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Slim exceptional set for sums of mixed powers of primes

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Abstract. Let N be a sufficiently large integer. In this paper, it is proved that, with at most $O(N^{4/27+\varepsilon})$ exceptions, all even positive integers up to N can be represented in the form $p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^6 + p_6^6$, where $p_1, p_2, p_3, p_4, p_5, p_6$ are prime numbers. This gives a large improvement of a recent result $O(N^{127/288+\varepsilon})$ due to Liu (*Proc. Indian Acad. Sci.* (*Math. Sci.*) **130(1)** (2020) Article ID. 8).

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1. Introduction and main result

Let $N, k_1, k_2, \ldots, k_s$ be natural numbers such that $2 \leqslant k_1 \leqslant k_2 \leqslant \cdots \leqslant k_s$, $N > s$. Waring's problem of mixed powers concerns the representation of N as the form

$$N = x_1^{k_1} + x_2^{k_2} + \cdots + x_s^{k_s}.$$

Almost since the invention of the circle method by Hardy and Littlewood nearly a century ago, it has been folklore that the method fails to establish the solubility of problems of Waring type when the sum of the reciprocals of the exponents does not exceed 2. As articulated in [3], this *convexity barrier* 'arises from the relative sizes of the product of local densities associated with the system, and the square–root of the available reservoir of variables that is a limiting feature of associated exponential sum estimates', and has been circumvented in very few cases by other devices. A variant of Waring's problem in which one considers mixed sums of squares, cubes and higher powers have provided a rich environment for testing methods designed to approach this theoretical limit of the circle method. A problem of this type that fails to be accessible to the circle method by the narrowest of margins is the notorious one of representing integers as sums of two squares, two positive integral cubes and two sixth powers. In 2013, Wooley [14] applied

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the method of Golubeva [5] to show, subject to the truth of the generalized Riemann hypothesis (GRH), that all large integers are thus represented. However, Wooley's work [14] employs representations of special type and fails to deliver the anticipated asymptotic formula for their total number. Also, Wooley [16] showed that, although the expected asymptotic formula may occasionally fail to hold, the set of such exceptional instances is extraordinarily sparse. Afterwards, Lü and Mu [8] refined the result of Wooley [16].

In view of the result of Wooley [14], it is reasonable to conjecture that, for every sufficiently large even integer N, the following Diophantine equation

$$N = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^6 + p_6^6 \tag{1.1}$$

is solvable. Here and below, the letter p, with or without subscript, always stands for a prime number. But this conjecture is perhaps out of reach at present. In 2020, Liu [7] considered the exceptional set of the representation (1.1). In [7], Liu mainly use the arguments of Wooley [15] and showed that $E(N) \ll N^{\frac{127}{288}+\varepsilon}$, where $E(N)$ denotes the number of positive even integers n up to N, which can not be represented as $p_1^2+p_2^2+p_3^3+p_4^3+p_5^6+p_6^6$.

In this paper, we shall continue to improve the estimate of the exceptional set for the problem (1.1) and establish the following result.

Theorem 1.1. *Let $E(N)$ denote the number of positive even integers n up to N, which can not be represented as*

$$n = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^6 + p_6^6. \tag{1.2}$$

Then, for any $\varepsilon > 0$, we have

$$E(N) \ll N^{\frac{4}{27}+\varepsilon}.$$

Remark. In order to compare the result of Theorem 1.1 with that of Liu [7], we list the numerical result as follows:

$$\frac{127}{288} = 0.440972222\cdots; \qquad \frac{4}{27} = 0.148148148\cdots.$$

We will establish Theorem 1.1 by using the method, which is created and developed by Kawada and Wooley [9], to bound the set of exceptional integers not represented by a given additive form in terms of the exceptional set corresponding to a subform. For the exceptional set corresponding to a subform, we shall employ pruning process into the Hardy–Littlewood circle method. In the treatment of the integrals over minor arcs, we will employ the methods, which is developed by Brüdern [4], combining with the new estimates for exponential sum over primes developed by Zhao [18] and Kumchev [10]. The full details will be explained in the following revelant sections.

Notation. Throughout this paper, let p, with or without subscripts, always denote a prime number; ε always denotes a sufficiently small positive constant, which may not be the same at different occurrences. As usual, we use $\varphi(n)$ and $d(n)$ to denote the Euler's function and Dirichlet's divisor function, respectively. Also, we use χ mod q to denote a Dirichlet character modulo q, and χ^0 mod q the principal character. $e(x) = e^{2\pi i x}$; $f(x) \ll g(x)$ means that $f(x) = O(g(x))$; $f(x) \asymp g(x)$ means that $f(x) \ll g(x) \ll f(x)$. N is a sufficiently large integer and $n \in [N/2, N]$, and thus $\log N \asymp \log n$. The letter c, with or without subscripts or superscripts, always denote a positive constant.

2. Preliminary and outline of the proof of Theorem 1.1

In order to better illustrate Lemma 2.1 and Lemma 2.2 below, we first introduce some notations and definitions. When $\mathcal{C} \subseteq \mathbb{N}$, we write $\overline{\mathcal{C}}$ for the complement $\mathbb{N} \setminus \mathcal{C}$ of $\mathcal{C}$ within $\mathbb{N}$. When a and b are non-negative integers, it is convenient to denote by $(\mathcal{C})_a^b$ the set $\mathcal{C} \cap (a, b]$, and by $|\mathcal{C}|_a^b$ the cardinality of $\mathcal{C} \cap (a, b]$. Next, when $\mathcal{C}, \mathcal{D} \subseteq \mathbb{N}$, we define

$$\mathcal{C} + \mathcal{D} = \{c + d : c \in \mathcal{C} \text{ and } d \in \mathcal{D}\}.$$

It is convenient, when k is a natural number, to describe a subset $\mathcal{Q}$ of $\mathbb{N}$ as being a *high-density subset of the k-th powers* when (i) one has $\mathcal{Q} \in \{n^k : n \in \mathbb{N}\}$, and (ii) for each positive number ε, whenever N is a natural number sufficiently large in terms of ε, then $|\mathcal{Q}|_0^N > N^{1/k-\varepsilon}$. Also, when $\theta > 0$, we shall refer to a set $\mathcal{R} \subseteq \mathbb{N}$ as having *complementary density growth exponent smaller than* θ when there exists a positive number δ with the property that, for all sufficiently large natural numbers N, one has $|\overline{\mathcal{R}}|_0^N < N^{\theta-\delta}$.

When q is a natural number and $\mathfrak{a} \in \{0, 1, \ldots, q-1\}$, we define $\mathcal{P}_{\mathfrak{a}} = \mathcal{P}_{\mathfrak{a},q}$ by

$$\mathcal{P}_{\mathfrak{a},q} = \{\mathfrak{a} + mq : m \in \mathbb{Z}\}.$$

Also, we describe a set $\mathcal{L}$ as being a *union of arithmetic progressions modulo q* when, for some subset $\mathfrak{L}$ of $\{0, 1, \ldots, q-1\}$, one has

$$\mathcal{L} = \bigcup_{\mathfrak{l} \in \mathfrak{L}} \mathcal{P}_{\mathfrak{l},q}.$$

In such circumstances, given a subset $\mathcal{C}$ of $\mathbb{N}$ and integers a and b, it is convenient to write

$$\langle \mathcal{C} \wedge \mathcal{L} \rangle_a^b = \min_{\mathfrak{l} \in \mathfrak{L}} \left| \mathcal{C} \cap \mathcal{P}_{\mathfrak{l},q} \right|_a^b.$$

Let $\mathcal{L}$ be a union of arithmetic progressions modulo q, for some natural number q. When k is a natural number, we describe a subset $\mathcal{Q}$ of $\mathbb{N}$ as being a *high-density subset of the k-th powers relative to* $\mathcal{L}$ when (i) one has $\mathcal{Q} \in \{n^k : n \in \mathbb{N}\}$, and (ii) for each positive number ε, whenever N is a natural number sufficiently large in terms of ε, then $\langle \mathcal{Q} \wedge \mathcal{L} \rangle_0^N \gg_q N^{1/k-\varepsilon}$. In addition, when $\theta > 0$, we shall refer to a set $\mathcal{R} \subseteq \mathbb{N}$ as having $\mathcal{L}$*-complementary density growth exponent smaller than* θ when there exists a positive number δ with the property that, for all sufficiently large natural numbers N, one has $|\overline{\mathcal{R}} \cap \mathcal{L}|_0^N < N^{\theta-\delta}$.

Lemma 2.1. *Let $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{N}$ be unions of arithmetic progressions modulo q, for some natural number q, and suppose that $\mathcal{N} \subseteq \mathcal{L} + \mathcal{M}$. Suppose also that $\mathcal{S}$ is a high-density subset of the squares relative to $\mathcal{L}$, and that $\mathcal{A} \subseteq \mathbb{N}$ has $\mathcal{M}$-complementary density growth exponent smaller than* 1. *Then, whenever $\varepsilon > 0$ and N is a natural number sufficiently large in terms of ε, one has*

$$\left|\overline{\mathcal{A} + \mathcal{S}} \cap \mathcal{N}\right|_{2N}^{3N} \ll_q N^{-\frac{1}{2}+\varepsilon} \left|\overline{\mathcal{A}} \cap \mathcal{M}\right|_N^{3N}.$$

Proof. See Theorem 2.2 of Kawada and Wooley [9]. □

Lemma 2.2. *Let $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{N}$ be unions of arithmetic progressions modulo q, for some natural number q, and suppose that $\mathcal{N} \subseteq \mathcal{L} + \mathcal{M}$. Suppose also that $\mathcal{C}$ is a high-density subset of the cubes relative to $\mathcal{L}$, and that $\mathcal{A} \subseteq \mathbb{N}$ has $\mathcal{M}$-complementary density growth exponent smaller than θ, for some positive number θ. Then, whenever $\varepsilon > 0$ and N is a natural number sufficiently large in terms of ε, without any condition on θ, one has*

$$|\overline{\mathcal{A}+\mathcal{C}} \cap \mathcal{N}|_{2N}^{3N} \ll_q N^{-\frac{1}{3}+\varepsilon}|\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N} + N^{-1+\varepsilon}(|\overline{\mathcal{A}} \cap \mathcal{M}|_N^{3N})^2.$$

Proof. See Theorem 4.1(a) of Kawada and Wooley [9]. □

In order to prove Theorem 1.1, we need the following proposition, whose proof will be given in Section 3.

