to \Gamma(n, a^2)$ defined by $\tau(i, j) = (ai, -j)$. So, without loss of generality, we assume that $a < b$, where $a, b \in \{2, \ldots, n - 2\}$.

On the other hand, let n be a positive integer such that a_1, b_1, a_2, b_2 are four elements of order 3 in $\mathbb{Z}_n^*$ with $a_1 b_1 \equiv 1 (\text{mod } n)$ and $a_2 b_2 \equiv 1 (\text{mod } n)$. Then, by the above argument, $\Gamma(n, a_1) \cong \Gamma(n, b_1)$ and $\Gamma(n, a_2) \cong \Gamma(n, b_2)$. However, $\Gamma(n, a_1)$ may not be isomorphic to $\Gamma(n, a_2)$. For example, if $n = 63$, we have $4 \cdot 16 \equiv 1 (\text{mod } 63)$ and $22 \cdot 43 \equiv 1 (\text{mod } 63)$, but $\Gamma(63, 4)$ is not isomorphic to $\Gamma(63, 22)$, as the odd girth of $\Gamma(63, 4)$ is 9, whereas that of $\Gamma(63, 22)$ is 21.

The definition of $\Gamma(n, a)$ requires that $3 | \varphi(n)$. We discuss the form of n for which this holds. Let $n = {p_1}^{\alpha_1} {p_2}^{\alpha_2} \cdots {p_k}^{\alpha_k}$, where the p_i are primes. Then $\varphi(n) = {p_1}^{\alpha_1 - 1} {p_2}^{\alpha_2 - 1} \cdots {p_k}^{\alpha_k - 1} (p_1 - 1)(p_2 - 1) \cdots (p_k - 1)$. As $3 | \varphi(n)$, either $3 | {p_i}^{\alpha_i - 1}$ or $3 | (p_i - 1)$ for some i, i.e., $9 | n$ or $p_i \equiv 1 (\text{mod } 3)$ for some i. Thus n is either of the form $9t$ or pt, where p is a prime of the form $1 (\text{mod } 3)$ and t is a positive integer.

At this junction, it is important to note the difference between our proof and the proof of [1].

(1) First, the proof techniques are entirely different: While their proof is built on semiregular automorphisms and blocks, ours is based on 6-cycles present in the graph.
(2) Second and most importantly, we prove that $\Gamma(n, a)$ is half-arc-transitive for all n except 7 and 14, i.e., n is not necessarily odd, so that we prove the result for a larger family of graphs.

In the next section, we prove the main results related to the automorphism group and half-arc-transitivity of $\Gamma(n, a)$. In the Appendix, we provide the SageMath [12] code for computing the automorphism group of $\Gamma(n, a)$.

2. Automorphisms of $\Gamma(n, a)$

Let $G = \mathsf{Aut}(\Gamma(n, a))$. We start by noting the following automorphisms of $\Gamma(n, a)$:

$$\alpha : (i, j) \mapsto (i + a^{-j}, j); \quad \beta : (i, j) \mapsto (i, j + 1); \quad \gamma : (i, j) \mapsto (-i, j).$$

It can be shown that $\alpha, \beta, \gamma \in G$ and $\circ(\alpha) = n$, $\circ(\beta) = 3$ and $\circ(\gamma) = 2$. Moreover, we have the following relations: $\alpha\beta = \beta\alpha^{a^2}$, $\alpha\gamma = \gamma\alpha^{-1}$ and $\beta\gamma = \gamma\beta$.

Theorem 2.1. *$\Gamma(n, a)$ is a Cayley graph.*

Proof. Let $H = \langle \alpha, \beta \rangle$. Clearly it forms a subgroup of G. Also, as $\circ(\alpha) = n$, $\circ(\beta) = 3$ and $\alpha\beta = \beta\alpha^{a^2}$, we have

$$H = \{\alpha^i \beta^j : 0 \le i \le n - 1, 0 \le j \le 2\} \quad \text{and} \quad |H| = 3n = |\Gamma(n, a)|.$$

We will show that H acts regularly on $\Gamma(n, a)$. As $|H| = |\Gamma(n, a)|$, it is enough to show that H acts transitively on $\Gamma(n, a)$. As $i \mapsto i + a^{-j}$ is a permutation of $\mathbb{Z}_n$ order n and $j \mapsto j + 1$ is a permutation of $\mathbb{Z}_3$ order 3, the action of H on $\Gamma(n, a)$ is transitive. □

Note that H is a semidirect product of $\langle\alpha\rangle$ and $\langle\beta\rangle$, as $\beta^{-1}\alpha\beta = \alpha^{a^2}$ and a^2 and n are coprime, and $\Gamma(n, a) = \mathrm{Cay}(H, S)$, where $S = \{\beta^2\alpha, \beta^2\alpha^{-1}, \beta\alpha^b, \beta\alpha^{-b}\}$. We now recall a result on Hamiltonicity of Cayley graphs.

Theorem 2.2 [11, Theorem 3.3]. *Every connected Cayley graph of a semidirect product of a cyclic group of prime order by an abelain group of odd order is Hamiltonian.* □

COROLLARY 2.1

If n is odd, then $\Gamma(n, a)$ is Hamiltonian.

Proof. By Theorem 2.1, we have $\Gamma(n, a)$ is a Cayley graph on H and H is a semidirect product of a cyclic group of order 3, namely $\langle\beta\rangle$, and another cyclic group of odd order n, namely $\langle\alpha\rangle$. Thus, by Theorem 2.2, $\Gamma(n, a)$ is Hamiltonian. □

Theorem 2.3. *$\Gamma(n, a)$ is edge-transitive.*

Proof. As $\Gamma(n, a)$ is Cayley, it is vertex-transitive. Hence, it is enough to show that any two edges incident with $(0, 0)$ can be permuted by an automorphism. As $\Gamma(n, a)$ is tetravalent, the four vertices adjacent to $(0, 0)$ are namely: $(1, 2)$, $(-1, 2)$, $(b, 1)$ and $(-b, 1)$. Let us name the following edges as

$$\begin{array}{ll} e_1 : (0, 0) \sim (1, 2) & e_2 : (0, 0) \sim (-1, 2) \\ e_3 : (0, 0) \sim (b, 1) & e_4 : (0, 0) \sim (-b, 1). \end{array}$$

It is to be noted that $\gamma(e_1) = e_2$, $\alpha\beta\gamma(e_1) = \overleftarrow{e_3}$ and $\gamma\alpha\beta\gamma(e_1) = \overleftarrow{e_4}$. The reverse arrow on top denotes that the orientation of the edge is changed. Hence, the theorem follows. □

For $n = 7, 14$, SageMath [12] computation shows that $\Gamma(n, a)$ is arc-transitive. Next, we prove that $\Gamma(n, a)$ is not arc-transitive if $n \neq 7, 14$. For that, we show that there does not exist any graph automorphism φ which maps the arc e_3 to e_1, i.e., $\varphi((0, 0)) = (0, 0)$ and $\varphi((b, 1)) = (1, 2)$.

The next theorem shows that there can not be an automorphism φ for which $\varphi((0, 0)) = (0, 0)$ and $\varphi((b, 1)) = (1, 2)$ because $\varphi((1, 2))$ should be one of $\{(b, 1), (-b, 1), (-1, 2)\}$.

