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Müntz–Szàsz analogues for compact extensions of Heisenberg groups

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Abstract. Given a Müntz–Szàsz sequence of positive real numbers $(\lambda_k)_k$ and a bounded interval $I \subset \mathbb{R}$, Müntz–Szàsz theorem for completeness of the monomials $\{x^{\lambda_k}\}_k$ in $L^2(I)$ can be extended to a class of compact extensions of Heisenberg groups. The idea is to define infinitely many coordinate functions of generic unitary representations which are shown to be compactly supported and whose Fourier transforms extend analytically on the complex plane with suitable exponential domination. The representation theory and a Plancherel formula of reducible generic representations play an important role in the proofs.

Keywords. Muntz–Szasz theorem; Heisenberg groups; representation; compact extension; Fourier transform.

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1. Introduction

The classic Müntz–Szàsz theorem proved way back in 1914, states that for a strictly increasing sequence of positive real numbers $0 \leq \lambda_0 < \lambda_1 < \cdots$, the family $\{x^{\lambda_k}, k \in \mathbb{N}\}$ is total in $L^2([0, 1])$, if and only if $\sum_{k=0}^{+\infty} \frac{1}{\lambda_k} = +\infty$ (cf. [1] and [6]). Several other versions of this result have been widely developed by many authors (cf. [1], [4] and [16] and some references therein). This was inspired by the original result of Müntz and Szàsz in the case of $C([0, 1])$, the space of complex continuous function on $[0, 1]$ (cf. [14] and [16]). The purpose here is to generate and prove some analogues of the upshot for some classes of non-commutative Lie groups, for which the main point is to define what the equivalent of monomials should be.

In [5], Cook stated and proved an analogue (in an appropriate sense) of the result above for $L^2_c(\mathbb{R}^n)$, the set of compactly supported square integrable functions on $\mathbb{R}^n$. He also treated the case of a restrictive class of nilpotent Lie groups having a fixed abelian polarizer for the open set of linear forms in general position. He showed a one way analogue of Müntz–Szàsz theorem making use of Müntz–Szàsz sequences, which will be defined in the next section.

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In [2], we treated the context of Euclidean motion groups $G := K \ltimes \mathbb{R}^n$, where $K = SO(n)$ and $n \geqslant 2$. For this, we make use of the representation theory of G to define analogues of Müntz–Szàsz theorem for matrix coefficients. The present paper aims to tackle the context of compact extensions of Heisenberg groups. We will propose a Müntz–Szàsz analogue for square integrable compactly supported functions of the semi direct product $G_K = K \ltimes H_{2n+1}$, where K is a compact subgroup of automorphisms of the Heisenberg group H_{2n+1}. To do so, we are submitted to rephrase the condition of Müntz–Szàsz theorem above as an integral against a monomial of a family of coordinate functions depending upon the parameters involved in the spectrum of the Plancherel measure of G.

The outline of the paper is as follows. The next section is devoted to recall Müntz–Szàsz theorem on the Euclidean space and record some basic facts about the Heisenberg group H_{2n+1}, the class of their compact extensions, their unitary representations, and their related Plancherel formula. In Section 3, when K is connected we prove a first analogue of Müntz–Szàsz theorem for G_K using the coordinates function of a compactly supported function on G_K associated to the Garding vectors of G_K (cf. Theorem 3.6), whose Fourier transform are shown to admit an analytic continuation on the whole complex plane with an exponential domination (cf. Theorem 3.2). A second variant of Müntz–Szàsz theorem for arbitrary compact subgroup K is also proved in Section 3 (cf. Theorem 3.9). As a direct application, the setting of Heisenberg groups is treated (cf. Subsection 3.4). We do hope that our study could go farther to encompass other more general contexts.

2. Backgrounds

2.1 *Müntz–Szàsz theorem on* $\mathbb{R}^n$

We first introduce the following symbols:

- $\mathbb{N}$ is the set of non-negative integers.
- $\mathbb{N}^*$ is the set of positive integers.
- $\mathbb{R}^*$ is the set of positive real numbers.
- $L^2_c(X)$ is the vector space of square integrable compactly supported functions on X, where X is a locally compact measure space.

DEFINITION 2.1

(1) Let $0 \leq \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ be an increasing sequence of positive real numbers. Let $[\lambda_k]$ designate the integer part of λ_k. Let $\mathscr{E} = \{k \in \mathbb{N} : [\lambda_k] \text{ is even}\}$ and $\mathscr{O} = \{k \in \mathbb{N} : [\lambda_k] \text{ is odd}\}$. Then $(\lambda_k)_k$ is said to be a real Müntz–Szàsz's sequence abbreviated as MS, if $\mathscr{E}$ and $\mathscr{O}$ are both infinite sets and

$$\sum_{k \in \mathscr{O}} \frac{1}{\lambda_k} = \sum_{k \in \mathscr{E}} \frac{1}{\lambda_k} = +\infty.$$

(2) Let λ be a positive real number. Define the function $\underline{x}^\lambda$ as

$$\underline{x}^\lambda = \begin{cases} x^\lambda & (x \geq 0) \\ (-1)^{[\lambda]}(-x)^\lambda & (x < 0), \end{cases}$$

with the convention that $\underline{x}^0 = 1$. We have the following (cf. [3]).