PROPOSITION 2.3

Let $E_1(N)$ denote the number of positive integers n up to N, which satisfies $n \equiv 0 \pmod 2$, $n \equiv \pm 1 \pmod 3$, and can not be represented as $p_1^2 + p_2^3 + p_3^6 + p_4^6$. Then, for any $\varepsilon > 0$, we have

$$E_1(N) \ll N^{1-\frac{1}{54}+\varepsilon}.$$

Proof of Theorem 1.1. Let

$$\begin{aligned}
\mathcal{A}_1 &= \{p_1^2 + p_2^3 + p_3^6 + p_4^6 : p_j' \text{ s are primes}, \ j = 1, 2, 3, 4\},\\
\mathcal{A}_2 &= \{p_1^2 + p_2^2 + p_3^3 + p_4^6 + p_5^6 : p_j' \text{ s are primes}, \ j = 1, 2, 3, 4, 5\},\\
\mathcal{S}_1 &= \{p^2 : p \text{ is a prime}\}, \quad \mathcal{S}_2 = \{p^3 : p \text{ is a prime}\},\\
\mathcal{M}_1 &= \{n \in \mathbb{N} : n \equiv 0 \pmod 2, \ n \equiv \pm 1 \pmod 3\},\\
\mathcal{N}_1 &= \{n \in \mathbb{N} : n \equiv 1 \pmod 2, \ n \equiv 0, 2 \pmod 3\}, \quad \mathcal{N}_2 = \{n \in \mathbb{N} : n \equiv 0 \pmod 2\},\\
\mathcal{L}_1 &= \{n \in \mathbb{N} : n \equiv 1 \pmod{24}\}, \quad \mathcal{L}_2 = \{n \in \mathbb{N} : n \equiv 1 \pmod 2, \ n \equiv \pm 1 \pmod 3\},\\
\mathscr{E}_1 &= \{n \in \mathbb{N} : n \equiv 0 \pmod 2, n \equiv \pm 1 \pmod 3, n \neq p_1^2 + p_2^3 + p_3^6 + p_4^6, \ p_j' \text{ s are primes}\},\\
\mathscr{E}_2 &= \{n \in \mathbb{N} : n \equiv 1 \pmod 2, n \equiv 0, 2 \pmod 3, n \neq p_1^2 + p_2^2 + p_3^3 + p_4^6 + p_5^6,\\
&\qquad p_j' \text{ s are primes}\},\\
\mathscr{E} &= \{n \in \mathbb{N} : n \equiv 0 \pmod 2, n \neq p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^6 + p_6^6, \ p_j' \text{ s are primes}\}.
\end{aligned}$$

Thus, we have $E_1(N) = |\mathscr{E}_1|_0^N$ and $E(N) = |\mathscr{E}|_0^N$. Also, we write $E_2(N) = |\mathscr{E}_2|_0^N$. Then $\mathcal{L}_1$ is a union of arithmetic progression modulo 24, $\mathcal{N}_2$ is a union of arithmetic progression modulo 2, and $\mathcal{M}_1, \mathcal{N}_1$ and $\mathcal{L}_2$ are unions of arithmetic progressions modulo 6, satisfying the condition that $\mathcal{N}_1 \subseteq \mathcal{L}_1 + \mathcal{M}_1$ and $\mathcal{N}_2 \subseteq \mathcal{L}_2 + \mathcal{N}_1$. Moreover, it follows from the Prime Number Theorem in arithmetic progression that

$$\langle \mathcal{S}_1 \wedge \mathcal{L}_1 \rangle_0^N \gg N^{\frac{1}{2}}(\log N)^{-1} \quad \text{and} \quad \langle \mathcal{S}_2 \wedge \mathcal{L}_2 \rangle_0^N \gg N^{\frac{1}{3}}(\log N)^{-1}.$$

Therefore, $\mathcal{S}_1$ is a high-density subset of the squares relative to $\mathcal{L}_1$, and $\mathcal{S}_2$ is a high-density subset of the cubes relative to $\mathcal{L}_2$. By Proposition 2.3, it is easy to see that

$$\left|\overline{\mathcal{A}_1} \cap \mathcal{M}_1\right|_0^N = |\mathscr{E}_1|_0^N = E_1(N) \ll N^{1-\frac{1}{54}+\varepsilon},$$

and thus $\overline{\mathcal{A}_1}$ has $\mathcal{M}_1$ complementary density growth exponent smaller than 1. From Lemma 2.1, we know that

$$\begin{aligned}\left|\mathscr{E}_2\right|_{2N}^{3N} = \left|\overline{\mathcal{A}_1 + \mathcal{S}_1} \cap \mathcal{L}_2\right|_{2N}^{3N} &\ll N^{-\frac{1}{2}+\varepsilon}\left|\overline{\mathcal{A}_1} \cap \mathcal{M}_1\right|_N^{3N} \\ &\ll N^{-\frac{1}{2}+\varepsilon} \cdot E_1(3N) \ll N^{\frac{1}{2}-\frac{1}{54}+\varepsilon}.\end{aligned}$$

Let the integers N_j for $j \geqslant 0$ by means of the iterative formula

$$N_0 = \left\lceil \frac{1}{2}N \right\rceil, \qquad N_{j+1} = \left\lceil \frac{2}{3}N_j \right\rceil, \quad (j \geqslant 0), \tag{2.1}$$

where $\lceil N \rceil$ denotes the least integer not smaller than N. Moreover, we define J to be the least positive integer with the property that $N_j \leqslant 10$, then $J \ll \log N$. Therefore, there holds

$$E_2(N) \leqslant 10 + \sum_{j=1}^{J} \left|\mathscr{E}_2\right|_{2N_j}^{3N_j} \ll N^{\frac{1}{2}-\frac{1}{54}+\varepsilon}. \tag{2.2}$$

By (2.2), we know that

$$\left|\overline{\mathcal{A}_2} \cap \mathcal{N}_1\right|_0^N = \left|\mathscr{E}_2\right|_0^N = E_2(N) \ll N^{\frac{1}{2}-\frac{1}{54}+\varepsilon},$$

and thus $\mathcal{A}_2$ has $\mathcal{N}_1$-complementary density growth exponent smaller than $\frac{1}{2}$. From Lemma 2.2, we obtain

$$\begin{aligned}|\mathscr{E}|_{2N}^{3N} = \left|\overline{\mathcal{A}_2 + \mathcal{S}_2} \cap \mathcal{N}_2\right|_{2N}^{3N} &\ll N^{-\frac{1}{3}+\varepsilon}\left|\overline{\mathcal{A}_2} \cap \mathcal{N}_1\right|_N^{3N} + N^{-1+\varepsilon}\left(\left|\overline{\mathcal{A}_2} \cap \mathcal{N}_1\right|_N^{3N}\right)^2 \\ &\ll N^{-\frac{1}{3}+\varepsilon} \cdot E_2(3N) + N^{-1+\varepsilon}\big(E_2(3N)\big)^2 \\ &\ll N^{\frac{1}{6}-\frac{1}{54}+\varepsilon} \ll N^{\frac{4}{27}+\varepsilon}.\end{aligned}$$

Therefore, with the same notation as in (2.1), we deduce that

$$E(N) \leqslant 10 + \sum_{j=1}^{J} |\mathscr{E}|_{2N_j}^{3N_j} \ll N^{\frac{4}{27}+\varepsilon},$$

which completes the proof of Theorem 1.1.

3. Outline of the proof of Proposition 2.3

In this section, we shall give the outline of the proof of Proposition 2.3. Let N be a sufficiently large positive integer. For $k = 2, 3, 6$, we define

$$f_k(\alpha) = \sum_{X_k < p \leqslant 2X_k} (\log p) e(p^k \alpha),$$

where $X_k = (N/16)^{\frac{1}{k}}$. Let

$$\mathscr{R}(n) = \sum_{\substack{n = p_1^2 + p_2^3 + p_3^6 + p_4^6 \\ X_2 < p_1 \leqslant 2X_2,\ X_3 < p_2 \leqslant 2X_3 \\ X_6 < p_3, p_4 \leqslant 2X_6}} (\log p_1)(\log p_2)(\log p_3)(\log p_4).$$

Then for any $Q > 0$, it follows from orthogonality that

$$\begin{aligned}\mathscr{R}(n) &= \int_0^1 f_2(\alpha) f_3(\alpha) f_6^2(\alpha) e(-n\alpha) \mathrm{d}\alpha \\ &= \int_{\frac{1}{Q}}^{1+\frac{1}{Q}} f_2(\alpha) f_3(\alpha) f_6^2(\alpha) e(-n\alpha) \mathrm{d}\alpha.\end{aligned}$$

In order to apply the circle method, we set

$$P = N^{\frac{1}{24} - 2\varepsilon}, \qquad Q = N^{\frac{23}{24} + \varepsilon}. \tag{3.1}$$

By Dirichlet's lemma on rational approximation (for instance, see Lemma 2.1 of Vaughan [12]), each $\alpha \in [1/Q, 1 + 1/Q]$ can be written in the form

$$\alpha = \frac{a}{q} + \lambda, \qquad |\lambda| \leqslant \frac{1}{qQ},$$

for some integers a, q with $1 \leqslant a \leqslant q \leqslant Q$ and $(a, q) = 1$. Then we define the major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m}$ as follows:

$$\mathfrak{M} = \bigcup_{1 \leqslant q \leqslant P} \bigcup_{\substack{1 \leqslant a \leqslant q \\ (a,q)=1}} \mathfrak{M}(q, a), \quad \mathfrak{m} = \left[\frac{1}{Q}, 1 + \frac{1}{Q}\right] \Big\backslash \mathfrak{M}, \tag{3.2}$$

where

$$\mathfrak{M}(q, a) = \left[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}\right].$$

Then one has

$$\mathscr{R}(n) = \left\{ \int_{\mathfrak{M}} + \int_{\mathfrak{m}} \right\} f_2(\alpha) f_3(\alpha) f_6^2(\alpha) e(-n\alpha) \mathrm{d}\alpha.$$

In order to prove Proposition 2.3, we need the two following propositions, whose proofs will be given in Sects. 4 and 6, respectively.