Theorem 2.4. *If φ is an automorphism of $\Gamma(n, a)$ such that $n \neq 7, 14$ and $\varphi((0, 0) = (0, 0)$ and $\varphi((b, 1)) = (1, 2)$, then $\varphi((1, 2)) \notin \{(b, 1), (-b, 1), (-1, 2)\}$.* □

Thus, from Theorems 2.1, 2.3 and 2.4, we obtain the following result.

Theorem 2.5. *$\Gamma(n, a)$ is half-arc-transitive if $n \neq 7, 14$.* □

3. Proof of Theorem 2.4

To prove Theorem 2.4, we prove a lemma and three theorems. Throughout this section, φ denotes an automorphism of $\Gamma(n, a)$ and G denotes the full automorphsim group of $\Gamma(n, a)$.

Lemma 3.1. *The following relations can not hold*:

(1) $2a - 4b = 0$,
(2) $2a + 4b = 0$ *except for* $n = 9$,
(3) $4a - 2b = 0$ *except for* $n = 7, 14$,
(4) $4a + 2b = 0$ *except for* $n = 18$,
(5) $2a - 2b = 0$,
(6) $2a + 2b = 0$,
(7) $4a + 4 = 0$,
(8) $2a + 6 = 0$,
(9) $2(a + b - 1) = 0$,
(10) $2(b - a + 1) = 0$,
(11) $2(a - b + 1) = 0$,
(12) $2(a + b + 2) = 0$,
(13) $2(a - b - 2) = 0$.

Proof.

(1) $2a - 4b = 0$, i.e., $8 = 64$, i.e., $56 = 0$, i.e., $n \mid 56$ and hence $n \in \{7, 14, 28, 56\}$ and the possible values of b are 4, 11, 25, 25, respectively. In all these cases $2b \neq 4$, which is a contradiction.
(2) $2a + 4b = 0$, i.e., $8 = -64$, i.e., $72 = 0$, i.e., $n \mid 72$ and hence $n \in \{9, 18, 36, 72\}$ and the possible values of b are 7, 13, 25, 49. But the relation holds only if $n = 9$ and $b = 7$.
(3) $4a - 2b = 0$, $8 = 64$, i.e., $56 = 0$, i.e., $n \mid 56$ and hence $n \in \{7, 14, 28, 56\}$ and the possible values of (a, b) are $(2, 4)$, $(9, 11)$, $(9, 25)$, $(9, 25)$, respectively. But the relation holds only if $n \in \{7, 14\}$.

(4) $4a + 2b = 0$, i.e., $64 = -8$, i.e., $72 = 0$, i.e., $n \mid 72$ and hence $n \in \{9, 18, 36, 72\}$ and the possible values of (a, b) are $(4, 7)$, $(7, 13)$, $(13, 25)$, $(25, 49)$, respectively. But the relation holds only if $n = 18$.

(5) $2a - 2b = 0$, i.e., $2a \equiv 2 (\mathrm{mod}\ n)$. If n being odd, then $a = 1$, which is impossible. Let n be even and $n = 2m$. Then we have $m \mid a - 1$, i.e., $a = mt + 1$, for some $t \in \mathbb{Z}$. As $a \neq 1$, so $a = m + 1$, i.e., $a^3 - 1 = m(m^2 + 3m + 3)$. Note that irrespective of m is odd or even, $(m^2 + 3m + 3)$ is odd, say $(2s + 1)$, for some $s \in \mathbb{Z}$. So we have $a^3 - 1 = m(2s + 1)$, i.e., $a^3 - 1 \equiv m (\mathrm{mod}\ n)$, which is a contradiction.

(6) The proof is the same as (5)

(7) $4a + 4 = 0$, i.e., $4a \equiv -4 (\mathrm{mod}\ n)$. If n is odd, then $a = -1$, which is impossible. If n is even and $n = 2m$, then we have $m \mid 2(a + 1)$, i.e., $2a = mt - 2$, for some $t \in \mathbb{Z}$. As $2a \neq -2$, so $2a = m - 2$, i.e., $8(a^3 - 1) = m(m^2 - 6m + 12) - 16$. If m is even, then $(m^2 - 6m + 12)$ is even, say $2u$, for some $u \in \mathbb{Z}$. So we have $8(a^3 - 1) = 2mu - 16$, i.e., $8(a^3 - 1) \equiv -16 (\mathrm{mod}\ n)$, which is a contradiction. If m is odd, then $(m^2 - 6m + 12)$ is odd, say $2v + 1$, for some $v \in \mathbb{Z}$. So we have $8(a^3-1) = m(2v+1)-16$, i.e., $8(a^3-1) \equiv m-16 (\mathrm{mod}\ n)$, which is a contradiction as $m \neq 16$.

(8) $2a + 6 = 0$, i.e., $8 = -216$, i.e., $224 = 0$, i.e., $n \mid 224$, i.e., $n \in \{7, 14, 28, 56, 112, 224\}$. However, in all these cases, the possible values of a does not allow $2a + 6 = 0$.

(9) $2(a + b - 1) = 0$, i.e., $2(1 + a - b) = 0$, i.e., $4a = 0$, contradicting that a is a unit.

(10) The proof is the same as (9).

(11) $2(a - b + 1) = 0$, i.e., $2(1 - a + b) = 0$, i.e., $2(a - b + 1) + 2(1 - a + b) = 0$, i.e., $4 = 0$, which is a contradiction.

(12) $2(a+b+2) = 0$, i.e., $2(1+a+2b) = 0$, i.e., $4(a+b+2)-2(1+a+2b) = 0$, i.e., $2(a+3) = 0$, i.e., $8 = -216$, i.e., $224 = 0$, i.e., $n \mid 224$, i.e., $n \in \{7, 14, 28, 56, 112, 224\}$. In all these cases, $2a + 6 \neq 0$, which is a contradiction.

(13) $2(a - b - 2) = 0$, i.e., $2(1 - a - 2b) = 0$, i.e., $2(a - b - 2) + 2(1 - a - 2b) = 0$, i.e., $6b + 2 = 0$, i.e., $2a + 6 = 0$. The rest of the proof is the same as (8). $\square$

Theorem 3.1. *If $\varphi \in G$ and $\varphi((0, 0)) = (0, 0)$, $\varphi((b, 1)) = (1, 2)$, $\varphi((1, 2)) = (-1, 2)$ then $n = 7$ or 14.*

Proof. Consider the cycle $C : (0, 0) \sim (b, 1) \sim (a+b, 2) \sim (1+a+b, 0) \sim (a+1, 1) \sim (1, 2) \sim (0, 0)$. Then $\varphi(C) : (0, 0) \sim (1, 2) \sim \varphi((a + b, 2)) \sim \varphi((1 + a + b, 0)) \sim \varphi((a + 1, 1)) \sim (-1, 2) \sim (0, 0)$. As $\varphi((a + b, 2)) \sim (1, 2)$ and $\varphi((0, 0)) = (0, 0)$, so $\varphi((a+b, 2)) \in \{(2b, 0), (a\pm 1, 1)\}$. Again, $\varphi((a+1, 1)) \sim (-1, 2)$ and $\varphi((0, 0)) = (0, 0)$ imply $\varphi((a + 1, 1)) \in \{(-2b, 0), (-a \pm 1, 1)\}$. Also, $\varphi((1 + a + b, 0)) \sim \varphi((a + b, 2))$ and $\varphi((b, 1)) = (1, 2)$ imply

$$\varphi((1 + a + b, 0)) \in \{(2a \pm b, 1), (3, 2), (1 + 2b, 2), (b + a \pm 1, 0), (1 - 2b, 2), (b - a \pm 1, 0)\}. \tag{1}$$