Theorem 2.2. *Let $(\lambda_i(k))_k$, $(i \in \{1, \ldots, n\})$ be n strictly increasing sequences of positive real numbers. Let $b_1, \ldots, b_n$ be n fixed positive numbers. Then the family*

$$\{\underline{x}_1^{\lambda_1(k_1)} \cdots \underline{x}_n^{\lambda_n(k_n)}; (k_1, \ldots, k_n) \in \mathbb{N}^n\}$$

spans $L^2(\prod_{i=1}^n [-b_i, b_i])$ if and only if $(\lambda_i(k))_k$ is MS *for any $i \in \{1, \ldots, n\}$.*

This result can be easily generalized to encompass $L_c^2(\mathbb{R}^n)$ as follows.

COROLLARY 2.3

Let $(\lambda_i(k))_k$, $(i \in \{1, \ldots, n\})$ be n strictly increasing sequences of positive real numbers. Then the null-function is the unique function in $L_c^2(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \underline{x}_1^{\lambda_1(k_1)} \cdots \underline{x}_n^{\lambda_n(k_n)} f(x_1, \ldots, x_n) \mathrm{d}x_1 \cdots \mathrm{d}x_n = 0$ for every $(k_1, \ldots, k_n) \in \mathbb{N}^n$, if and only if each of the sequences $(\lambda_i(k))_k$ is MS, *for every $i \in \{1, \ldots, n\}$.*

Remark 2.4. Let $(\lambda_i(k))_k$, $(i \in \{1, \ldots, n\})$ be n strictly increasing sequence of positive integers. We can rephrase the statement of Corollary 2.3 as follows: Suppose that $f \in L_c^2(\mathbb{R}^n)$ and s is a fixed element of $\mathbb{C}^n$. Then $f = 0$ is the unique function such that

$$\frac{\partial^{\lambda_1(k_1)+\cdots+\lambda_n(k_n)}}{\partial s_1^{\lambda_1(k_1)} \cdots \partial s_n^{\lambda_n(k_n)}} \hat{f}(s_1, \ldots, s_n) = 0$$

for every $k_1, \ldots, k_n \in \mathbb{N}^n$, if and only if each $(\lambda_i(k))_k$ is MS, for any $i \in \{1, \ldots, n\}$. Here,

$$\hat{f}(s) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x) \mathrm{e}^{-ix\cdot s} \mathrm{d}x \tag{2.1}$$

designates the Fourier transform of $f \in L^1(\mathbb{R}^n)$, where $x \cdot s$ denotes the usual inner product on $\mathbb{R}^n$.

2.2 *On Heisenberg groups*

2.2.1 *Generalities.* For a positive integer n, let $\mathfrak{h} := \mathfrak{h}_{2n+1}$ be the $(2n+1)$-dimensional Heisenberg Lie algebra endowed with the basis $\mathscr{B} = \{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\}$, whose pairwise brackets equal to zero, except the following:

$$[X_i, Y_i] = Z, \quad i \in \{1, \ldots, n\}.$$

It is a two-step nilpotent Lie algebra. We denote by H_{2n+1} its Lie group and $\mathfrak{h}^*$ its vector dual space. Any element of H_{2n+1} is written through the basis $\mathscr{B}$ as

$$(x, y, z) := \exp(x_1 X_1) \cdots \exp(x_n X_n) \exp(y_1 Y_1) \cdots \exp(y_n Y_n) \exp(zZ),$$

with $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $z \in \mathbb{R}$, where $\exp : \mathfrak{h} \to H_{2n+1}$ designates the exponential mapping. The law group of H_{2n+1} reads as

$$(x, y, z) * (x', y', z') = (x + x', y + y', z + z' + x \cdot y').$$

2.2.2 *The unitary dual of* H_{2n+1}. Let $\widehat{H_{2n+1}}$ denote the unitary dual of H_{2n+1} and let $\pi \in \widehat{H_{2n+1}}$ be a unitary irreducible representation of infinite dimension. Using the Kirillov theory, π is equivalent to $\pi_\ell := \mathrm{Ind}_M^{H_{2n+1}} \chi_\ell$, where $\ell \in \mathfrak{h}^*$, $M = \exp(\mathfrak{m})$, $\mathfrak{m} := \mathbb{R}$-span$\{Z, Y_1, Y_2, \ldots, Y_n\}$ is a real polarization for ℓ, and χ_ℓ is a unitary character of M given by

$$\chi_\ell(\exp X) = \mathrm{e}^{-i\langle \ell, X\rangle}; \quad X \in \mathfrak{m}.$$

Actually, one can take $\ell = \lambda Z^*$, for some $\lambda \in \mathbb{R}^* = \mathbb{R}\diagdown\{0\}$. We denote from now on π_λ instead of π_ℓ. Any infinite dimensional irreducible unitary representation is equivalent to a representation π_λ for some $\lambda \in \mathbb{R}^*$.

Let $\mathscr{H}'_{\pi_\lambda}$ be the space of all continuous functions on H_{2n+1} with compact support modulo M and which satisfy the following covariance relation:

$$\xi(g * h) = \xi(g)\chi_\ell^{-1}(h), h \in M \quad \text{and} \quad g \in H_{2n+1}. \tag{2.2}$$

Then, the representation π_λ is realized as a left translation on the completion of $\mathscr{H}'_{\pi_\lambda}$ with respect to the norm $\int_{H_{2n+1}/M} |\xi(g)|^2 \mathrm{d}\dot{g} < +\infty$, where $\mathrm{d}\dot{g}$ is a H_{2n+1}-invariant measure on H_{2n+1}/M.