PROPOSITION 3.1

Let the major arcs $\mathfrak{M}$ *be defined as in* (3.2) *with* P *and* Q *defined in* (3.1). *Then, for* $n \in (N/2, N]$ *and any* $A > 0$, *there holds*

$$\int_{\mathfrak{M}} f_2(\alpha) f_3(\alpha) f_6^2(\alpha) e(-n\alpha)\mathrm{d}\alpha = \frac{1}{216}\mathfrak{S}(n)\mathfrak{J}(n) + O\big(N^{\frac{1}{6}}L^{-A}\big),$$

where $\mathfrak{S}(n)$ *is the singular series defined in* (4.1) *below, which is absolutely convergent and satisfies*

$$(\log\log n)^{-c^*} \ll \mathfrak{S}(n) \ll d(n) \tag{3.3}$$

for any integer n satisfying $n \equiv 0 \pmod 2$ *and* $n \equiv \pm 1 \pmod 3$, *and some fixed constant* $c^* > 0$; *while* $\mathfrak{J}(n)$ *is defined by* (4.9) *and satisfies*

$$\mathfrak{J}(n) \asymp N^{\frac{1}{6}}.$$

For the properties (3.3) of singular series, we shall give the proof in Section 5.

PROPOSITION 3.2

Let the minor arcs $\mathfrak{m}$ *be defined as in* (3.2) *with* P *and* Q *defined in* (3.1). *Then we have*

$$\int_{\mathfrak{m}} \big|f_2^2(\alpha) f_3^2(\alpha) f_6^4(\alpha)\big|\mathrm{d}\alpha \ll N^{\frac{1}{3}+1-\frac{1}{54}+\varepsilon}.$$

The remaining part of this section is devoted to establishing Proposition 2.3 by using Propositions 3.1 and 3.2.

Proof of Proposition 2.3. Let $\mathscr{U}(N)$ denote the set of integers $n \in (N/2, N]$ such that

$$\left|\int_{\mathfrak{m}} f_2(\alpha) f_3(\alpha) f_6^2(\alpha) e(-n\alpha)\mathrm{d}\alpha\right| \gg N^{\frac{1}{6}}L^{-A}.$$

Then we have

$$\begin{aligned} N^{\frac{1}{3}}L^{-2A}|\mathscr{U}(N)| &\ll \sum_{n\in\mathscr{U}(N)} \left|\int_{\mathfrak{m}} f_2(\alpha) f_3(\alpha) f_6^2(\alpha) e(-n\alpha)\mathrm{d}\alpha\right|^2 \\ &\ll \sum_{\frac{N}{2}<n\leqslant N} \left|\int_{\mathfrak{m}} f_2(\alpha) f_3(\alpha) f_6^2(\alpha) e(-n\alpha)\mathrm{d}\alpha\right|^2. \end{aligned} \tag{3.4}$$

By Bessel's inequality, we have

$$\sum_{\frac{N}{2}<n\leqslant N}\left|\int_{\mathfrak{m}} f_2(\alpha)f_3(\alpha)f_6^2(\alpha)e(-n\alpha)\mathrm{d}\alpha\right|^2 \leqslant \int_{\mathfrak{m}} \left|f_2^2(\alpha)f_3^2(\alpha)f_6^4(\alpha)\right|\mathrm{d}\alpha. \tag{3.5}$$

Combining (3.4), (3.5) and Proposition 3.2, we have

$$|\mathscr{U}(N)| \ll N^{1-\frac{1}{54}+\varepsilon}.$$

Therefore, with at most $O(N^{1-1/54+\varepsilon})$ exceptions, all the integers $n \in (N/2, N]$ satisfy

$$\left|\int_{\mathfrak{m}} f_2(\alpha)f_3(\alpha)f_6^2(\alpha)e(-n\alpha)\mathrm{d}\alpha\right| \ll N^{\frac{1}{6}}L^{-A}.$$

Using the above equation and Proposition 3.1, we deduce that, with at most $O(N^{1-1/54+\varepsilon})$ exceptions, all the positive integers $n \in (N/2, N]$ satisfying $n \equiv 0\,(\mathrm{mod}\,2)$ and $n \equiv \pm 1\,(\mathrm{mod}\,3)$ can be represented in the form $p_1^2 + p_2^3 + p_3^6 + p_4^6$, where p_1, p_2, p_3, p_4 are prime numbers. By a splitting argument, we obtain

$$E_1(N) \ll N^{1-\frac{1}{54}+\varepsilon}.$$

This completes the proof of Proposition 2.3.

4. Proof of Proposition 3.1

In this section, we shall concentrate on proving Proposition 3.1. We first introduce some notations. For a Dirichlet character χ mod q and $k \in \{2, 3, 6\}$, we define

$$C_k(\chi, a) = \sum_{h=1}^{q} \overline{\chi(h)}e\left(\frac{ah^k}{q}\right), \qquad C_k(q, a) = C_k(\chi^0, a),$$

where χ^0 is the principal character modulo q. Let $\chi_2, \chi_3, \chi_6^{(1)}, \chi_6^{(2)}$ be Dirichlet characters modulo q. Define

$$B\big(n, q, \chi_2, \chi_3, \chi_6^{(1)}, \chi_6^{(2)}\big) = \sum_{\substack{a=1\\(a,q)=1}}^{q} C_2(\chi_2, a)C_3(\chi_3, a)C_6\big(\chi_6^{(1)}, a\big) C_6\big(\chi_6^{(2)}, a\big)e\left(-\frac{an}{q}\right), \; B(n, q) = B\big(n, q, \chi^0, \chi^0, \chi^0, \chi^0\big),$$

and write

$$A(n, q) = \frac{B(n, q)}{\varphi^4(q)}, \qquad \mathfrak{S}(n) = \sum_{q=1}^{\infty} A(n, q). \tag{4.1}$$

Lemma 4.1. *For* $(a, q) = 1$ *and any Dirichlet character* $\chi \bmod q$, *there holds*

$$|C_k(\chi, a)| \leqslant 2q^{1/2} d^{\beta_k}(q)$$

with $\beta_k = (\log k)/\log 2$.

Proof. See Problem 14 of Chapter VI of [13]. □

Lemma 4.2. *The singular series* $\mathfrak{S}(n)$ *satisfies* (3.3).

The proof of Lemma 4.2 is given in Section 5.

Lemma 4.3. *Let* $f(x)$ *be a real differentiable function in the interval* $[a, b]$. *If* $f'(x)$ *is monotonic and satisfies* $|f'(x)| \leqslant \theta < 1$, *then we have*

$$\sum_{a<n\leqslant b} e^{2\pi i f(n)} = \int_a^b e^{2\pi i f(x)} \mathrm{d}x + O(1).$$

Proof. See Lemma 4.8 of [11]. □

Lemma 4.4. *Let* $\chi_2 \bmod r_2$, $\chi_3 \bmod r_3$ *and* $\chi_6^{(i)} \bmod r_6^{(i)}$ *with* $i = 1, 2$ *be primitive characters*, $r_0 = \left[r_2, r_3, r_6^{(1)}, r_6^{(2)}\right]$, *and* χ^0 *the principal character modulo* q. *Then there holds*

$$\sum_{\substack{q\leqslant x\\ r_0|q}} \frac{1}{\varphi^4(q)} \left|B\left(n, q, \chi_2\chi^0, \chi_3\chi^0, \chi_6^{(1)}\chi^0, \chi_6^{(2)}\chi^0\right)\right| \ll r_0^{-1+\varepsilon} \log^{257} x. \tag{4.2}$$

Proof. By Lemma 4.1, we have

$$\begin{aligned}
&\left|B\left(n, q, \chi_2\chi^0, \chi_3\chi^0, \chi_6^{(1)}\chi^0, \chi_6^{(2)}\chi^0\right)\right| \\
&\ll \sum_{\substack{a=1\\ (a,q)=1}}^{q} \left|C_2\left(\chi_2\chi^0, a\right) C_3\left(\chi_3\chi^0, a\right) C_6\left(\chi_6^{(1)}\chi^0, a\right) C_6\left(\chi_6^{(2)}\chi^0, a\right)\right| \\
&\ll q^2 \varphi(q) d^8(q).
\end{aligned}$$

Therefore, the left-hand side of (4.2) is

$$\begin{aligned}
&\ll \sum_{\substack{q\leqslant x\\ r_0|q}} \frac{q^2\varphi(q)d^8(q)}{\varphi^4(q)} = \sum_{t\leqslant \frac{x}{r_0}} \frac{r_0^2 t^2 d^8(r_0 t)}{\varphi^3(r_0 t)} \ll r_0^{-1+\varepsilon} (\log x) \sum_{t\leqslant x} \frac{d^8(t)}{t}. \\
&\ll r_0^{-1+\varepsilon} \log^{257} x.
\end{aligned}$$

This completes the proof of Lemma 4.4. □

Write

$$V_k(\lambda) = \sum_{X_k < m \leqslant 2X_k} e(m^k\lambda),$$
$$W_k(\chi, \lambda) = \sum_{X_k < p \leqslant 2X_k} (\log p)\chi(p)e(p^k\lambda) - \delta_\chi \sum_{X_k < m \leqslant 2X_k} e(m^k\lambda), \tag{4.3}$$

where $\delta_\chi = 1$ or 0 according to whether χ is principal or not. Then by the orthogonality of Dirichlet characters, for $(a, q) = 1$, we have

$$f_k\left(\frac{a}{q} + \lambda\right) = \frac{C_k(q, a)}{\varphi(q)} V_k(\lambda) + \frac{1}{\varphi(q)} \sum_{\chi \bmod q} C_k(\chi, a) W_k(\chi, \lambda).$$