If $\varphi((a + 1, 1)) = (-2b, 0)$, then $\varphi((1 + a + b, 0)) \sim \varphi((a + 1, 1))$ and $\varphi((1, 2)) = (-1, 2))$ imply

$$\varphi((1 + a + b, 0)) \in \{(-3, 2), (-2a \pm b, 1)\}. \tag{2}$$

From Equations (1) and (2), we have

- either $-3 = 3$, i.e., $6 = 0$, i.e., $n = 6$ which is impossible.
- or $-3 = 1 + 2b$, i.e., $2a + 4b = 0$, which is possible only when $n = 9$ (by Lemma 3.1). However, direct SageMath computation for $n = 9$ shows that such φ does not exist.
- or $-3 = 1 - 2b$, i.e., $2a - 4b = 0$, which is impossible by Lemma 3.1.
- or $-2a \pm b = 2a \pm b$, i.e., $4a = 0$ or $4a - 2b = 0$ or $4a + 2b = 0$. Though the first one is impossible, the other two can hold only if $n \in \{7, 14, 18\}$ (by Lemma 3.1). However, direct SageMath computation for $n = 7$, 14 and 18 shows that such a φ does not exist.

Hence $\varphi((a + 1, 1)) \neq (-2b, 0)$.

If $\varphi((a+1, 1)) = (-a+1, 1)$, then $\varphi((1+a+b, 0)) \sim \varphi((a+1, 1))$ and $\varphi((1, 2)) = (-1, 2)$ imply

$$\varphi((1 + a + b, 0)) \in \{(-1 + 2b, 2), (-b + a \pm 1, 0)\}. \tag{3}$$

From Equations (1) and (3), we have

- either $-b+a\pm1 = b-a\pm1$, i.e., $2(a-b) = 0$ or $2(b-a+1) = 0$ or $2(b-a-1) = 0$, all of which are impossible by Lemma 3.1.
- or $-b + a \pm 1 = b + a \pm 1$, i.e., $2b = 0$ or$2a - 2b = 0$ or $2a + 2b = 0$, all of which are impossible by Lemma 3.1.
- or $-1 + 2b = 3$, i.e., $2a - 4b = 0$, which is impossible by Lemma 3.1.
- or $-1 + 2b = 1 + 2b$, i.e., $2 = 0$, which is a contradiction.
- or $-1 + 2b = 1 - 2b$, i.e., $4a - 2b = 0$ which can hold only if $n = 7$ or 14. (by Lemma 3.1.) However, direct SageMath computation for $n = 7, 14$ shows that such φ does not exist.

Hence $\varphi((a + 1, 1)) \neq (-a + 1, 1)$.

If $\varphi((a+1, 1)) = (-a-1, 1)$, then $\varphi((1+a+b, 0)) \sim \varphi((a+1, 1))$ and $\varphi((1, 2)) = (-1, 2))$ imply

$$\varphi((1 + a + b, 0)) \in \{(-1 - 2b, 2), (-b - a \pm 1, 0)\} \tag{4}$$

From Equations (1) and (4), we have

- either $-1 - 2b = 3$, i.e., $2a + 4b = 0$, which can hold only if $n = 9$ (by Lemma 3.1). However, direct SageMath computation rules out this possibility.
- or $-1 - 2b = 1 + 2b$, i.e., $4a + 2b = 0$, which can hold only if $n = 18$. (by Lemma 3.1). However, direct SageMath computation rules out this possibility.
- or $-1 - 2b = 1 - 2b$, i.e., $2 = 0$, which is a contradiction.
- or $-b - a \pm 1 = b - a \pm 1$, i.e., $2b = 0$, or $2a + 2b = 0$, or $2a - 2b = 0$ all of which are impossible by Lemma 3.1.
- or $-b - a \pm 1 = b + a \pm 1$, i.e., $2(b + a - 1) = 0$ or $2(b + a) = 0$ (which are impossible by Lemma 3.1), but $2(1 + a + b) = 0$ may hold.

Therefore we have $\varphi((1 + a + b, 0)) = (1 + a + b, 0)$, $\varphi((a + 1, 1)) = (-a - 1, 1)$, $\varphi((a + b, 2)) = (a + 1, 1)$ with $2(a + b + 1) = 0$.

Consider the cycle $C' : (1 + a + b, 0) \sim (a + 1, 1) \sim (1 + 2b, 2) \sim (2a, 0) \sim (b + 2, 1) \sim (a + b, 2) \sim (1 + a + b, 0)$. Then $\varphi(C') : (1 + a + b, 0) \sim (-a - 1, 1) \sim \varphi((1 + 2b, 2)) \sim \varphi((2a, 0)) \sim \varphi((b + 2, 1)) \sim (a + 1, 1) \sim (1 + a + b, 0)$. Now

$\varphi((b+2,1)) \sim (a+1,1)$, $\varphi((1+a+b,0)) = (1+a+b,0)$ and $\varphi((b,1)) = (1,2)$ imply $\varphi((b+2,1)) \in \{(1+2b,2),(b+a-1,0)\}$. Again $\varphi((1+2b,2)) \sim (-a-1,1)$, $\varphi((a+b+1,0)) = (a+b+1,0) = (-a-b-1,0)$ and $\varphi((1,2)) = (-1,2)$ imply $\varphi((1+2b,2)) \in \{(-1-2b,2),(-b-a+1,0)\}$. Also $\varphi((2a,0)) \sim \varphi((b+2,1))$ and $\varphi((a+b,2)) = (a+1,1)$ imply

$$\varphi((2a,0)) \in \{(b+2a\pm b,0),(a+3,1),(a+1-2b,1),(1+b-a\pm 1,2)\}. \tag{5}$$

Let $\varphi((1+2b,2)) = (-1-2b,2)$. Then $\varphi((2a,0)) \sim \varphi((1+2b,2))$ and $\varphi((a+1,1)) = (-a-1,1)$ imply

$$\varphi((2a,0)) \in \{(-b-2a\pm b,0),(-a-3,1). \tag{6}$$

From Equations (5) and (6), we have

- either $-b-2a\pm b = b+2a\pm b$, i.e., $4a+4=0$ or $4a=0$ (which are impossible by Lemma 3.1) or $4a+2b=0$, which can hold only if $n=18$. However, direct SageMath computation rules out this possibility.
- or $-a-3 = a+3$, i.e., $2a+6=0$, which is impossible by Lemma 3.1.
- or $-a-3 = a+1-2b$, i.e., $2(a-b+2)=0$. Also, we had $2(a+b+1)=0$ previously. This yields $2a=4$, i.e., $n=7$ or 14.

Hence $\varphi((1+2b,2)) = (-1-2b,2)$ is possible only if $n=7$ or 14. Moreover, direct SageMath computation for $n=7$ and 14 confirms the possibility.