For $\xi \in L^2(\mathbb{R}^n, \lambda) \simeq L^2(H_{2n+1}/M, \lambda)$, we have by a direct computation,

$$\begin{aligned}\pi_\lambda(p, q, t)\xi(x, 0, 0) &= \xi((p, q, t)^{-1} * (x, 0, 0)) \\ &= \xi(x - p, -q, -t + p \cdot q),\end{aligned}$$

where $p, q, x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Using the covariance relation (2.2), we obtain for $\xi \in \mathscr{H}'_{\pi_\lambda}$,

$$\begin{aligned}\pi_\lambda(p, q, t)\xi(x, 0, 0) &= \xi(x - p, -q, -t + p \cdot q) \\ &= \xi(x - p, 0, 0)e^{-i\lambda(t - q\cdot x)}.\end{aligned}$$

The group Fourier transform of an integrable function f on H_{2n+1} is defined to be the operator-valued function:

$$\pi_\lambda(f) = \hat{f}(\pi_\lambda) = \int_{H_{2n+1}} f(p, q, t)\pi_\lambda(p, q, t)\mathrm{d}p\mathrm{d}q\mathrm{d}t.$$

The Plancherel measure is supported on the subset of $\widehat{H_{2n+1}}$ given by $\{\pi_\lambda; \lambda \in \mathbb{R}^*\}$, and the Plancherel formula for H_{2n+1} precisely reads as follows:

$$\|f\|_2^2 = \int_{\mathbb{R}^*} \|\pi_\lambda(f)\|_{\mathrm{HS}}^2 |\lambda|^{\frac{n}{2}} \mathrm{d}\lambda, \tag{2.3}$$

where $\|\pi_\lambda(f)\|_{\mathrm{HS}}^2 = \mathrm{trace}(\pi_\lambda(f * f^*))$ denotes the Hilbert–Schmidt norm of the operator $\pi_\lambda(f)$ (cf. [7], [11] and [17]).

2.3 *On the semi direct product* $K \ltimes H_{2n+1}$

2.3.1 *Generalities.* Let K be a compact subgroup of $\mathrm{Aut}(H_{2n+1})$, the group of automorphisms of H_{2n+1}. It acts on H_{2n+1} via $k \odot (p, q, t) = (k \cdot (p, q), t)$, where K acts naturally on $\mathbb{R}^{2n}$. A maximal compact connected group of automorphisms of H_{2n+1} is given by the unitary group $U(n)$. Conjugating by an automorphism of H_{2n+1} if necessary, we can always assume that $K \subset U(n)$, the unitary group of $\mathbb{C}^n$. It is well known that $(U(n), H_{2n+1})$ is a Gelfand pair (this means that the convolution algebra of K-invariant L^1-functions on H_{2n+1} is commutative) and there are many proper subgroups K of $U(n)$ for which (K, H_{2n+1}) form a Gelfand pair. Let us fix K and opt for the notation $G = K \ltimes H_{2n+1}$ instead of G_K for the semi-direct product of H_{2n+1} and K with the group law

$$(k, x, y, t)(h, u, v, s) = (kh, (x, y, t) * (k \cdot (u, v), s)),$$

where $(x, y, t), (u, v, s) \in H_{2n+1}$ and $k, h \in K$. For $K = U(n)$, $K \ltimes H_{2n+1}$ is known as the Heisenberg motion group (cf. [15]). Moreover, for each $k \in K \subseteq U(n)$, $\vartheta : (x, y, t) \mapsto (k \cdot (x, y), t)$ is an automorphism of H_{2n+1}, because $U(n)$ preserves the symplectic form $x \cdot u - y \cdot v$.

2.3.2 *Representation theory of* G. Taking a representation ρ of H_{2n+1} and using the automorphism ϑ considered below, we can define the representation ρ_k by $\rho_k(x, y, t) = \rho(k \cdot (x, y), t)$ which coincides with ρ at the center. If we take ρ to be the Schrödinger representation π_λ, then by Stone-von Neumann theorem, $(\pi_\lambda)_k$ is unitarily equivalent to π_λ and we get a unitary intertwining operator μ_λ such that

$$k \cdot \pi_\lambda(x, y, t) = \pi_\lambda(k^{-1} \cdot (x, y), t) = \mu_\lambda(k)^{-1} \pi_\lambda(x, y, t) \mu_\lambda(k)$$

for $k \in K$ and $(x, y, t) \in H_{2n+1}$. The operator-valued function μ_λ can be chosen so that it becomes a unitary representation of K on $L^2(\mathbb{R}^n)$ and is called a metaplectic representation. For more details, see [7].

For $\lambda \neq 0$ and $(\pi_\lambda, L^2(\mathbb{R}^n, \lambda))$ the irreducible unitary representation of H_{2n+1}, the Mackey induction machinery (cf. [13]) allows to define the induced representation

$$\Theta_\lambda = \mathrm{Ind}_{H_{2n+1}}^G \pi_\lambda \simeq \mathrm{Ind}_{H_{2n+1}}^G \mathrm{Ind}_M^{H_{2n+1}} \chi_\lambda \simeq \mathrm{Ind}_M^G \chi_\lambda$$

which acts on the space $\mathscr{H}_{\Theta_\lambda}$, the completion of the set of compactly supported continuous functions $\xi : K \to L^2(\mathbb{R}^n, \lambda)$ satisfying the covariance relation: $\xi(k \cdot (p, q), t) = \pi_\lambda(p, q, t)^{-1}[\xi(k)]$ with respect to the norm

$$\|\xi\|_2^2 = \int_K \|\xi(k)\|_{L^2(\mathbb{R}^n)}^2 \mathrm{d}k,$$

where $\mathrm{d}k$ denotes a normalized measure on K. Actually, Θ_λ is defined by

$$\Theta_\lambda(k, (p, q, t))\xi(s) := \pi_\lambda(s^{-1} \cdot (p, q), t)[\xi(k^{-1} \cdot s)],$$