For $j = 1, 2, \ldots, 12$, we define the sets $\mathscr{S}_j$ as follows:

$$\mathscr{S}_j = \begin{cases} \{2,3,6,6\}, \text{ if } j=1; \ \{3,6,6\}, \text{ if } j=4; \ \{3,6\}, \text{ if } j=7; \ \{3\}, \text{ if } j=10; \\ \{2,3,6\}, \text{ if } j=2; \quad \{2,3\}, \text{ if } j=5; \quad \{6,6\}, \text{ if } j=8; \ \{6\}, \text{ if } j=11; \\ \{2,6,6\}, \text{ if } j=3; \quad \{2,6\}, \text{ if } j=6; \quad \{2\}, \text{ if } j=9; \quad \phi, \text{ if } j=12. \end{cases}$$

Also, we write $\overline{\mathscr{S}_j} = \{2, 3, 6, 6\} \setminus \mathscr{S}_j$. Then we have

$$\int_{\mathfrak{M}} f_2(\alpha) f_3(\alpha) f_6^2(\alpha) e(-n\alpha) \mathrm{d}\alpha =: I_1 + 2I_2 + I_3 + I_4 + I_5 + 2I_6 + 2I_7 + I_8 + I_9 + I_{10} + 2I_{11} + I_{12}, \tag{4.4}$$

where

$$I_j = \sum_{q \leqslant P} \frac{1}{\varphi^4(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left(\prod_{k \in \mathscr{S}_j} C_k(q, a)\right) e\left(-\frac{an}{q}\right) \times \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} \left(\prod_{k \in \mathscr{S}_j} V_k(\lambda)\right) \left(\prod_{k \in \overline{\mathscr{S}_j}} \sum_{\chi \bmod q} C_k(\chi, a) W_k(\chi, \lambda)\right) e(-n\lambda) \mathrm{d}\lambda.$$

In the following content of this section, we shall prove that I_1 produces the main term, while the others contribute to the error term.

For $k = 2, 3, 6$, applying Lemma 4.3 to $V_k(\lambda)$, we have

$$V_k(\lambda) = \int_{X_k}^{2X_k} e(u^k\lambda) \mathrm{d}u + O(1) = \frac{1}{k} \int_{X_k^k}^{(2X_k)^k} v^{\frac{1}{k}-1} e(v\lambda) \mathrm{d}v + O(1)$$
$$= \frac{1}{k} \sum_{X_k^k < m \leqslant (2X_k)^k} m^{\frac{1}{k}-1} e(m\lambda) + O(1). \tag{4.5}$$

Putting (4.5) into I_1, we see that

$$\begin{aligned} I_1 &= \frac{1}{216} \sum_{q \leqslant P} \frac{B(n,q)}{\varphi^4(q)} \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} \Bigg(\sum_{X_2^2 < m \leqslant (2X_2)^2} m^{-\frac{1}{2}} e(m\lambda) \Bigg) \\ &\quad \times \Bigg(\sum_{X_3^3 < m \leqslant (2X_3)^3} m^{-\frac{2}{3}} e(m\lambda) \Bigg) \Bigg(\sum_{X_6^6 < m \leqslant (2X_6)^6} m^{-\frac{5}{6}} e(m\lambda) \Bigg)^2 e(-n\lambda) \mathrm{d}\lambda \\ &\quad + O\Bigg(\sum_{q \leqslant P} \frac{|B(n,q)|}{\varphi^4(q)} \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} \Bigg| \sum_{X_2^2 < m \leqslant (2X_2)^2} m^{-\frac{1}{2}} e(m\lambda) \Bigg| \\ &\quad \times \Bigg| \sum_{X_3^3 < m \leqslant (2X_3)^3} m^{-\frac{2}{3}} e(m\lambda) \Bigg| \Bigg| \sum_{X_6^6 < m \leqslant (2X_6)^6} m^{-\frac{5}{6}} e(m\lambda) \Bigg| \mathrm{d}\lambda \Bigg). \end{aligned} \tag{4.6}$$

By using the elementary estimate

$$\sum_{X_k^k < m \leqslant (2X_k)^k} m^{\frac{1}{k}-1} e(m\lambda) \ll N^{\frac{1}{k}-1} \min\left(N, \frac{1}{|\lambda|} \right), \tag{4.7}$$

and Lemma 4.4 with $r_0 = 1$, the O-term in (4.6) can be estimated as

$$\ll \sum_{q \leqslant P} \frac{|B(n,q)|}{\varphi^4(q)} \left(\int_0^{\frac{1}{N}} N \mathrm{d}\lambda + \int_{\frac{1}{N}}^{\infty} N^{-2} \cdot \frac{1}{\lambda^3} \mathrm{d}\lambda \right) \ll L^{257} \ll N^{\frac{1}{6}} L^{-A}.$$

If we extend the interval of the integral in the main term of (4.6) to $[-1/2, 1/2]$, then from (3.1) we can see that the resulting error is

$$\ll L^{257} \int_{\frac{1}{qQ}}^{\frac{1}{2}} N^{-\frac{17}{6}} \cdot \frac{\mathrm{d}\lambda}{\lambda^4} \ll N^{-\frac{17}{6}} q^3 Q^3 L^{257} \ll N^{-\frac{17}{6}} (PQ)^3 L^{257} \ll N^{\frac{1}{6}-\varpi}$$

for some $\varpi > 0$. Therefore, by Lemma 4.2, (4.6) becomes

$$I_1 = \frac{1}{216} \mathfrak{S}(n) \mathfrak{J}(n) + O\big(N^{\frac{1}{6}} L^{-A}\big), \tag{4.8}$$

where

$$\mathfrak{J}(n) := \sum_{\substack{m_1+m_2+m_3+m_4=n \\ X_2^2 < m_1 \leqslant (2X_2)^2,\ X_3^3 < m_2 \leqslant (2X_3)^3 \\ X_6^6 < m_3, m_4 \leqslant (2X_6)^6}} m_1^{-\frac{1}{2}} m_2^{-\frac{2}{3}} (m_3 m_4)^{-\frac{5}{6}} \asymp N^{\frac{1}{6}}. \tag{4.9}$$

In order to estimate the contribution of I_j for $j = 2, 3, \ldots, 12$, we shall need the following three preliminary lemmas, i.e. Lemmas 4.5–4.7, whose proofs are exactly the same as Lemmas 3.5–3.7 in Zhang and Li [17], so we omit the details herein. In view of this, for $k = 2, 3, 6$, we recall the definition of $W_k(\chi, \lambda)$ in (4.3) and write

$$J_k(g) = \sum_{r \leqslant P} [g, r]^{-1+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \max_{|\lambda| \leqslant \frac{1}{rQ}} \left|W_k(\chi, \lambda)\right|,$$

and

$$K_k(g) = \sum_{r \leqslant P} [g, r]^{-1+\varepsilon} \sideset{}{^*}\sum_{\chi \bmod r} \left(\int_{-\frac{1}{rQ}}^{\frac{1}{rQ}} \left|W_k(\chi, \lambda)\right|^2 \mathrm{d}\lambda \right)^{\frac{1}{2}}.$$

Here and below, Σ^* indicates that the summation is taken over all primitive characters.

Lemma 4.5. *Let P, Q be defined as in (3.1). For $k = 6$, we have*

$$K_6(g) \ll g^{-1+\varepsilon} N^{-\frac{1}{3}} L^c.$$

Lemma 4.6. *Let P, Q be defined as in (3.1). Then we have*

$$J_3(g) \ll g^{-1+\varepsilon} N^{\frac{1}{3}} L^c.$$

Lemma 4.7. *Let P, Q be defined as in (3.1). Then, for any $A > 0$, we have*

$$J_2(1) \ll N^{\frac{1}{2}} L^{-A}.$$

Now, we concentrate on estimating the terms I_j for $j = 2, 3, \ldots, 12$. We begin with the term I_{12}, which is the most complicated one. Reducing the Dirichlet characters in I_{12} into primitive characters, we have

$$\begin{aligned}
|I_{12}| &= \left| \sum_{q \leqslant P} \frac{1}{\varphi^4(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(-\frac{an}{q}\right) \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} \left(\sum_{\chi_2 \bmod q} C_2(\chi_2, a) W_2(\chi_2, \lambda) \right) \right. \\
&\quad \left. \times \left(\sum_{\chi_3 \bmod q} C_3(\chi_3, a) W_3(\chi_3, \lambda) \right) \left(\sum_{\chi_6 \bmod q} C_6(\chi_6, a) W_6(\chi_6, \lambda) \right)^2 e(-n\lambda) \mathrm{d}\lambda \right| \\
&= \left| \sum_{q \leqslant P} \sum_{\chi_2 \bmod q} \sum_{\chi_3 \bmod q} \sum_{\chi_6^{(1)} \bmod q} \sum_{\chi_6^{(2)} \bmod q} \frac{1}{\varphi^4(q)} \cdot B\left(n, q, \chi_2, \chi_3, \chi_6^{(1)}, \chi_6^{(2)}\right) \right. \\
&\quad \left. \times \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} W_2(\chi_2, \lambda) W_3(\chi_3, \lambda) W_6(\chi_6^{(1)}, \lambda) W_6(\chi_6^{(2)}, \lambda) e(-n\lambda) \mathrm{d}\lambda \right| \\
&\leqslant \sum_{r_2 \leqslant P} \sum_{r_3 \leqslant P} \sum_{r_6^{(1)} \leqslant P} \sum_{r_6^{(2)} \leqslant P} \sideset{}{^*}\sum_{\chi_2 \bmod r_2} \sideset{}{^*}\sum_{\chi_3 \bmod r_3} \sideset{}{^*}\sum_{\chi_6^{(1)} \bmod r_6^{(1)}} \sideset{}{^*}\sum_{\chi_6^{(2)} \bmod r_6^{(2)}}
\end{aligned}$$