Let $\varphi((1+2b,2)) = (-b-a+1,0)$. Then $\varphi((2a,0)) \sim \varphi((1+2b,2))$ and $\varphi((a+1,1)) = (-a-1,1)$ imply

$$\varphi((2a,0)) \in \{(-a-1+2b,1),(-1-b+a\pm 1,2). \tag{7}$$

From Equations (5) and (7), we have

- either $-1-b+a\pm 1 = 1+b-a\pm 1$, i.e., $2(b-a+1)=0$ or $2(a-b)=0$ or $2(a-b-2)=0$, all of which are impossible by Lemma 3.1.
- or $-a-1+2b = a+1-2b$, i.e., $2(a+1-2b)=0$, i.e., $2(a+b-2)=0$. Hence combining $2(a+b+1)=0$ and $2(a+b-2)=0$, we have $6=0$, which is impossible.
- or $-a-1+2b = a+3$, i.e., $2(a-b+2)=0$. Therefore, from $2(a+b+1)=0$ and $2(a-b+2)=0$, we have $2a=4$, i.e., $n=7$ or 14.

Hence $\varphi((1+2b,2)) = (-b-a+1,0)$ may be possible if $n=7$ or 14. Moreover, direct SageMath computation for $n=7$ and 14 confirms the possibility.

Therefore, for $\varphi \in G$, we can have $\varphi((0,0)) = (0,0)$, $\varphi((b,1)) = (1,2)$, $\varphi((1,2)) = (-1,2)$ only if $n=7$ or 14. □

Theorem 3.2. *If $\varphi \in G$ and $\varphi((0,0)) = (0,0)$, $\varphi((b,1)) = (1,2)$, then $\varphi((1,2)) \neq (-b,1)$.*

Proof. Suppose that $\varphi \in G$ and $\varphi((0,0)) = (0,0)$, $\varphi((b,1)) = (1,2)$ and $\varphi((1,2)) = (-b,1)$. Consider the cycle $C : (0,0) \sim (b,1) \sim (a+b,2) \sim (1+a+b,0) \sim (a+1,1) \sim$

$(1,2) \sim (0,0)$. Then $\varphi(C) : (0,0) \sim (1,2) \sim \varphi((a+b,2)) \sim \varphi((1+a+b,0)) \sim \varphi((a+1,1)) \sim (-b,1) \sim (0,0)$. As $\varphi((a+b,2)) \sim (1,2)$ and $\varphi((0,0)) = (0,0)$, then $\varphi((a+b,2)) \in \{(2b,0),(a\pm 1,1)\}$. Again, $\varphi((a+1,1)) \sim (-b,1)$ and $\varphi((0,0)) = (0,0)$ imply $\varphi((a+1,1)) \in \{(-2,0),(-a\pm b,2)\}$. Now as $\varphi((1+a+b,0)) \sim \varphi((a+b,2))$ and $\varphi((b,1)) = (1,2)$, we have

$$\varphi((1+a+b,0)) \in \{(2a\pm b,1),(3,2),(1+2b,2),(b+a\pm 1,0),\\ (1-2b,2),(b-a\pm 1,0)\}. \tag{8}$$

Depending upon the value of $\varphi((a+1,1))$, one of the following three cases must hold, namely:

Case A: $\varphi((a+1,1)) = (-2,0))$,
Case B: $\varphi((a+1,1)) = (-a-b,2))$ or
Case C: $\varphi((a+1,1)) = (-a+b,2))$.

However, before resolving these three cases, we prove a claim which will be crucial in the following proof.

Claim. $\varphi((-1-a-b,0)) \in \{(-2a\pm b,1),(-3,2),(-1+2b,2),(-b+a\pm 1,0),(-1-2b,2),(-b-a\pm 1,0),(2a\pm 1,2),(3b,1),(b+2,1),(1+a\pm b,0),(b-2,1),(1-a\pm b,0)\}$.

Proof of Claim. As $(-b,1) \sim (0,0)$ and $(-1,2) \sim (0,0)$, we have $\varphi((-b,1)), \varphi((-1,2)) \in \{(b,1),(-1,2)\}$.

Case 1: Let $\varphi((-b,1)) = (-1,2)$ and $\varphi((-1,2)) = (b,1)$. Consider the cycle C' : $(0,0) \sim (-b,1) \sim (-a-b,2) \sim (-1-a-b,0) \sim (-a-1,1) \sim (-1,2) \sim (0,0)$, then $\varphi(C') : (0,0) \sim (-1,2) \sim \varphi((-a-b,2)) \sim \varphi((-1-a-b,0)) \sim \varphi((-a-1,1)) \sim (b,1) \sim (0,0)$. As $\varphi((-a-b,2)) \sim (-1,2)$ and $\varphi((0,0)) = (0,0)$ then $\varphi((-a-b,2)) \in \{(-2b,0),(-a\pm 1,1)\}$. $\varphi((-a-1,1)) \sim (b,1)$ and $\varphi((0,0)) = (0,0)$ imply $\varphi((-a-1,1)) \in \{(2,0),(a\pm b,2)\}$. Now $\varphi((-1-a-b,0)) \sim \varphi((-a-b,2))$ and $\varphi((-b,1)) = (-1,2)$ imply

$$\varphi((-1-a-b,0)) \in \{(-2a\pm b,1),(-3,2),(-1+2b,2),(-b+a\pm 1,0),\\ (-1-2b,2),(-b-a\pm 1,0)\}. \tag{9}$$

Case 2: Let $\varphi((-b,1)) = (b,1)$ and $\varphi((-1,2)) = (-1,2)$. Consider the cycle C' : $(0,0) \sim (-b,1) \sim (-a-b,2) \sim (-1-a-b,0) \sim (-a-1,1) \sim (-1,2) \sim (0,0)$. Then $\varphi(C') : (0,0) \sim (b,1) \sim \varphi((-a-b,2)) \sim \varphi((-1-a-b,0)) \sim \varphi((-a-1,1)) \sim (b,1) \sim (0,0)$. As $\varphi((-a-b,2)) \sim (b,1)$ and $\varphi((0,0)) = (0,0)$, then $\varphi((-a-b,2)) \in \{(-2,0),(a\pm b,2)\}$. $\varphi((-a-1,1)) \sim (-1,2)$ and $\varphi((0,0)) = (0,0)$ imply $\varphi((-a-1,1)) \in \{(-2b,0),(-a\pm 1,1)\}$. Now $\varphi((-1-a-b,0)) \sim \varphi((-a-b,2))$ and $\varphi((-b,1)) = (b,1)$ imply

$$\varphi((-1-a-b,0)) \in \{(2a\pm 1,2),(3b,1),(b+2,1),(1+a\pm b,0),\\ (b-2,1),(1-a\pm b,0)\}. \tag{10}$$

Combining Cases (1) and (2), the claim follows.

We now turn towards the three cases mentioned earlier.