for $\xi \in \mathscr{H}_{\Theta_\lambda}$, $k, s \in K$ and $(p, q, t) \in H_{2n+1}$. In this context, $\mathscr{H}_{\Theta_\lambda}$ is identified (and so far noted) to the space $L^2(K \times \mathbb{R}^n, \lambda)$ of the representation $\mathrm{Ind}_M^G \chi_\lambda$. As the space $L^2(K) \otimes L^2(\mathbb{R}^n)$ turns out to be dense in $L^2(K \times \mathbb{R}^n)$, for the sake of simplicity, we are very often submitted to pick up a basis $\{e_i \otimes \Psi_\gamma\}_{(i,\gamma)\in A\times\Gamma} \subset L^2(K)\otimes L^2(\mathbb{R}^n)$, where A and Γ are countable sets and $\{\Psi_\gamma\}_{\gamma\in\Gamma}$ are of compact supports (one can take, for instance, a basis of wavelets of compact support included in some dyadic cubes, cf. [12]). Note that Θ_λ is not irreducible.

2.3.3 *A variant of the Plancherel formula on* G. For many technical reasons, we found it hard to manipulate the operators $(\mu_\lambda)_{\lambda\neq 0}$. Thus, we will later generate our analogue of Müntz–Szàsz theorem making use of the following "non-irreducible" version of the Plancherel theorem on G. The group Fourier transform of an integrable function f on G is given by the operator-valued function

$$\Theta_\lambda(f) = \hat{f}(\Theta_\lambda) = \int_K \int_{H_{2n+1}} f(k, p, q, t)\Theta_\lambda(k, p, q, t)\mathrm{d}k\mathrm{d}p\mathrm{d}q\mathrm{d}t.$$

Thanks to [9] and [10], a variant of the Plancherel formula for the group G is given for $f \in L^1 \cap L^2(G)$ by

$$\|f\|_2^2 = \int_{\mathbb{R}^*} \|\Theta_\lambda(f)\|_{\mathrm{HS}}^2 |\lambda|^{\frac{n}{2}}\mathrm{d}\lambda \tag{2.4}$$

and this finds back formula (2.3), when $K = \{e\}$.

3. Müntz–Szàsz's analogues for $K \ltimes H_{2n+1}$

Our objective in this section is to generate an analogue of Müntz–Szàsz theorem in the context of the semi-direct product $G = K \ltimes H_{2n+1}$, where K is a compact subgroup of $\mathrm{Aut}(H_{2n+1})$.

3.1 *First preliminary results*

3.1.1 *The coordinates functions.* Let $f \in L_c^2(G)$. Let $\{e_i \otimes \Psi_\gamma\}_{(i,\gamma)\in A\times\Gamma}$ be a fixed orthonormal basis of $L^2(K)\otimes L^2(\mathbb{R}^n)$ as above. For two pairs $I = (i, \gamma)$, $I' = (i', \gamma') \in A \times \Gamma$ and $u \in \mathbb{R}$, we define the coordinate function:

$$\begin{aligned} f_{I',I}(u) = \int_{\mathbb{R}^n}\int_{K^2}\int_{\mathbb{R}^{2n}} & f(s\cdot k^{-1}, s\cdot(p,q), u + q\cdot x)e_{i'} \\ & (k)\Psi_{\gamma'}(x-p)\bar{e}_i(s)\bar{\Psi}_\gamma(x) \\ & \mathrm{d}p\mathrm{d}q\mathrm{d}s\mathrm{d}k\mathrm{d}x. \end{aligned} \tag{3.1}$$

We have the following.

Lemma 3.1. *For any $f \in L^1(G)$ and any $I, I' \in A\times\Gamma$, the function $f_{I',I}$ is well defined and it is of compact support whenever f is.*

Proof. We have

$$\begin{aligned}\int_{\mathbb{R}} |f_{I',I}(u)|\mathrm{d}u \leq & \int_{\mathbb{R}}\int_{\mathbb{R}^n}\int_{K^2}\int_{\mathbb{R}^{2n}} |f(s\cdot k^{-1}, s\cdot(p,q), u+q\cdot x)e_{i'} \\ & (k)\Psi_{\gamma'}(x-p)| \\ & |\bar{e}_i(s)\bar{\Psi}_\gamma(x)|\mathrm{d}p\mathrm{d}q\mathrm{d}s\mathrm{d}k\mathrm{d}x\mathrm{d}u \\ \leq & \int_{\mathbb{R}^{3n}}\int_{K^2}\int_{\mathbb{R}} |f(s\cdot k^{-1}, s\cdot(p,q), u)e_{i'} \\ & (k)\Psi_{\gamma'}(x-p)\bar{e}_i(s)\bar{\Psi}_\gamma(x)| \\ & \mathrm{d}p\mathrm{d}q\mathrm{d}s\mathrm{d}k\mathrm{d}x\mathrm{d}u.\end{aligned}$$