$$\times \sum_{\substack{q\leqslant P\\ r_0|q}} \frac{\left|B\left(n,q,\chi_2\chi^0,\chi_3\chi^0,\chi_6^{(1)}\chi^0,\chi_6^{(2)}\chi^0\right)\right|}{\varphi^4(q)}$$
$$\times \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} \left|W_2\left(\chi_2\chi^0,\lambda\right)W_3\left(\chi_3\chi^0,\lambda\right)W_6\left(\chi_6^{(1)}\chi^0,\lambda\right)W_6\left(\chi_6^{(2)}\chi^0,\lambda\right)\right|\mathrm{d}\lambda,$$

where χ^0 is the principal character modulo q and $r_0 = [r_2, r_3, r_6^{(1)}, r_6^{(2)}]$. For $q \leqslant P$ and $X_k < p \leqslant 2X_k$ with $k = 2, 3, 6$, we have $(q, p) = 1$. From this and the definition of $W_k(\chi,\lambda)$, we obtain $W_2\left(\chi_2\chi^0,\lambda\right) = W_2\left(\chi_2,\lambda\right)$, $W_3\left(\chi_3\chi^0,\lambda\right) = W_3\left(\chi_3,\lambda\right)$ and $W_6\left(\chi_6^{(i)}\chi^0,\lambda\right) = W_6\left(\chi_6^{(i)},\lambda\right)$ for primitive characters $\chi_2, \chi_3, \chi_6^{(i)}$ with $i = 1, 2$. Therefore, by Lemma 4.4, we obtain

$$|I_{12}| \leqslant \sum_{r_2\leqslant P}\sum_{r_3\leqslant P}\sum_{r_6^{(1)}\leqslant P}\sum_{r_6^{(2)}\leqslant P}\ \sideset{}{^*}\sum_{\chi_2 \bmod r_2}\ \sideset{}{^*}\sum_{\chi_3 \bmod r_3}\ \sideset{}{^*}\sum_{\chi_6^{(1)} \bmod r_6^{(1)}}\ \sideset{}{^*}\sum_{\chi_6^{(2)} \bmod r_6^{(2)}}$$
$$\times \int_{-\frac{1}{r_0Q}}^{\frac{1}{r_0Q}} \left|W_2(\chi_2,\lambda)W_3(\chi_3,\lambda)W_6\left(\chi_6^{(1)},\lambda\right)W_6\left(\chi_6^{(2)},\lambda\right)\right|\mathrm{d}\lambda$$
$$\times \sum_{\substack{q\leqslant P\\ r_0|q}} \frac{\left|B\left(n,q,\chi_2\chi^0,\chi_3\chi^0,\chi_6^{(1)}\chi^0,\chi_6^{(2)}\chi^0\right)\right|}{\varphi^4(q)}$$
$$\ll L^{257}\cdot \sum_{r_2\leqslant P}\sum_{r_3\leqslant P}\sum_{r_6^{(1)}\leqslant P}\sum_{r_6^{(2)}\leqslant P} r_0^{-1+\varepsilon}\ \sideset{}{^*}\sum_{\chi_2 \bmod r_2}\ \sideset{}{^*}\sum_{\chi_3 \bmod r_3}\ \sideset{}{^*}\sum_{\chi_6^{(1)} \bmod r_6^{(1)}}\ \sideset{}{^*}\sum_{\chi_6^{(2)} \bmod r_6^{(2)}}$$
$$\times \int_{-\frac{1}{r_0Q}}^{\frac{1}{r_0Q}} \left|W_2(\chi_2,\lambda)W_3(\chi_3,\lambda)W_6\left(\chi_6^{(1)},\lambda\right)W_6\left(\chi_6^{(2)},\lambda\right)\right|\mathrm{d}\lambda.$$

In the last integral, we pick out $\left|W_2(\chi_2,\lambda)\right|$ and $\left|W_3(\chi_3,\lambda)\right|$, and then use Cauchy's inequality to derive that

$$|I_{12}| \ll L^{257}\Bigg(\sum_{r_2\leqslant P}\ \sideset{}{^*}\sum_{\chi_2 \bmod r_2}\max_{|\lambda|\leqslant\frac{1}{r_2Q}}\left|W_2(\chi_2,\lambda)\right|\Bigg)\Bigg(\sum_{r_3\leqslant P}\ \sideset{}{^*}\sum_{\chi_3 \bmod r_3}\max_{|\lambda|\leqslant\frac{1}{r_3Q}}\left|W_3(\chi_3,\lambda)\right|\Bigg)$$
$$\times \sum_{r_6^{(1)}\leqslant P}\ \sideset{}{^*}\sum_{\chi_6^{(1)} \bmod r_6^{(1)}}\left(\int_{-\frac{1}{r_6^{(1)}Q}}^{\frac{1}{r_6^{(1)}Q}}\left|W_6\left(\chi_6^{(1)},\lambda\right)\right|^2\mathrm{d}\lambda\right)^{\frac{1}{2}}$$
$$\times \sum_{r_6^{(2)}\leqslant P} r_0^{-1+\varepsilon}\ \sideset{}{^*}\sum_{\chi_6^{(2)} \bmod r_6^{(2)}}\left(\int_{-\frac{1}{r_6^{(2)}Q}}^{\frac{1}{r_6^{(2)}Q}}\left|W_6\left(\chi_6^{(2)},\lambda\right)\right|^2\mathrm{d}\lambda\right)^{\frac{1}{2}}. \tag{4.10}$$

Now we introduce the iterative procedure to bound the sums over $r_6^{(2)}, r_6^{(1)}, r_3, r_2$, consecutively. We first estimate the above sum over $r_6^{(2)}$ in (4.10) via Lemma 4.5. Since

$$r_0 = \left[r_2, r_3, r_6^{(1)}, r_6^{(2)}\right] = \left[\left[r_2, r_3, r_6^{(1)}\right], r_6^{(2)}\right],$$

the sum over $r_6^{(2)}$ is

$$= \sum_{r_6^{(2)}\leqslant P} \big[[r_2, r_3, r_6^{(1)}], r_6^{(2)}\big]^{-1+\varepsilon} \sideset{}{^*}\sum_{\chi_6^{(2)} \bmod r_6^{(2)}} \left(\int_{-\frac{1}{r_6^{(2)}Q}}^{\frac{1}{r_6^{(2)}Q}} \left|W_6\big(\chi_6^{(2)}, \lambda\big)\right|^2 \mathrm{d}\lambda \right)^{\frac{1}{2}}$$
$$= K_6\Big(\big[r_2, r_3, r_6^{(1)}\big]\Big) \ll \big[r_2, r_3, r_6^{(1)}\big]^{-1+\varepsilon} N^{-\frac{1}{3}} L^c. \tag{4.11}$$

Again, by Lemma 4.5, the contribution of the quantity on the right-hand side of (4.11) to the sum over $r_6^{(1)}$ in (4.10) is

$$\ll N^{-\frac{1}{3}} L^c \cdot \sum_{r_6^{(1)}\leqslant P} \big[[r_2, r_3], r_6^{(1)}\big]^{-1+\varepsilon} \sideset{}{^*}\sum_{\chi_6^{(1)} \bmod r_6^{(1)}} \left(\int_{-\frac{1}{r_6^{(1)}Q}}^{\frac{1}{r_6^{(1)}Q}} \left|W_6\big(\chi_6^{(1)}, \lambda\big)\right|^2 \mathrm{d}\lambda \right)^{\frac{1}{2}}$$
$$= N^{-\frac{1}{3}} L^c \cdot K_6\big([r_2, r_3]\big) \ll [r_2, r_3]^{-1+\varepsilon} N^{-\frac{2}{3}} L^c. \tag{4.12}$$

By Lemma 4.6, the contribution of the quantity on the right-hand side of (4.12) to the sum over r_3 in (4.10) is

$$\ll N^{-\frac{2}{3}} L^c \cdot \sum_{r_3\leqslant P} [r_2, r_3]^{-1+\varepsilon} \sideset{}{^*}\sum_{\chi_3 \bmod r_3} \max_{|\lambda|\leqslant \frac{1}{r_3 Q}} \left|W_3\big(\chi_3, \lambda\big)\right|$$
$$= N^{-\frac{2}{3}} L^c \cdot J_3(r_2) \ll r_2^{-1+\varepsilon} N^{-\frac{1}{3}} L^c. \tag{4.13}$$

Finally, from Lemma 4.7, inserting the bound on the right-hand side of (4.13) to the sum over r_2 in (4.10), we get

$$|I_{12}| \ll N^{-\frac{1}{3}} L^c \cdot \sum_{r_2\leqslant P} [1, r_2]^{-1+\varepsilon} \sideset{}{^*}\sum_{\chi_2 \bmod r_2} \max_{|\lambda|\leqslant \frac{1}{r_2 Q}} \left|W_2\big(\chi_2, \lambda\big)\right|$$
$$= N^{-\frac{1}{3}} L^c \cdot J_2(1) \ll N^{\frac{1}{6}} L^{-A}. \tag{4.14}$$

For the estimation of the terms $I_2, I_3, \ldots, I_{11}$, by noting (4.5) and (4.7), we obtain

$$\left(\int_{-\frac{1}{Q}}^{\frac{1}{Q}} |V_k(\lambda)|^2 \mathrm{d}\lambda \right)^{\frac{1}{2}} \ll \left(\int_{-\frac{1}{Q}}^{\frac{1}{Q}} N^{\frac{2}{k}-2} \min\left(N, \frac{1}{|\lambda|}\right)^2 \mathrm{d}\lambda + \frac{1}{Q} \right)^{\frac{1}{2}}$$
$$\ll N^{\frac{1}{k}-1} \left(\int_0^{\frac{1}{N}} N^2 \mathrm{d}\lambda + \int_{\frac{1}{N}}^{\frac{1}{Q}} \frac{\mathrm{d}\lambda}{\lambda^2} \right)^{\frac{1}{2}} + \frac{1}{Q^{1/2}} \ll N^{\frac{1}{k}-\frac{1}{2}}.$$

Using this estimate and the upper bound of $V_k(\lambda)$, which derives from (4.5) and (4.7), that $V_k(\lambda) \ll N^{\frac{1}{k}}$, we can argue similarly to the treatment of I_{12} and obtain

$$\sum_{j=2}^{11} I_j \ll N^{\frac{1}{6}} L^{-A}. \tag{4.15}$$

Combining (4.4), (4.8), (4.14) and (4.15), we can derive the conclusion of Proposition 3.1.