Case A: If $\varphi((a+1, 1)) = (-2, 0)$, then $\varphi((1+a+b, 0)) \sim \varphi((a+1, 1))$ and $\varphi((1, 2)) = (-b, 1)$ imply

$$\varphi((1 + a + b, 0)) \in \{(-3b, 1), (-2a \pm 1, 2)\}. \tag{11}$$

From Equations (8) and (11), we have

- either $-3b = 2a \pm b$, i.e., $-2b = 2a$ or $2b = -4$. By Lemma 3.1, this can hold only if $n = 9$. However, direct SageMath computation for $n = 9$ shows that such a φ does not exist.
- or $-2a \pm 1 = 3$, i.e., $-2a = 4$ or $-2a = 2$, i.e., $4a + 2b = 0$ or $2a + 2b = 0$. By Lemma 3.1, $2a + 2b = 0$ can not hold and $4a + 2b = 0$ can hold only if $n = 18$. However, direct SageMath computation for $n = 18$ shows that such a φ does not exist.
- or $-2a \pm 1 = 1 - 2b$, i.e., $2a = 2b$, $2(b - a - 1) = 0$, both of which are impossible by Lemma 3.1.
- or $-2a \pm 1 = 1 + 2b$, i.e., $2a + 2b = 0$ (which is impossible by Lemma 3.1) but

$$2a + 2b + 2 = 0 \text{ may hold.} \tag{12}$$

When $\varphi((a+1, 1)) = (-2, 0)$, $\varphi((1+a+b, 0)) = (1+2b, 2)$, $\varphi((a+b, 2)) = (a+1, 1)$, then we have Equation (12). As $2a + 2b + 2 = 0$, i.e., $a + b + 1 = -a - b - 1$, then $\varphi((-1 - a - b, 0)) = (1 + 2b, 2)$. But from Equations (9) and (10), we have $\varphi((-1 - a - b, 0)) \neq (1 + 2b, 2)$, which is a contradiction. Hence $\varphi((a + 1, 1)) \neq (-2, 0)$ and Case A can not hold.

Case B: If $\varphi((a + 1, 1)) = (-a - b, 2)$, then $\varphi((1 + a + b, 0)) \sim \varphi((a + 1, 1))$ and $\varphi((1, 2)) = (-b, 1))$ imply

$$\varphi((1 + a + b, 0)) \in \{(-b - 2, 1), (-1 - a \pm b, 0)\}. \tag{13}$$

From Equations (8) and (13), we have as follows:

Case B(1): $-b-2 = 2a \pm b$, i.e., $2a + 2b = 0$, which is impossible by the Lemma 3.1, or

$$2a + 2b + 2 = 0 \text{ may hold.} \tag{14}$$

Case B(2): $-1 - a \pm b = b + a \pm 1$, i.e., $2a + 2b = 0$ or $2a = 0$ (which are impossible by Lemma 3.1), but $-1 - a - b = b + a + 1$, i.e.,

$$2a + 2b + 2 = 0 \text{ may hold.} \tag{15}$$

Case B(3): $-1-a \pm b = b-a \pm 1$. This gives rise to four equations, out of which three are impossible by Lemma 3.1, namely $2 = 0$, $2b = 0$ and $2a + 2b = 0$. The only possibility which remains is $-1 - a + b = b - a - 1$ and it is an identity.

So assuming this identity, we have $\varphi((a + 1, 1)) = (-a - b, 2)$, $\varphi((a + b + 1, 0)) = (b - a - 1, 0)$ and $\varphi((a + b, 2)) = (a - 1, 1)$. Similarly, we can show that $\varphi((a - 1, 1)) = (-a + b, 2)$, $\varphi((b - a + 1, 0)) = (b + a - 1, 0)$ and $\varphi((a - b, 2)) = (a + 1, 1)$. Now $\varphi((a+b, 2)) = (a-1, 1)$, $\varphi(((a-b, 2)) = (a+1, 1)$ and $\varphi((2, 0)) \sim \varphi((b, 1)) = (1, 2)$ imply $\varphi((2, 0)) = (2b, 0)$.

Now, consider the cycle $C_2 : (a + b + 1, 0) \sim (a + 1, 1) \sim (1, 2) \sim (2b, 0) \sim (2a+b, 1) \sim (a+b+2, 2) \sim (a+b+1, 0)$. So $\varphi(C_2) : (b-a-1, 0) \sim (-a-b, 2) \sim (-b, 1) \sim \varphi((2b, 0)) \sim \varphi((2a + b, 1)) \sim \varphi((a + b + 2, 2)) \sim (b - a - 1, 0)$.

Again $\varphi((0,0)) = (0,0)$, $\varphi((a+1,1)) = (-a-b,2)$ and $\varphi((2b,0)) \sim (-b,1)$ imply $\varphi((2b,0)) \in \{(-a+b,2), (-2,0)\}$ and $\varphi(((a+b,2)) = (a-1,1)$, $\varphi((a+1,1)) = (-a-b,2)$ and $\varphi((a+b+2,2)) \sim (b-a-1,0)$ imply $\varphi((a+b+2,2)) \in \{(-b-a+2,2), (-1-2b+a,1)\}$. $\varphi((1,2)) = (-b,1)$ and $\varphi((2a+b,1)) \sim \varphi((2b,0))$ imply

$$\varphi((2a+b,1)) \in \{(-1+a \pm b,0), (-b+2,1), (-3b,1), (-2a \pm 1,2). \quad (16)$$

Case B(3)(*a*): If $\varphi((a+b+2,2)) = (-b-a+2,2)$, then $\varphi((a+b+1,0)) = (b-a-1,0)$ and $\varphi((2a+b,1)) \sim \varphi((a+b+2,2))$ imply

$$\varphi((2a+b,1)) \in \{(a-1+3b,0), (-1-b+2a \pm 1,1)\}. \quad (17)$$

From Equations (16) and (17), we have

- either $-1-b+2a \pm 1 = -b+2$, which implies either $2a-2b=0$ which is impossible by Lemma 3.1 or $4a-2b=0$ which is possible only for $n=7$ or 14. However, direct SageMath computation for $n=7$ and 14 shows that such a φ does not exist.
- or $a-1+3b = -1-a \pm b$, i.e., $2a+4b=0$ which is possible only for $n=9$ or $2a+2b=0$, which is impossible by Lemma 3.1. And finally direct SageMath computation for $n=9$ shows that such φ does not exist.
- or $-1-b+2a \pm 1 = -3b$, i.e., $2a+2b=0$ or $2a+2b-2=0$, both of which are impossible by Lemma 3.1.

Hence $\varphi((a+b+2,2)) \neq (-b-a+2,2)$.

Case B(3)(*b*): If $\varphi((a+b+2,2)) = (a-1-2b,1)$, then $\varphi((a+b+1,0) = (b-a-1,0)$ and $\varphi((2a+b,1)) \sim \varphi((a+b+2,2))$ imply

$$\varphi((2a+b,1)) \in \{(b-a-3,0), (1-b-2a \pm b,2)\}. \quad (18)$$

From Equations (16) and (18), we have

- either $b-a-3 = -1+a \pm b$, i.e., $2a+2b=0$ or $2a-2b+2=0$, both of which are impossible by Lemma 3.1.
- or $1-b-2a \pm b = -2a \pm 1$. These give rise to four equations, out of which three are impossible, by Lemma 3.1, namely $2=0$, $2b=0$ and $2a-2b=0$. The only possibility which remains is $1-b-2a+b = -2a+1$ and it is an identity.

So assuming this to be the case, we have $\varphi((2b,0)) = (-2,0)$, $\varphi((a+b+2,2)) = (a-1-2b,1)$ and $\varphi((2a+b,1)) = (-2a+1,2)$.