By applying successive Hölder inequalities, we get

$$\begin{aligned}\int_{\mathbb{R}} |f_{I',I}(u)|\mathrm{d}u \leq & \int_{\mathbb{R}^{2n}}\int_{K^2}\int_{\mathbb{R}} |f(s\cdot k^{-1}, s\cdot(p,q), u)e_{i'}(k)\bar{e}_i(s)| \\ & \left(\int_{\mathbb{R}^n} |\Psi_{\gamma'}(x-p)|^2 dx\right)^{\frac{1}{2}} \\ & \left(\int_{\mathbb{R}^n} |\bar{\Psi}_\gamma(x)|^2\mathrm{d}x\right)^{\frac{1}{2}}\mathrm{d}u\mathrm{d}s\mathrm{d}k\mathrm{d}p\mathrm{d}q \\ \leq & \|\Psi_\gamma\|_2\|\Psi_{\gamma'}\|_2\int_{\mathbb{R}^{2n}}\int_K\int_{\mathbb{R}} |f(k,p,q,u)| \\ & \left(\int_K |e_{i'}(k^{-1}s)|^2\mathrm{d}s\right)^{\frac{1}{2}} \\ & \left(\int_K |\bar{e}_i(s)|^2\mathrm{d}s\right)^{\frac{1}{2}}\mathrm{d}k\mathrm{d}u\mathrm{d}p\mathrm{d}q \\ \leq & \|\Psi_\gamma\|_2\|\Psi_{\gamma'}\|_2\|e_{i'}\|_2\|e_i\|_2\|f\|_1 < +\infty.\end{aligned}$$

Assume now that $\mathrm{supp}(f) \subset K \times \mathscr{B}(0,\alpha) \times [-\alpha,\alpha]$, where $\mathscr{B}(0,\alpha)$ designates the ball of $\mathbb{R}^{2n}$ of radius α centered at the origin for some positive real number α. As $q\cdot x$ lies in a compact set of $\mathbb{R}$ as x belongs to the support of $\Psi_{\gamma'}$ which is compact, it appears clear that $f_{I',I}$ is compactly supported on $\mathbb{R}$.

□

3.2 *From a basis of $L^2(\mathbb{R}^n)$ to a basis of $L^2(\mathbb{R}^n,\lambda)$*

There is a way to construct a function $\tilde{\xi} \in L^2(\mathbb{R}^n,\lambda)$ starting from $\xi \in L^2(\mathbb{R}^n)$. Let $g \in H_{2n+1}$, there exists a unique $x \in \mathbb{R}^n$ and a unique $b \in M$ such that $g = \tilde{x} * b$ where $\tilde{x} = (x,0,0)$. Define then $\tilde{\xi}$ by $\tilde{\xi}(g) = \xi(x)\chi_\lambda^{-1}(b)$. Thus $\tilde{\xi}$ obviously satisfies the covariance relation (2.2) and belongs to the space $L^2(\mathbb{R}^n,\lambda)$.

We are now ready to prove the following.

Theorem 3.2. *Let $f \in L^2_c(G)$ and $\{e_i \otimes \Psi_\gamma\}_{(i,\gamma)\in A\times\Gamma}$ be an orthonormal basis of $L^2(K) \otimes L^2(\mathbb{R}^n)$ as above. Let $\{\tilde{\Psi}_\gamma\}_{\gamma\in\Gamma}$ be the corresponding orthonormal basis of $L^2(\mathbb{R}^n,\lambda)$ defined as in Section 3.2. Then for $\lambda \neq 0$, the operator valued $\Theta_\lambda(f)$ expands* *as* $\Theta_\lambda(f)(e_i \otimes \tilde{\Psi}_\gamma) = \sum_{I'=(i',\gamma')\in A\times\Gamma} \widehat{f_{I',I}}(\lambda)(e_{i'} \otimes \tilde{\Psi}_{\gamma'})$, $I = (i,\gamma) \in A \times \Gamma$ *and* $\widehat{f_{I',I}}$ *admits an analytic continuation to the whole complex plane with the property that for some positive constant* α_γ, $\widehat{f_{I',I}}(z) = O(e^{\alpha_\gamma |z|})$ *for any* $z \in \mathbb{C}$. *Here* $\widehat{f_{I',I}}$ *denotes the Fourier transform of* $f_{I',I}$ *as in formula* (2.1).

Proof. We first prove the following.

Lemma 3.3. *Consider the orthonormal basis* $\{e_i \otimes \tilde{\Psi}_\gamma\}_{(i,\gamma)\in A\times\Gamma} \subset L^2(K) \otimes L^2(\mathbb{R}^n, \lambda)$. *Then* $\langle \Theta_\lambda(f)(e_{i'} \otimes \tilde{\Psi}_{\gamma'}), e_i \otimes \tilde{\Psi}_\gamma\rangle = \widehat{f_{I',I}}(\lambda)$ *for any* $f \in L^1(G) \cap L^2(G)$, $I', I \in A \times \Gamma$, *and any* $\lambda \neq 0$.

Proof. We have the following:

$$
\begin{aligned}
&\langle \Theta_\lambda(f)(e_{i'} \otimes \tilde{\Psi}_{\gamma'}), e_i \otimes \tilde{\Psi}_\gamma\rangle \\
&= \int_K \int_{\mathbb{R}^n} \Theta_\lambda(f) e_{i'}(s)\tilde{\Psi}_{\gamma'}(\tilde{x})\bar{e}_i(s)\bar{\tilde{\Psi}}_\gamma(\tilde{x})\mathrm{d}x\mathrm{d}s \\
&= \int_K \int_{\mathbb{R}^n} \Theta_\lambda(f) e_{i'}(s)\Psi_{\gamma'}(x)\bar{e}_i(s)\bar{\Psi}_\gamma(x)\mathrm{d}x\mathrm{d}s \\
&= \int_{\mathbb{R}^n} \int_{K^2} \int_{H_{2n+1}} f(k,p,q,t)\pi_\lambda(s^{-1}\cdot(p,q,t))e_{i'}(k^{-1}\cdot s)\Psi_{\gamma'} \\
&\quad (x)\bar{e}_i(s)\bar{\Psi}_\gamma(x)\mathrm{d}p\mathrm{d}q\mathrm{d}t\mathrm{d}k\mathrm{d}s\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \int_{K^2} \int_{H_{2n+1}} f(s\cdot k^{-1}, s\cdot(p,q),t)\pi_\lambda(p,q,t)\Psi_{\gamma'}(x)e_{i'} \\
&\quad (k)\bar{e}_i(s)\bar{\Psi}_\gamma(x)\mathrm{d}p\mathrm{d}q\mathrm{d}t\mathrm{d}k\mathrm{d}s\mathrm{d}x \\
&= \int_{\mathbb{R}^n} \int_{K^2} \int_{H_{2n+1}} f(s\cdot k^{-1}, s\cdot(p,q),t)e_{i'} \\
&\quad (k)\Psi_{\gamma'}(x-p)\mathrm{e}^{-i\lambda(t-q\cdot x)}\bar{e}_i(s)\bar{\Psi}_\gamma(x)\mathrm{d}p\mathrm{d}q\mathrm{d}t\mathrm{d}k\mathrm{d}s\mathrm{d}x \\
&= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}^n} \int_{K^2} \int_{\mathbb{R}^{2n}} f(s\cdot k^{-1}, s\cdot(p,q), u+q\cdot x)e_{i'} \\
&\quad (k)\Psi_{\gamma'}(x-p)\bar{e}_i(s)\bar{\Psi}_\gamma(x)\mathrm{d}p\mathrm{d}q\mathrm{d}s\mathrm{d}k\mathrm{d}x \Big) \\
&\qquad \mathrm{e}^{-i\lambda u}\mathrm{d}u \\
&= \widehat{f_{I',I}}(\lambda).
\end{aligned}
$$

□

Now, fix $f \in L^2_c(G)$ meeting the assumption of the theorem. We have the following.

Lemma 3.4. *The function* $f_{I',I}$ *belongs to* $L^2_c(\mathbb{R})$ *for any* $I', I \in A \times \Gamma$.

Proof. We have

$$
|f_{I',I}(u)| \leq \int_{\mathbb{R}^n} \int_{K^2} \int_{\mathbb{R}^{2n}} |f(s\cdot k^{-1}, s\cdot(p,q), u+q\cdot x)|
$$

$$|e_{i'}(k)\Psi_{\gamma'}(x-p)||\bar{e}_i(s)\bar{\Psi}_\gamma(x)|\mathrm{d}p\mathrm{d}q\mathrm{d}k\mathrm{d}s\mathrm{d}x$$

(we use Hölder's inequality)

$$\leq \int_{\mathbb{R}^n}\int_{K^2}\int_{\mathbb{R}^n}|\bar{\Psi}_\gamma(x)\bar{e}_i(s)e_{i'}(k)|\left(\int_{\mathbb{R}^n}|f(s\cdot k^{-1},s\cdot(p,q),u+q\cdot x)|^2\mathrm{d}p\right)^{\frac{1}{2}}$$
$$\left(\int_{\mathbb{R}^n}|\Psi_{\gamma'}(x-p)|^2dp\right)^{\frac{1}{2}}\mathrm{d}q\mathrm{d}k\mathrm{d}s\mathrm{d}x.$$

Since $f,\ e_{i'},\ \Psi_{\gamma'},\ e_i,\ \Psi_\gamma$ are compactly supported and due to successive Hölder inequalities, we obtain:

$$|f_{I',I}(u)| \leq M_1\int_{\mathbb{R}^n}\int_{K^2}|\bar{\Psi}_\gamma(x)\bar{e}_i(s)e_{i'}(k)|$$
$$\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|f(s\cdot k^{-1},s\cdot(p,q),u+q\cdot x)|^2\mathrm{d}p\mathrm{d}q\right)^{\frac{1}{2}}$$
$$\left(\int_{\mathbb{R}^n}|\Psi_{\gamma'}(x-p)|^2\mathrm{d}p\right)^{\frac{1}{2}}\mathrm{d}k\mathrm{d}s\mathrm{d}x$$
$$\leq M_2\Big(\int_{\mathbb{R}^n}\int_{K^2}\int_{\mathbb{R}^{2n}}|f(s\cdot k^{-1},s\cdot(p,q),u+q\cdot x)|^2|$$
$$\bar{e}_i(s)\bar{\Psi}_\gamma(x)|^2\mathrm{d}p\mathrm{d}q\mathrm{d}k\mathrm{d}s\mathrm{d}x\Big)^{\frac{1}{2}}\Big(\int_K\int_{\mathbb{R}^{2n}}|e_{i'}(k)\Psi_{\gamma'}(x-p)|^2\mathrm{d}p\mathrm{d}k\mathrm{d}x\Big)^{\frac{1}{2}}$$
$$\leq M_3\|e_{i'}\|_2\|\Psi_{\gamma'}\|_2\Big(\int_{\mathbb{R}^n}\int_{K^2}\int_{\mathbb{R}^{2n}}|f(s\cdot k^{-1},s\cdot(p,q),u+q\cdot x)|^2|$$
$$\bar{e}_i(s)\bar{\Psi}_\gamma(x)|^2\mathrm{d}p\mathrm{d}q\mathrm{d}k\mathrm{d}s\mathrm{d}x\Big)^{\frac{1}{2}}.$$