5. The singular series

In this section, we shall investigate the properties of the singular series which appear in Proposition 3.1.

Lemma 5.1. *Let p be a prime and $p^\alpha \| k$. For $(a, p) = 1$, if $\ell \geqslant \gamma(p)$, we have $C_k(p^\ell, a) = 0$, where*

$$\gamma(p) = \begin{cases} \alpha + 2, & \text{if } p \neq 2 \text{ or } p = 2,\ \alpha = 0; \\ \alpha + 3, & \text{if } p = 2,\ \alpha > 0. \end{cases}$$

Proof. See Lemma 8.3 of [6]. □

For $k \geqslant 1$, we define

$$S_k(q, a) = \sum_{m=1}^{q} e\left(\frac{am^k}{q}\right).$$

Lemma 5.2. *Suppose that $(p, a) = 1$. Then*

$$S_k(p, a) = \sum_{\chi \in \mathscr{A}_k} \overline{\chi(a)} \tau(\chi),$$

where $\mathscr{A}_k$ denotes the set of non-principal characters χ modulo p for which χ^k is principal, and $\tau(\chi)$ denotes the Gauss sum

$$\sum_{m=1}^{p} \chi(m) e\left(\frac{m}{p}\right).$$

Also, there hold $|\tau(\chi)| = p^{1/2}$ and $|\mathscr{A}_k| = (k, p-1) - 1$.

Proof. See Lemma 4.3 of [12]. □

Lemma 5.3. *For $(p, n) = 1$, we have*

$$\left| \sum_{a=1}^{p-1} \frac{S_2(p, a) S_3(p, a) S_6^2(p, a)}{p^4} e\left(-\frac{an}{p}\right) \right| \leqslant 50 p^{-\frac{3}{2}}. \tag{5.1}$$

Proof. We denote by $\mathcal{S}$ the left-hand side of (5.1). By Lemma 5.2, we have

$$\mathcal{S} = \frac{1}{p^4} \sum_{a=1}^{p-1} \left(\sum_{\chi_2 \in \mathscr{A}_2} \overline{\chi_2(a)} \tau(\chi_2) \right) \left(\sum_{\chi_3 \in \mathscr{A}_3} \overline{\chi_3(a)} \tau(\chi_3) \right) \times \left(\sum_{\chi_6 \in \mathscr{A}_6} \overline{\chi_6(a)} \tau(\chi_6) \right)^2 e\left(-\frac{an}{p}\right).$$

If $|\mathscr{A}_k| = 0$ for some $k \in \{2, 3, 6\}$, then $\mathcal{S} = 0$. If this is not the case, then

$$\mathcal{S} = \frac{1}{p^4} \sum_{\chi_2 \in \mathscr{A}_2} \sum_{\chi_3 \in \mathscr{A}_3} \sum_{\chi_6^{(1)} \in \mathscr{A}_6} \sum_{\chi_6^{(2)} \in \mathscr{A}_6} \tau(\chi_2)\tau(\chi_3)\tau\big(\chi_6^{(1)}\big)\tau\big(\chi_6^{(2)}\big)$$
$$\times \sum_{a=1}^{p-1} \overline{\chi_2(a)\chi_3(a)\chi_6^{(1)}(a)\chi_6^{(2)}(a)}\, e\Big(-\frac{an}{p}\Big).$$

From Lemma 5.2, the quadruple outer sums have not more than 50 terms. In each of these terms, we have

$$\Big|\tau(\chi_2)\tau(\chi_3)\tau\big(\chi_6^{(1)}\big)\tau\big(\chi_6^{(2)}\big)\Big| = p^2.$$

Since in any one of these terms $\overline{\chi_2(a)\chi_3(a)\chi_6^{(1)}(a)\chi_6^{(2)}(a)}$ is a Dirichlet character $\chi \pmod p$, the inner sum is

$$\sum_{a=1}^{p-1} \chi(a) e\Big(-\frac{an}{p}\Big) = \overline{\chi(-n)} \sum_{a=1}^{p-1} \chi(-an) e\Big(-\frac{an}{p}\Big) = \overline{\chi(-n)}\tau(\chi).$$

From the fact that $\tau(\chi^0) = -1$ for principal character $\chi^0 \bmod p$, we have

$$\big|\overline{\chi(-n)}\tau(\chi)\big| \leqslant p^{\frac{1}{2}}.$$

By the above arguments, we obtain

$$|\mathcal{S}| \leqslant \frac{1}{p^4} \cdot 50 \cdot p^2 \cdot p^{\frac{1}{2}} = 50 p^{-\frac{3}{2}}.$$

This completes the proof of Lemma 5.3. □

Lemma 5.4. *Let $\mathscr{L}(p, n)$ denote the number of solutions of the following congruence*;

$$x_1^2 + x_2^3 + x_3^6 + x_4^6 \equiv n \pmod p, \qquad 1 \leqslant x_1, x_2, x_3, x_4 \leqslant p - 1.$$

Then, for $n \equiv 0 \pmod 2$ and $n \equiv \pm 1 \pmod 3$, we have $\mathscr{L}(p, n) > 0$.

Proof. We have

$$p \cdot \mathscr{L}(p, n) = \sum_{a=1}^{p} C_2(p, a) C_3(p, a) C_6^2(p, a) e\Big(-\frac{an}{p}\Big) = (p-1)^4 + E_p,$$

where

$$E_p = \sum_{a=1}^{p-1} C_2(p, a) C_3(p, a) C_6^2(p, a) e\Big(-\frac{an}{p}\Big).$$

By Lemma 5.2, we obtain

$$|E_p| \leqslant (p-1)(\sqrt{p}+1)(2\sqrt{p}+1)(5\sqrt{p}+1)^2.$$

It is easy to check that $|E_p| < (p-1)^4$ for $p \geqslant 67$. Therefore, we obtain $\mathscr{L}(p,n) > 0$ for $p \geqslant 67$. For $p = 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61$, we can check $\mathscr{L}(p,n) > 0$ directly. This completes the proof of Lemma 5.4. □

Lemma 5.5. *$A(n,q)$ is multiplicative in q.*

Proof. By the definition of $A(n,q)$ in (4.1), we only need to show that $B(n,q)$ is multiplicative in q. Suppose $q = q_1q_2$ with $(q_1,q_2) = 1$. Then we have

$$\begin{aligned} B(n,q_1q_2) &= \sum_{\substack{a=1\\(a,q_1q_2)=1}}^{q_1q_2} C_2(q_1q_2,a)C_3(q_1q_2,a)C_6^2(q_1q_2,a)e\left(-\frac{an}{q_1q_2}\right)\\ &= \sum_{\substack{a_1=1\\(a_1,q_1)=1}}^{q_1}\sum_{\substack{a_2=1\\(a_2,q_2)=1}}^{q_2} C_2(q_1q_2,a_1q_2+a_2q_1)C_3(q_1q_2,a_1q_2+a_2q_1)\\ &\quad\times C_6^2(q_1q_2,a_1q_2+a_2q_1)e\left(-\frac{a_1n}{q_1}\right)e\left(-\frac{a_2n}{q_2}\right). \end{aligned} \tag{5.2}$$

For $(q_1,q_2) = 1$ and $k \in \{2,3,6\}$, there holds

$$\begin{aligned} C_k(q_1q_2,a_1q_2+a_2q_1) &= \sum_{\substack{m=1\\(m,q_1q_2)=1}}^{q_1q_2} e\left(\frac{(a_1q_2+a_2q_1)m^k}{q_1q_2}\right)\\ &= \sum_{\substack{m_1=1\\(m_1,q_1)=1}}^{q_1}\sum_{\substack{m_2=1\\(m_2,q_2)=1}}^{q_2} e\left(\frac{(a_1q_2+a_2q_1)(m_1q_2+m_2q_1)^k}{q_1q_2}\right)\\ &= \sum_{\substack{m_1=1\\(m_1,q_1)=1}}^{q_1} e\left(\frac{a_1(m_1q_2)^k}{q_1}\right)\sum_{\substack{m_2=1\\(m_2,q_2)=1}}^{q_2} e\left(\frac{a_2(m_2q_1)^k}{q_2}\right)\\ &= C_k(q_1,a_1)C_k(q_2,a_2). \end{aligned} \tag{5.3}$$

Putting (5.3) into (5.2), we deduce that

$$\begin{aligned} B(n,q_1q_2) &= \sum_{\substack{a_1=1\\(a_1,q_1)=1}}^{q_1} C_2(q_1,a_1)C_3(q_1,a_1)C_6^2(q_1,a_1)e\Big(-\frac{a_1n}{q_1}\Big) \\ &\quad\times \sum_{\substack{a_2=1\\(a_2,q_2)=1}}^{q_2} C_2(q_2,a_2)C_3(q_2,a_2)C_6^2(q_2,a_2)e\Big(-\frac{a_2n}{q_2}\Big) \\ &= B(n,q_1)B(n,q_2). \end{aligned}$$

This completes the proof of Lemma 5.5. □

Lemma 5.6. *Let $A(n,q)$ be as defined in* (4.1)*.*

(i) *We have*

$$\sum_{q>Z} |A(n,q)| \ll Z^{-\frac{1}{2}+\varepsilon} d(n),$$

and thus the singular series $\mathfrak{S}(n)$ is absolutely convergent and satisfies $\mathfrak{S}(n) \ll d(n)$; (ii) *There exists an absolute positive constant $c^* > 0$, such that, for $n \equiv 0 \pmod 2$ and $n \equiv \pm 1 \pmod 3$,*

$$\mathfrak{S}(n) \gg (\log\log n)^{-c^*}.$$

Proof. From Lemma 5.5, we know that $B(n,q)$ is multiplicative in q. Therefore, there holds

$$B(n,q) = \prod_{p^t\|q} B(n,p^t) = \prod_{p^t\|q} \sum_{\substack{a=1\\(a,p)=1}}^{p^t} C_2(p^t,a)C_3(p^t,a)C_6^2(p^t,a)e\Big(-\frac{an}{p^t}\Big). \tag{5.4}$$