Now consider the cycle $C_3 : (2b,0) \sim (2a+b,1) \sim (a+b+2,2) \sim (1+a+3b,0) \sim (3a+1,1) \sim (3,2) \sim (2b,0)$. So $\varphi(C_3) : (-2,0) \sim (-2a+1,2) \sim (a-1-2b,1) \sim \varphi((1+a+3b,0)) \sim \varphi((3a+1,1)) \sim \varphi((3,2)) \sim (-2,0)$. Then $\varphi((2b,0)) = (-2,0)$, $\varphi((2a+b,1)) = (-2a+1,2)$ and $\varphi((3,2)) \sim (-2,0)$ imply $\varphi((3,2)) \in \{(-3b,1), (-2a-1,2)\}$. Again $\varphi((2a+b,1)) = (-2a+1,2)$, $\varphi((a+b+1,0)) = (b-a-1,0)$ and $\varphi((1+a+3b,0)) \sim (a-1-2b,1)$ imply $\varphi((1+a+3b,0)) \in \{(1-2a-2b,2), (b-a-3,0)\}$. Finally $\varphi((2b,0)) = (-2,0)$ and $\varphi((3a+1,1)) \sim \varphi((3,2))$ imply

$$\varphi((3a+1,1)) \in \{(-4,0), (-3a \pm b,2), (-2-2b,0), (-2b-a \pm 1,1)\}. \quad (19)$$

Case B(3)(*b*)(1): If $\varphi((1+a+3b,0)) = (1-2a-2b,2)$, then $\varphi((3a+1,1)) \sim \varphi((1+a+3b,0))$ implies

$$\varphi((3a+1,1)) \in \{(b-2a-2\pm b,0), (a-2-2b\pm 1,1)\}. \tag{20}$$

From Equations (19) and (20), we have

- either $b-2a-2\pm b = -4$, i.e., $2a-2b=0$ or $2a-2b-2=0$, both of which are impossible by Lemma 3.1.
- or $b-2a-2\pm b = -2-2b$, i.e., $2a-2b=0$ or $2a-4b=0$, both of which are impossible by Lemma 3.1.
- or $a-2-2b\pm 1 = -2b-a\pm 1$, i.e. $2a-2b=0$ or $2a=0$ or $4a-2b=0$. By Lemma 3.1, the first two are impossible and the third one may hold only for $n=7$ or 14. However, direct SageMath computation for $n=7$ and 14 shows that such a φ does not exist.

So we have $\varphi((1+a+3b,0)) \neq (1-2b-2a,2)$.

Case B(3)(*b*)(2): If $\varphi((1+a+3b,0)) = (b-a-3,0)$, then $\varphi((3a+1,1)) \sim \varphi((1+a+3b,0))$ implies

$$\varphi((3a+1,1)) \in \{(a-1-3b\pm b,1), (b-1-3a\pm 1,2)\}. \tag{21}$$

From Equations (19) and (21), we have

- either $a-1-3b\pm b = -2b-a\pm 1$, i.e., $2a=0$ or $2a-2b=0$ or $2a-2b-2=0$, all of which are impossible by Lemma 3.1.
- or $1-b-3a\pm 1 = -3a\pm b$. Out of the four relations that we get, three of them (namely, $2=0$, $2a-2b=0$ and $2b=0$) are invalid, by Lemma 3.1 and the fourth is an identity, i.e., $1-b-3a-1=-3a-b$.

So we have $\varphi((3a+1,1)) = (-3a-b,2)$, $\varphi((3,2)) = (-3b,1)$, $\varphi((a+1+3b,0)) = (b-a-3,0)$. Similarly we can show that $\varphi((3a-1,1)) = (-3a+b,2)$ and $\varphi((-a+1+3b,0)) = (a+b-3,0)$.

Now $\varphi((2b,0)) = (-2,0)$, $\varphi((3a+1,1)) = (-3a-b,2)$, $\varphi((3a-1,1)) = (-3a+b,2)$ and $\varphi((4b,0)) \sim \varphi((3,2)) = (-3b,1)$ imply $\varphi((4b,0)) = (-4,0)$.

Proceeding in this way, we can show that $\varphi((2kb,0)) = (-2k,0)$ for all $k\in\mathbb{Z}$. So we have $\varphi((2,0)) = (-2a,0)$, where $k=a$, which is a contradiction as we have shown earlier that $\varphi((2,0)) = (2b,0)$ and $2b\neq -2a$. Therefore, $\varphi((a+b+1,0)) \neq (b-a-1,0)$.

Case B(1): When $\varphi((a+1,1)) = (-a-b,2)$, $\varphi((1+a+b,0)) = (2a+b,1)$, $\varphi((a+b,2)) = (2b,0)$, we have Equation (14). As $2a+2b+2=0$, i.e., $a+b+1 = -a-b-1$, then $\varphi((-1-a-b,0)) = (2a+b,1)$. But from Equations (9) and (10), we have $\varphi((-1-a-b,0)) \neq (2a+b,1)$, which is a contradiction. Hence $\varphi((1+a+b,0)) \neq (2a+b,1)$.

Case B(2): Now when $\varphi((a+1,1)) = (-a-b,2)$, $\varphi((a+b+1,0)) = (a+b+1,0)$, $\varphi((a+b,2)) = (a+1,1)$, then we have Equation (15). Consider the cycle $C_1 : (a+b+1,0) \sim (a+1,1) \sim (1+2b,2) \sim (2a,0) \sim (b+2,1) \sim (a+b,2) \sim (a+b+1,0)$. Then $\varphi(C_1) : (a+b+1,0) \sim (-a-b,2) \sim \varphi((1+2b,2)) \sim \varphi((2a,0)) \sim \varphi((b+2,1)) \sim (a+1,1) \sim (a+b+1,0)$. Now $\varphi((a+b+1,0)) = (a+b+1,0) = (-a-b-1,0)$,

$\varphi((1,2)) = (-b,1)$ and $\varphi((1+2b,2)) \sim (-a-b,2)$ imply $\varphi((1+2b,2)) \in \{(-1-a+b,0),(-b-2,1)\}$. Again, $\varphi((b,1)) = (1,2)$, $\varphi((a+b+1,0)) = (a+b+1,0)$ and $\varphi((b+2,1)) \sim \varphi((a+1,1))$ imply $\varphi((b+2,1)) \in \{(b+a-1,0),(1+2b,2)\}$. Also, $\varphi((a+1,1)) = (-a-b,2)$ and $\varphi((2a,0)) \sim \varphi((1+2b,2))$ imply

$$\varphi((2a,0)) \in \{(-b-1+a\pm b,1),(-a-b+2,2),(-1-2a\pm 1,0),\\ (-a-3b,2)\}. \tag{22}$$

Case B(2)(*a*): Now if $\varphi((b+2,1)) = (b+a-1,0)$, then $\varphi((a+b,2)) = (a+1,1)$ and $\varphi((2a,0)) \sim \varphi((b+2,1))$ imply

$$\varphi((2a,0)) \in \{(a+1-2b,1),(1+b-a\pm 1,2)\}. \tag{23}$$

From Equations (22) and (23), we have

- either $1+b-a\pm 1 = -a-b+2$, i.e., $2b = 0$ or $2a-2b = 0$, both of which are impossible by Lemma 3.1.
- or $1+b-a\pm 1 = -a-3b$, i.e., $4b = 0$ which is impossible or $4a+2b = 0$, which, by Lemma 3.1, holds only if $n = 18$. However, direct SageMath computation for $n = 18$ shows that such a φ does not exist.
- or $a+1-2b = -b-1+a\pm b$, i.e., $2 = 0$ or $2a-2b = 0$, both of which are impossible by Lemma 3.1.

Hence $\varphi((b+2,1)) \neq (b+a-1,0)$.