Finally, we get

$$\int_{\mathbb{R}}|f_{I',I}(u)|^2\mathrm{d}u \leq M_4\|e_{i'}\|_2^2\|\Psi_{\gamma'}\|_2^2\left(\int_{\mathbb{R}^n}\int_K|\bar{e}_i(s)\bar{\Psi}_\gamma(x)|^2\mathrm{d}s\mathrm{d}x\right)$$
$$\left(\int_{\mathbb{R}}\int_K\int_{\mathbb{R}^{2n}}|f(k,p,q,t)|^2\mathrm{d}p\mathrm{d}q\mathrm{d}k\mathrm{d}t\right)$$
$$\leq M_4\|e_{i'}\|_2^2\|\Psi_{\gamma'}\|_2^2\|e_i\|_2^2\|\Psi_\gamma\|_2^2\|f\|_2^2 < +\infty,$$

for some positive constants M_1, M_2, M_3 and M_4 and this finishes the proof. □

Using the theorem of holomorphy under the integral sign, one can show that the function $\widehat{f_{I',I}}$ extends to an entire function on the whole complex plane. On the other hand, the support of $f_{I',I}$ is sitting inside $[-\alpha_\gamma,\alpha_\gamma]$. Indeed, suppose that $\mathrm{supp}(f)\subset K\times\mathscr{B}(0,\alpha)\times[-\alpha,\alpha]$ and assume that $u+q\cdot x\in[-\alpha,\alpha]$ (otherwise $f_{I',I}$ is identically zero), then we get $|u|\leq\alpha+|q\cdot x|\leq\alpha+\alpha\|x\|_2$. Here, x belongs to the support of $\overline{\Psi_\gamma}$ which is a compact of $\mathbb{R}^n$, supposed to be included in a ball $\mathscr{B}(0,B_\gamma)$. We obtain conclusively that $|u|\leq\alpha+\alpha B_\gamma:=\alpha_\gamma$. Therefore, we get

$$|\widehat{f_{I',I}}(z)| \leq \int_{\mathbb{R}}\int_{\mathbb{R}^{3n}}\int_{K^2}|\mathrm{e}^{-iuz}||f(s\cdot k^{-1},s\cdot(p,q),u+q\cdot x)|$$

$$|e_{i'}(k)\Psi_{\gamma'}(x-p)||\bar{e}_i(s)\bar{\Psi}_\gamma(x)|\mathrm{d}p\mathrm{d}q\mathrm{d}x\mathrm{d}k\mathrm{d}s\mathrm{d}u$$
$$\leq \mathrm{e}^{\alpha_\gamma \Im m(z)}\int_{\mathbb{R}}|f_{I',I}(u)|\mathrm{d}u \leq \mathrm{e}^{\alpha_\gamma |z|}\|f_{I',I}\|_1,$$

where $\Im m(z)$ denotes the imaginary part of z. □

3.2.1 *Analogue of Müntz–Szàsz theorem on* G. First, we have the following:

Lemma 3.5. *Let K be a connected compact Lie group and $\mathfrak{k}$ its Lie algebra. Then there exists a compact set C in $\mathfrak{k}$ such that the exponential mapping from C to K is surjective.*

Proof. Since $\exp:\mathfrak{k}\to K$ is a local diffeomorphism, one can write $\mathfrak{k}=\bigcup_{x\in\mathfrak{k}}V_x$, where each V_x is an open set of $\mathfrak{k}$ and $\exp: V_x\to\exp(V_x)$ is a diffeomorphism. Since K is compact and connected, then $\exp:\mathfrak{k}\to K$ is surjective (cf. [8]), hence $K=\exp(\mathfrak{k})=\bigcup_{x\in\mathfrak{k}}\exp(V_x)$, where each $\exp(V_x)$ is an open set of K. We obtain therefore that $K=\bigcup_{i=1}^N\exp(V_{x_i})$ for some positive integer N. Finally the exponential mapping from $C=\overline{\bigcup_{i=1}^N V_{x_i}}$ to $\bigcup_{i=1}^N\exp(V_{x_i})=K$ is surjective. □

We assume from now on that K is a connected compact subgroup of $O(2n)$. Then its Lie algebra $\mathfrak{k}$ is a subspace of $\mathscr{A}_{2n}(\mathbb{R})$, the Lie algebra of $O(2n)$. With the above in mind, define for $f\in L^2_c(G)$, the following function on $C_n\times H_{2n+1}\subset\mathbb{R}^m\times H_{2n+1}$ by

$$\tilde{f}(T,(p,q,u))=f(\exp(T),(p,q,u)),$$

where m is the dimension of $\mathfrak{k}$ and C_n is the compact set as in Lemma 3.5. Remark here that $\tilde{f}$ belongs to $L^2_c(\mathbb{R}^m\times H_{2n+1})$ whenever $f\in L^2_c(G)$.

Let now $s\in K$, $\mathbf{k}=(k_1,\ldots,k_m,k_{m+1})\in\mathbb{N}^{m+1}$, $\mathbf{t}=(t_1,\ldots,t_m)$ and $\underline{\mathbf{t}}^{\lambda_\mathbf{k}}:=\underline{t_1}^{\lambda_{k_1}}\cdots\underline{t_m}^{\lambda_{k_m}}$, we define

$$I^\lambda_\mathbf{k}(f)(s,p,x)=\int_{\mathbb{R}^m}\int_{\mathbb{R}^{n+1}}\underline{\mathbf{t}}^{\lambda_\mathbf{k}}\underline{u}^{\lambda_{k_{m+1}}}\tilde{f}(\mathbf{t},s\cdot(p,q),u+q\cdot x)\mathrm{d}q\mathrm{d}u\mathrm{d}\mathbf{t}. \quad (3.2)$$

We now suggest the following first analogue for Müntz–Szàsz theorem for G.