From (5.4) and Lemma 5.1, we deduce that $B(n,q) = \prod_{p\|q} B(n,p)$ or 0 according to q being square-free or not. Thus, one has

$$\sum_{q=1}^{\infty} A(n,q) = \sum_{\substack{q=1\\ q \text{ square-free}}}^{\infty} A(n,q). \tag{5.5}$$

Write

$$\mathscr{V}(p,a) := C_2(p,a)C_3(p,a)C_6^2(p,a) - S_2(p,a)S_3(p,a)S_6^2(p,a).$$

Then

$$A(n,p) = \frac{1}{(p-1)^4}\sum_{a=1}^{p-1} S_2(p,a)S_3(p,a)S_6^2(p,a)e\Big(-\frac{an}{p}\Big) + \frac{1}{(p-1)^4}\sum_{a=1}^{p-1}\mathscr{V}(p,a)e\Big(-\frac{an}{p}\Big). \tag{5.6}$$

Applying Lemma 4.1 and noticing that $S_k(p,a) = C_k(p,a)+1$, we get $S_k(p,a) \ll p^{\frac{1}{2}}$, and thus $\mathscr{V}(p,a) \ll p^{\frac{3}{2}}$. Therefore, the second term in (5.6) is $\leqslant c_1 p^{-\frac{3}{2}}$. On the other hand, from Lemma 5.3, we can see that the first term in (5.6) is $\leqslant 2^4\cdot 50p^{-\frac{3}{2}} = 800p^{-\frac{3}{2}}$. Let $c_2 = c_1 + 800$. Then we have proved that, for $p \nmid n$, there holds

$$|A(n,p)| \leqslant c_2 p^{-\frac{3}{2}}. \tag{5.7}$$

Moreover, if we use Lemma 4.1 directly, it follows that

$$\begin{aligned}|B(n,p)| &= \Big|\sum_{a=1}^{p-1} C_2(p,a)C_3(p,a)C_6^2(p,a)e\Big(-\frac{an}{p}\Big)\Big| \\ &\leqslant \sum_{a=1}^{p-1}\big|C_2(p,a)C_3(p,a)C_6^2(p,a)\big| \\ &\leqslant (p-1)\cdot 2^4\cdot p^2\cdot 216 = 3456p^2(p-1),\end{aligned}$$

and therefore,

$$|A(n,p)| = \frac{|B(n,p)|}{\varphi^4(p)} \leqslant \frac{3456p^2}{(p-1)^3} \leqslant \frac{2^3\cdot 3456p^2}{p^3} = \frac{27648}{p}. \tag{5.8}$$

Let $c_3 = \max(c_2, 27648)$. Then, for square-free q, we have

$$\begin{aligned}\big|A(n,q)\big| &= \Big(\prod_{\substack{p|q\\ p\nmid n}}\big|A(n,p)\big|\Big)\Big(\prod_{\substack{p|q\\ p|n}}\big|A(n,p)\big|\Big) \leqslant \Big(\prod_{\substack{p|q\\ p\nmid n}}\big(c_3p^{-\frac{3}{2}}\big)\Big)\Big(\prod_{\substack{p|q\\ p|n}}\big(c_3p^{-1}\big)\Big) \\ &= c_3^{\omega(q)}\Big(\prod_{p|q}p^{-\frac{3}{2}}\Big)\Big(\prod_{p|(n,q)}p^{\frac{1}{2}}\Big) \ll q^{-\frac{3}{2}+\varepsilon}(n,q)^{\frac{1}{2}}.\end{aligned}$$

Hence, by (5.5), we obtain

$$\begin{aligned}\sum_{q>Z}|A(n,q)| &\ll \sum_{q>Z}q^{-\frac{3}{2}+\varepsilon}(n,q)^{\frac{1}{2}} = \sum_{d|n}\sum_{q>\frac{Z}{d}}(dq)^{-\frac{3}{2}+\varepsilon}d^{\frac{1}{2}} \\ &= \sum_{d|n}d^{-1+\varepsilon}\sum_{q>\frac{Z}{d}}q^{-\frac{3}{2}+\varepsilon} \ll \sum_{d|n}d^{-1+\varepsilon}\Big(\frac{Z}{d}\Big)^{-\frac{1}{2}+\varepsilon}\end{aligned}$$

$$= Z^{-\frac{1}{2}+\varepsilon} \sum_{d|n} d^{-\frac{1}{2}+\varepsilon} \ll Z^{-\frac{1}{2}+\varepsilon} d(n).$$

This proves (i) of Lemma 5.6.

To prove (ii) of Lemma 5.6, by Lemma 5.5, we first note that

$$\begin{aligned}\mathfrak{S}(n) &= \prod_p \Big(1 + \sum_{t=1}^{\infty} A\big(n, p^t\big)\Big) = \prod_p \big(1 + A(n, p)\big) \\ &= \Big(\prod_{p \leqslant c_3} \big(1 + A(n, p)\big)\Big)\Big(\prod_{\substack{p > c_3 \\ p \nmid n}} \big(1 + A(n, p)\big)\Big)\Big(\prod_{\substack{p > c_3 \\ p | n}} \big(1 + A(n, p)\big)\Big).\end{aligned} \tag{5.9}$$

From (5.7), we have

$$\prod_{\substack{p > c_3 \\ p \nmid n}} \big(1 + A(n, p)\big) \geqslant \prod_{p > c_3} \Big(1 - \frac{c_3}{p^{3/2}}\Big) \geqslant c_4 > 0. \tag{5.10}$$

By (5.8), we know that there exists $c_5 > 0$ such that

$$\prod_{\substack{p > c_3 \\ p | n}} \big(1 + A(n, p)\big) \geqslant \prod_{\substack{p > c_3 \\ p | n}} \Big(1 - \frac{c_3}{p}\Big) \geqslant \prod_{p | n} \Big(1 - \frac{c_3}{p}\Big) \gg (\log\log n)^{-c_5}. \tag{5.11}$$

On the other hand, it is easy to see that

$$1 + A(n, p) = \frac{p \cdot \mathscr{L}(p, n)}{\varphi^4(p)}. \tag{5.12}$$

By Lemma 5.4, we know that $\mathcal{L}(p, n) > 0$ for all p with $n \equiv 0 \pmod 2$ and $n \equiv \pm 1 \pmod 3$, and thus $1 + A(n, p) > 0$. Therefore, there holds

$$\prod_{p \leqslant c_3} \big(1 + A(n, p)\big) \geqslant c_6 > 0. \tag{5.13}$$

Combining the estimates (5.9)–(5.11) and (5.13), and taking $c^* = c_5 > 0$, we derive that

$$\mathfrak{S}(n) \gg (\log\log n)^{-c^*}.$$

This completes the proof of Lemma 5.6. □

6. Proof of Proposition 3.2

In this section, we first present some lemmas that will be used to prove Proposition 3.2.

Lemma 6.1. *Suppose that* α *is a real number, and that there exist integers* $a \in \mathbb{Z}$ *and* $q \in \mathbb{N}$ *satisfying*

$$(a, q) = 1, \qquad 1 \leqslant q \leqslant \Xi, \qquad |q\alpha - a| \leqslant \Xi^{-1},$$

where

$$\Xi = \begin{cases} N^{\frac{3}{4}}, & \text{if } k = 2, \\ N^{\frac{1}{2}}, & \text{if } k = 3. \end{cases}$$

Then, for $k \in \{2, 3\}$, *we have*

$$f_k(\alpha) \ll X_k^{1-\eta_k+\varepsilon} + \frac{X_k^{1+\varepsilon}}{\sqrt{q(1 + N|\alpha - a/q|)}},$$

where

$$\eta_2 = \frac{1}{8}, \qquad \eta_3 = \frac{1}{12}.$$

Proof. For the proof of the upper bound of $f_2(\alpha)$, one can see Theorem 3 of [10]; while for the proof of the upper bound of $f_3(\alpha)$, one can see Lemma 2.3 of [18]. □

Lemma 6.2. *Let* $2 \leqslant k_1 \leqslant k_2 \leqslant \cdots \leqslant k_s$ *be natural numbers such that*

$$\sum_{i=j+1}^{s} \frac{1}{k_i} \leqslant \frac{1}{k_j}, \quad 1 \leqslant j \leqslant s - 1.$$

Then we have

$$\int_0^1 \left| \prod_{i=1}^{s} f_{k_i}(\alpha) \right|^2 \mathrm{d}\alpha \ll N^{\frac{1}{k_1}+\cdots+\frac{1}{k_s}+\varepsilon}.$$

Proof. See Lemma 1 of [1]. □

Lemma 6.3. *Let* $f_k(\alpha)$ *be defined as above. Then we have*

$$\int_0^1 \left| f_2^2(\alpha) f_6^8(\alpha) \right| \mathrm{d}\alpha \ll N^{\frac{4}{3}+\varepsilon}.$$

Proof. The conclusion can be deduced by counting the number of solutions of the underlying Diophantine equation:

$$x_1^2 - x_2^2 = y_1^6 + y_2^6 + y_3^6 + y_4^6 - y_5^6 - y_6^6 - y_7^6 - y_8^6$$

with $X_2 < x_1, x_2 \leqslant 2X_2$ and $X_6 < y_i \leqslant 2X_6$ for $i = 1, 2, \ldots, 8$. If $x_1 \neq x_2$, the contribution is bounded by $X_6^{8+\varepsilon}$. If $x_1 = x_2$, the contribution is bounded by

$$\ll X_2 \cdot \int_0^1 |f_6(\alpha)|^8 \mathrm{d}\alpha.$$

By Lemma 2.5 of [12], we have

$$\int_0^1 |f_6(\alpha)|^8 \mathrm{d}\alpha \ll X_6^{5+\varepsilon},$$

and thus the contribution with $x_1 = x_2$ is $\ll X_2 \cdot X_6^{5+\varepsilon}$. Combining the above two cases, we deduce that

$$\int_0^1 \left| f_2^2(\alpha) f_6^8(\alpha) \right| \mathrm{d}\alpha \ll X_6^{8+\varepsilon} + X_2 \cdot X_6^{5+\varepsilon} \ll N^{\frac{4}{3}+\varepsilon}.$$