Case B(2)(*b*): Now if $\varphi((b+2,1)) = (1+2b,2)$, then $\varphi((a+b,2)) = (a+1,1)$ and $\varphi((2a,0)) \sim \varphi((b+2,1))$ imply

$$\varphi((2a,0)) \in \{(a+3,1),(b+2a\pm b,0)\}. \tag{24}$$

From Equations (22) and (24), we have

- either $a+3 = -b-1+a\pm b$, i.e., $4 = 0$ or $2a+4b = 0$, which, by Lemma 3.1, can hold only if $n = 9$. However, direct SageMath computation for $n = 9$ shows that such a φ does not exist.
- or $b+2a\pm b = -1-2a\pm 1$, i.e., $2(a+b+2) = 0$ or $4a = 0$ or $2a+4b = 0$ or $4a+2b = 0$. By Lemma 3.1, the first two are impossible and the next two can hold only if $n = 9$ or 18. But those are also ruled out by SageMath computation.

Hence $\varphi((b+2,1)) \neq (1+2b,2)$ and hence $\varphi((a+1,1)) \neq (-a-b,2)$, i.e., Case B can not hold.

Case C: If $\varphi((a+1,1)) = (-a+b,2)$, then $\varphi((1,2)) = (-b,1)$ and $\varphi((a+b+1,0)) \sim \varphi((a+1,1))$ imply

$$\varphi((1+a+b,0)) \in \{(-b+2,1),(-1+a\pm b,0)\}. \tag{25}$$

From Equations (8) and (25), we have

- either $-b+2 = 2a\pm b$, i.e., $2a-2b = 0$ or $2(a+b-1) = 0$, both of which are impossible by Lemma 3.1.

- or $-1+a\pm b = b-a\pm 1$, i.e., $2a=0$ or $2a-2b=0$ or $2(a-b-1)=0$ and all of them are ruled out by Lemma 3.1.
- or $-1+a\pm b = b+a\pm 1$. This gives rise two four conditions, out of which three (namely, $2=0$, $2a+2b=0$ and $2b=0$) are ruled out by Lemma 3.1 and the fourth one is the identity $-1+a+b=b+a-1$.

So we have $\varphi((a+b+1,0)) = (a-1+b,0)$, $\varphi(((a+b,2)) = (a+1,1)$ and $\varphi((a+1,1)) = (-a+b,2)$.

Similarly we can show that $\varphi((-a+b+1,0)) = (-a-1+b,0)$, $\varphi(((a-b,2)) = (a-1,1)$ and $\varphi((a-1,1)) = (-a-b,2)$.

Now $\varphi((a+b,2)) = (a+1,1)$, $\varphi(((a-b,2)) = (a-1,1)$ and $\varphi((2,0)) \sim \varphi((b,1)) = (1,2)$ imply $\varphi((2,0)) = (2b,0)$.

Consider the cycle $C_2 : (a+b+1,0) \sim (a+1,1) \sim (1,2) \sim (2b,0) \sim (2a+b,1) \sim (a+b+2,2) \sim (a+b+1,0)$. So $\varphi(C_2) : (a-1+b,0) \sim (-a+b,2) \sim (-b,1) \sim \varphi((2b,0)) \sim \varphi((2a+b,1)) \sim \varphi((a+b+2,2)) \sim (a-1+b,0)$. Then $\varphi((0,0)) = (0,0)$, $\varphi((a+1,0)) = (-a+b,2)$ and $\varphi((2b,0)) \sim (-b,1)$ imply $\varphi((2b,0)) \in \{(-a-b,2),(-2,0)\}$. Again, $\varphi(((a+b,2)) = (a+1,1)$, $\varphi((a+1,1)) = (-a+b,2)$ and $\varphi((a+b+2,2)) \sim (a-1+b,0)$ imply $\varphi((a+b+2,2)) \in \{(b-a+2,2),(1-2b+a,1)\}$. Also, $\varphi((1,2)) = (-b,1)$ and $\varphi((2a+b,1)) \sim \varphi((2b,0))$ imply

$$\varphi((2a+b,1)) \in \{(-1-a\pm b,0),(-b-2,1),(-3b,1),(-2a\pm 1,2)\}. \tag{26}$$

Case $C(1)$: If $\varphi((a+b+2,2)) = (b-a+2,2)$, then $\varphi((a+b+1,0)) = (a+b-1,0)$ and $\varphi((2a+b,1)) \sim \varphi((a+b+2,2))$ imply

$$\varphi((2a+b,1)) \in \{(a-1+3b,0),(1-b+2a\pm 1,1)\}. \tag{27}$$

From Equations (26) and (27), we have

- either $1-b+2a\pm 1 = -b-2$, i.e., $2a+2b=0$ or $4a+2b=0$. By Lemma 3.1, the first is an impossibility and the second one can hold only if $n=18$. However, that is also ruled out by SageMath computation for $n=18$.
- or $a-1+3b = -1-a\pm b$, i.e., $2a+2b=0$ or $2a+4b=0$. By Lemma 3.1, the first is an impossibility and the second one can hold only if $n=9$. However, that is also ruled out by SageMath computation for $n=9$.
- or $1-b+2a\pm 1 = -3b$, i.e., $2a+2b=0$, which is impossible by Lemma 3.1 but,
$$2a+2b+2=0 \text{ may hold.} \tag{28}$$

When $\varphi((a+b+1,0)) = (a-1+b,0)$, $\varphi(((a+b,2)) = (a+1,1)$ and $\varphi((a+1,1)) = (-a+b,2)$, then we have Equation (28). As $2a+2b+2=0$, i.e., $a+b+1 = -a-b-1$, then $\varphi((-1-a-b,0)) = (a-1+b,0)$. But from Equations (9) and (10), we have $\varphi((-1-a-b,0)) \neq (a-1+b,0)$, which is a contradiction. Hence $\varphi((a+b+2,2)) \neq (b-a+2,2)$.

Case $C(2)$: If $\varphi((a+b+2,2)) = (a+1-2b,1)$, then $\varphi((a+b+1,0) = (a-1+b,0)$ and $\varphi((2a+b,1)) \sim \varphi((a+b+2,2))$ imply

$$\varphi((2a+b,1)) \in \{(b+a-3,0),(1+b-2a\pm b,2)\}. \tag{29}$$

From Equations (26) and (29), we have

- either $b+a-3=-1-a\pm b$, i.e., $2a-2b=0$ or $2a+2b-2=0$, both of which are impossible by Lemma 3.1.
- or $1+b-2a\pm b=-2a\pm 1$. This gives rise to four conditions, out of which three, namely $2b=0$, $2-0$ and $2a+2b=0$, are ruled out, by Lemma 3.1 and the fourth one is the identity $1+b-2a-b=-2a+1$.

So we have $\varphi((2a+b,1))=(-2a+1,2)$. Also previously, we had $\varphi((2b,0))=(-2,0)$ and $\varphi((a+b+2,2))=(1-2b+a,1)$.

Now consider the cycle $C_3:(2b,0)\sim(2a+b,1)\sim(a+b+2,2)\sim(1+a+3b,0)\sim(3a+1,1)\sim(3,2)\sim(2b,0)$. So $\varphi(C_3):(-2,0)\sim(-2a+1,2)\sim(1+a-2b,1)\sim\varphi((1+a+3b,0))\sim\varphi((3a+1,1))\sim\varphi((3,2))\sim(-2,0)$. Now, $\varphi((2b,0))=(-2,0)$, $\varphi((2a+b,1))=(-2a+1,2)$ and $\varphi((3,2))\sim(-2,0)$ imply $\varphi((3,2))\in\{(-3b,1),(-2a-1,2)\}$. Again $\varphi((2a+b,1))=(-2a+1,2)$, $\varphi((a+b+1,0))=(a+b-1,0)$ and $\varphi((1+a+3b,0))\sim(1+a-2b,1)$ imply $\varphi((1+a+3b,0))\in\{(2b+1-2a,2),(a+b-3,0)\}$. Also, $\varphi((2b,0))=(-2,0)$ and $\varphi((3a+1,1))\sim\varphi((3,2))$ imply

$$\varphi((3a+1,1))\in\{(-4,0),(-3a\pm b,2),(-2-2b,0),(-2b-a\pm 1,1)\}. \tag{30}$$

Case $C(2)(a)$: If $\varphi((1+a+3b,0))=(2b+1-2a,2)$, then $\varphi((3a+1,1))\sim\varphi((1+a+3b,0))$ implies

$$\varphi((3a+1,1))\in\{(2a+b-2\pm b,0),(2+a-2b\pm 1,1)\}. \tag{31}$$

From Equations (30) and (31), we have

- either $2+a-2b\pm 1=-2b-a\pm 1$, i.e. $2a+2b=0$ or $2a=0$ (which are impossible by Lemma 3.1) or $4a+2b=0$ which can hold only if $n=18$. But direct SageMath computation for $n=18$ ruled out this case.
- or $2a+b-2\pm b=-2-2b$, i.e., $2a+2b=0$ (impossible by Lemma 3.1) or $2a+4b=0$, which can hold only if $n=9$. But direct SageMath computation ruled out this possibility.
- or $2a+b-2\pm b=-4$, i.e., $2a+2b=0$, which is impossible by Lemma 3.1 but

$$2a+2b+2=0 \text{ may hold.} \tag{32}$$

Thus, if $\varphi((a+b+1,0))=(a-1+b,0)$, $\varphi(((a+b,2))=(a+1,1)$ and $\varphi((a+1,1))=(-a+b,2)$ holds, then we have $2a+2b+2=0$. As $2a+2b+2=0$, i.e., $a+b+1=-a-b-1$, then $\varphi((-1-a-b,0))=(a-1+b,0)$. But from Equations (9) and (10), we have $\varphi((-1-a-b,0))\neq(a-1+b,0)$, which is a contradiction. Thus, Equation (32) does not hold.

So we have $\varphi((1+a+3b,0))\neq(2b+1-2a,2)$.

Case $C(2)(b)$: If $\varphi((1+a+3b,0))=(a+b-3,0)$, then $\varphi((3a+1,1))\sim\varphi((1+a+3b,0))$ implies

$$\varphi((3a+1,1))\in\{(1+a-3b\pm b,1),(b+1-3a\pm 1,2)\}. \tag{33}$$

From Equations (30) and (33), we have

- either $1+a-3b \pm b = -2b-a \pm 1$, i.e., $2a = 0$ or $2a+2b = 0$ or $2a-2b = 0$ or $2a-2b+2 = 0$, all of which are impossible by Lemma 3.1.
- or $b+1-3a \pm 1 = -3a \pm b$. This gives rise to four conditions, out of which three, namely $2 = 0$, $2a+2b = 0$ and $2b = 0$, are ruled out, by Lemma 3.1 and fourth one is the identity $b+1-3a-1 = -3a+b$.

So we have $\varphi((3a+1, 1)) = (-3a+b, 2)$, $\varphi((3, 2)) = (-3b, 1)$, $\varphi((a+1+3b, 0)) = (a+b-3, 0)$.

Similarly we can show that $\varphi((3a-1, 1)) = (-3a-b, 2)$ and $\varphi((-a+1+3b, 0)) = (-a+b-3, 0)$.

Now $\varphi((2b, 0)) = (-2, 0)$, $\varphi((3a+1, 1)) = (-3a+b, 2)$, $\varphi((3a-1, 1)) = (-3a-b, 2)$ and $\varphi((4b, 0)) \sim \varphi((3, 2)) = (-3b, 1)$ imply $\varphi((4b, 0)) = (-4, 0)$.

Proceeding this way, we can show that $\varphi((2kb, 0)) = (-2k, 0)$, for all $k \in \mathbb{Z}$. So we have $\varphi((2, 0)) = (-2a, 0)$, where $k = a$, which is a contradiction as we have shown that $\varphi((2, 0)) = (2b, 0)$ and $2b \neq -2a$. Therefore, we have $\varphi((a+1, 1)) \neq (-a+b, 2)$ and Case C can not hold.

As none of the cases A, B and C hold, the assumption that $\varphi((1, 2)) = (-b, 1)$ is wrong. Hence the lemma follows. □

Theorem 3.3. *If $\varphi \in G$ and $\varphi((0, 0)) = (0, 0)$, $\varphi((b, 1)) = (1, 2)$, $\varphi((1, 2)) = (b, 1)$, then $n = 7$ or 14.*

Proof. The proof follows as in the proof of Theorem 3.1. For a detailed proof, one can see the proof of Theorem 4.3 [2]. □

Now, the proof of Theorem 2.4 follows from Theorems 3.1, 3.2 and 3.3.

4. Open issues

In this paper, we proved the half-arc-transitivity of an infinite family of tetravalent Cayley graphs. However, a few issues are still pending and can be topics for further research.

(1) *Full Automorphism Group.* It was shown that $\langle \alpha, \beta, \gamma \rangle$ is a subgroup of the full automorphism group. It remains to be shown (as observed in SageMath) that $G = \langle \alpha, \beta, \gamma \rangle$ for $n \neq 7, 14$.

(2) *Structural Properties of $\Gamma(n, a)$.* We have shown that $\Gamma(n, a)$ is Hamiltonian if n is odd. However, the Hamiltonicity for even values of n is still unresolved. Similarly, other structural properties like girth, diameter, domination number are few open issues.

Acknowledgements

The first author is supported by the Ph.D. Fellowship of CSIR (File No. 08/155(0086)/2020-EMR-I), Government of India. The second author acknowledges the funding of DST-SERB-SRG Sanction No. SRG/2019/000475, Govt. of India.

Appendix

Appendix A: Sage Code for $\Gamma(n, a)$ for $n = 7, a = 2$:

```
n=7
a=2
b=mod(a^2,n)
A=list(var('A_%d' % i) for i in range(n))
B=list(var('B_%d' % i) for i in range(3))
C = cartesian_product([A, B])
V=C.list()
E=[]
Gamma=Graph()
Gamma.add_vertices(V)
for i in range(n):
  for j in range(3):
    E.append(((A[i],B[j]),(A[mod(a*i+1,n)],B[mod(j-1,3)])))
    E.append(((A[i],B[j]),(A[mod(a*i-1,n)],B[mod(j-1,3)])))
    E.append(((A[i],B[j]),(A[mod(b*i+b,n)],B[mod(j+1,3)])))
    E.append(((A[i],B[j]),(A[mod(b*i-b,n)],B[mod(j+1,3)])))
Gamma.add_edges(E)
G=Gamma.automorphism_group()
for f in G:
   if f((A[0],B[0]))==(A[0],B[0]) and f((A[b],B[1]))==(A[1],
   B[2]) and
   f((A[1],B[2]))==(A[n-1],B[2]):
   print "sucess"
```

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COMMUNICATING EDITOR: Sukanta Pati