This completes the proof of Lemma 6.3. □

For the proof of Proposition 3.2, we define a general Hardy–Littlewood dissection employed in our application of the circle method. When X is a positive number with $X \leqslant \sqrt{N}$, we take $\mathfrak{N}(X)$ to be the union of the intervals

$$\mathfrak{N}(q, a, X) = \left\{ \alpha : |q\alpha - a| \leqslant XN^{-1} \right\},$$

with $1 \leqslant a \leqslant q \leqslant X$ and $(a, q) = 1$. Also, when $X \leqslant \sqrt{N}/2$, we put $\mathfrak{K}(X) = \mathfrak{N}(2X) \setminus \mathfrak{N}(X)$. Finally, we take

$$\mathfrak{m}_1 = \mathfrak{m} \cap \mathfrak{N}(N^{\frac{1}{8}}), \qquad \mathfrak{m}_2 = \mathfrak{m} \setminus \mathfrak{N}(N^{\frac{1}{8}}).$$

For $\alpha \in \mathfrak{m}_2$, by Dirichlet's lemma on rational approximation (for instance, see Lemma 2.1 of Vaughan [12]), there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying

$$1 \leqslant q \leqslant N^{\frac{1}{2}}, \qquad |q\alpha - a| \leqslant N^{-\frac{1}{2}}, \qquad (a, q) = 1.$$

Since $\alpha \in \mathfrak{m}_2$, we know that either $q > N^{\frac{1}{8}}$ or $N|q\alpha - a| > N^{\frac{1}{8}}$. Therefore, by Lemma 6.1, it is easy to see that

$$f_3(\alpha) \ll X_3^{1-\frac{1}{12}+\varepsilon} + \frac{X_3^{1+\varepsilon}}{\sqrt{N^{1/8}}} \ll N^{\frac{11}{36}+\varepsilon},$$

which combines Hölder's inequality, Lemma 6.2 and Lemma 6.3 yields

$$\begin{aligned}&\int_{\mathfrak{m}_2}\left|f_2^2(\alpha)f_3^2(\alpha)f_6^4(\alpha)\right|\mathrm{d}\alpha\\&\quad\ll\sup_{\alpha\in\mathfrak{m}_2}\left|f_3(\alpha)\right|^{\frac{2}{3}}\times\left(\int_0^1\left|f_2^2(\alpha)f_3^2(\alpha)f_6^2(\alpha)\right|\mathrm{d}\alpha\right)^{\frac{2}{3}}\left(\int_0^1\left|f_2^2(\alpha)f_6^8(\alpha)\right|\mathrm{d}\alpha\right)^{\frac{1}{3}}\\&\quad\ll\left(N^{\frac{11}{36}+\varepsilon}\right)^{\frac{2}{3}}\cdot(N^{1+\varepsilon})^{\frac{2}{3}}\cdot(N^{\frac{4}{3}+\varepsilon})^{\frac{1}{3}}\ll N^{\frac{1}{3}+1-\frac{1}{54}+\varepsilon}.\end{aligned}\tag{6.1}$$

We define the function $\Upsilon:[0,1]\to[0,1]$ by putting $\Upsilon(\alpha)=0$ for $\alpha\in[0,1]\setminus\mathfrak{N}(N^{\frac{1}{8}})$, and when $\alpha\in\mathfrak{N}(N^{\frac{1}{8}})\cap\mathfrak{N}(q,a,N^{\frac{1}{8}})$ we write

$$\Upsilon(\alpha)=(q+qN|\alpha-a/q|)^{-1}.$$

Define

$$\mathfrak{m}_3=\mathfrak{m}_1\cap\mathfrak{N}(N^{\frac{1}{18}}),\qquad\mathfrak{m}_4=\mathfrak{m}_1\setminus\mathfrak{N}(N^{\frac{1}{18}}).$$

By noting the fact that $\mathfrak{m}_4\subseteq\mathfrak{N}(N^{\frac{1}{8}})\setminus\mathfrak{N}(N^{\frac{1}{18}})$, hence for $\alpha\in\mathfrak{m}_4$, it follows from Lemma 6.1 that

$$|f_2(\alpha)|^2\ll N^{1+\varepsilon}\Upsilon(\alpha)\qquad\text{and}\qquad|f_3(\alpha)|^2\ll N^{\frac{11}{18}+\varepsilon},$$

which combined with the trivial estimate $f_6(\alpha)\ll N^{\frac{1}{6}+\varepsilon}$ yields

$$\begin{aligned}&\int_{\mathfrak{m}_4}\left|f_2^2(\alpha)f_3^2(\alpha)f_6^4(\alpha)\right|\mathrm{d}\alpha\\&\quad\ll\left(\sup_{\alpha\in\mathfrak{m}_4}|f_3(\alpha)|^2\right)\cdot N^{\frac{1}{3}+\varepsilon}\cdot N^{1+\varepsilon}\cdot\int_{\mathfrak{N}(N^{\frac{1}{8}})}\Upsilon(\alpha)|f_6(\alpha)|^2\mathrm{d}\alpha\\&\quad\ll N^{\frac{35}{18}+\varepsilon}\cdot\int_{\mathfrak{N}(N^{\frac{1}{8}})}\Upsilon(\alpha)|f_6(\alpha)|^2\mathrm{d}\alpha.\end{aligned}\tag{6.2}$$

By Lemma 2 of [2], we obtain

$$\int_{\mathfrak{N}(N^{\frac{1}{8}})}\Upsilon(\alpha)|f_6(\alpha)|^2\mathrm{d}\alpha\ll N^{-\frac{2}{3}+\varepsilon}.$$

Using the above estimate and (6.2), we conclude that

$$\int_{\mathfrak{m}_4}\left|f_2^2(\alpha)f_3^2(\alpha)f_6^4(\alpha)\right|\mathrm{d}\alpha\ll N^{\frac{1}{3}+1-\frac{1}{18}+\varepsilon}.\tag{6.3}$$

For $\alpha\in\mathfrak{m}_3$, by Lemma 6.1, we get

$$|f_2(\alpha)|^2\ll N^{1+\varepsilon}\Upsilon(\alpha)\qquad\text{and}\qquad|f_3(\alpha)|^2\ll N^{\frac{2}{3}+\varepsilon}\Upsilon(\alpha).$$

Hence, for $\alpha \in \mathfrak{m}_3$, there holds

$$\left|f_2^2(\alpha) f_3^2(\alpha)\right| \ll N^{\frac{5}{3}+\varepsilon} \Upsilon^2(\alpha),$$

which combined with the trivial estimate $f_6(\alpha) \ll N^{\frac{1}{6}+\varepsilon}$ yields

$$\int_{\mathfrak{m}_3} \left|f_2^2(\alpha) f_3^2(\alpha) f_6^4(\alpha)\right| \mathrm{d}\alpha \ll N^{2+\varepsilon} \cdot \int_{\mathfrak{m} \cap \mathfrak{N}(N^{\frac{1}{18}})} \Upsilon^2(\alpha) \left|f_6(\alpha)\right|^2 \mathrm{d}\alpha. \tag{6.4}$$

This leaves the set $\mathfrak{m} \cap \mathfrak{N}(N^{\frac{1}{18}})$ for treatment, and this set is covered by the union of sets $\mathfrak{R}(Y) = \mathfrak{N}(2Y) \setminus \mathfrak{N}(Y)$ as Y runs over the sequence $2^{-j} N^{\frac{1}{18}}$ with $P \ll Y \leqslant N^{\frac{1}{18}}/2$. Note that $\Upsilon(\alpha) \ll Y^{-1}$ for $\alpha \notin \mathfrak{N}(Y)$. Moreover, Lemma 2 of [2] supplies the following upper bound

$$\int_{\mathfrak{N}(2Y)} \Upsilon(\alpha) \left|f_6(\alpha)\right|^2 \mathrm{d}\alpha \ll Y N^{-\frac{5}{6}+\varepsilon} + N^{-\frac{2}{3}+\varepsilon},$$

which implies that

$$\begin{aligned} \int_{\mathfrak{R}(Y)} \Upsilon^2(\alpha) \left|f_6(\alpha)\right|^2 \mathrm{d}\alpha &\ll N^{-\frac{5}{6}+\varepsilon} + N^{-\frac{2}{3}+\varepsilon} Y^{-1} \\ &\ll N^{-\frac{5}{6}+\varepsilon} + N^{-\frac{2}{3}+\varepsilon} P^{-1} \ll N^{-\frac{17}{24}+\varepsilon}. \end{aligned} \tag{6.5}$$

By a splitting argument, from (6.4) and (6.5), we derive that

$$\begin{aligned} \int_{\mathfrak{m}_3} \left|f_2^2(\alpha) f_3^2(\alpha) f_6^4(\alpha)\right| \mathrm{d}\alpha &\ll N^{2+\varepsilon} \max_{P \ll Y \leqslant N^{\frac{1}{18}}/2} \int_{\mathfrak{R}(Y)} \Upsilon^2(\alpha) \left|f_6(\alpha)\right|^2 \mathrm{d}\alpha \\ &\ll N^{2+\varepsilon} \cdot N^{-\frac{17}{24}+\varepsilon} \ll N^{\frac{1}{3}+1-\frac{1}{24}+\varepsilon}. \end{aligned} \tag{6.6}$$

Combining (6.1), (6.3) and (6.6), we obtain the conclusion of Proposition 3.2.

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