Theorem 3.6. *Let $f\in L^2_c(G)$. For a strictly increasing sequence $(\lambda_k)_k$ of positive real numbers, $(\lambda_k)_{k\geq 0}$ is* MS, *if and only if, the trivial function is the unique function f satisfying $I^\lambda_{\boldsymbol{k}}(f)(s,p,x)=0$ for almost every $s\in K$, $p,x\in\mathbb{R}^n$ and for any $\mathbf{k}\in\mathbb{N}^{m+1}$.*

Proof. We first prove the following lemma.

Lemma 3.7. *Let $(\lambda_k)_{k\geq0}$ be* MS *and $f\in L^2_c(G)$. If $\int_{\mathbb{R}^{n+1}}\underline{u}^{\lambda_j}f(k,s\cdot(p,q),u+q\cdot x)\,dqdu=0$, for any $j\in\mathbb{N}$ and for almost every $(k,s,x,p)\in K^2\times\mathbb{R}^n\times\mathbb{R}^n$, then f vanishes on G.*

Proof. Let $I, I' \in A \times \Gamma$ and $f_{I',I}$ the coordinate function defined as in Section 3.1. Then we have

$$\begin{aligned}\int_{\mathbb{R}} \underline{u}^{\lambda_j} f_{I',I}(u)\mathrm{d}u &= \int_{\mathbb{R}}\int_{\mathbb{R}^n}\int_{K^2}\int_{\mathbb{R}^{2n}} \underline{u}^{\lambda_j} f(s\cdot k^{-1}, s\cdot(p,q), u+q\cdot x)e_{i'}(k)\\ &\quad \Psi_{\gamma'}(x-p)\bar{e}_i(s)\bar{\Psi}_\gamma(x)\mathrm{d}p\mathrm{d}q\mathrm{d}s\mathrm{d}k\mathrm{d}x\mathrm{d}u\\ &= \int_{\mathbb{R}^{2n}}\int_{K^2} F(p,x,s,k)\\ &\quad \Big(\int_{\mathbb{R}^{n+1}} \underline{u}^{\lambda_j} f(k, s\cdot(p,q), u+q\cdot x)\mathrm{d}q\mathrm{d}u\Big)\mathrm{d}s\mathrm{d}k\mathrm{d}p\mathrm{d}x\\ &\qquad\qquad (\text{where } F(p,x,s,k)\\ &= e_{i'}(k^{-1}\cdot s)\Psi_{\gamma'}(x-p)\bar{e}_i(s)\bar{\Psi}_\gamma(x))\\ &= 0 \quad (\text{by the assumption of the lemma}).\end{aligned}$$

As above, the function $f_{I',I}$ is compactly supported on $\mathbb{R}$. By applying Corollary 2.3, we obtain that $f_{I',I} = 0$ on $\mathbb{R}$ as $(\lambda_k)_k$ is MS. On the other hand, by Lemma 3.3, we have

$$\|\Theta_\lambda(f)\|^2_{\mathrm{HS}} = \sum_{I,I'\in A\times\Gamma} |\widehat{f_{I',I}}(\lambda)|^2 = 0.$$

Due to formula (2.4), we show that f vanishes on G. This achieves the proof. □

Back to the proof of Theorem 3.6, we first prove the "only if" part: If for almost every $s \in K$, $p, x \in \mathbb{R}^n$, $I^\lambda_{\mathbf{k}}(f)(s,p,x) = 0$ then by applying Corollary 2.3, we get that

$$\int_{\mathbb{R}^{n+1}} \underline{u}^{\lambda_{k_{m+1}}} \tilde{f}(\mathbf{t}, s\cdot(p,q), u+q\cdot x)\mathrm{d}q\mathrm{d}u = 0,$$

for almost every $\mathbf{t} \in \mathbb{R}^m$, which induces that

$$\int_{\mathbb{R}^{n+1}} \underline{u}^{\lambda_{k_{m+1}}} f(k, s\cdot(p,q), u+q\cdot x)\mathrm{d}q\mathrm{d}u = 0,$$

for almost every $k, s \in K$ and for any $k_{m+1} \in \mathbb{N}$. Lemma 3.7 applied to the function $(k,(p,q,u)) \mapsto f(k,(p,q,u))$ entails that $f = 0$ almost everywhere on G.

We look now for the "if" part: Suppose that $(\lambda_k)_k$ is not MS, then applying Corollary 2.3, there exists a non-zero function $\psi \in L^2_c(\mathbb{R}^m)$ satisfying

$$\int_{\mathbb{R}^m} \underline{t_1}^{\lambda_{k_1}} \cdots \underline{t_m}^{\lambda_{k_m}} \psi(\mathbf{t})\mathrm{d}\mathbf{t} = 0, \tag{3.3}$$

for all $(k_1,\ldots,k_m) \in \mathbb{N}^m$. Thus, considering $\tilde{f}(\mathbf{t},p,q,u) = \psi(\mathbf{t})1_B(p,q,u)$, where B is any compact set of H_{2n+1}, we get that $I^\lambda_{\mathbf{k}}(f)(s,p,x) = 0$ for almost every $s \in K$, $p, x \in \mathbb{R}^n$ and for any $\mathbf{k} \in \mathbb{N}^{m+1}$. Here, 1_B means the characteristic function of B. This ends the proof of the theorem.

□

It is somehow possible to reformulate the statement of Theorem 3.6 making use of an integral over the group G. Let $\mathbf{j}, j' \in \mathbb{N}^m \times \mathbb{N}$ and consider

$$J^\lambda_{\mathbf{j},j'}(f)(s,x) = \int_{\mathbb{R}^n} p^{\lambda_{j'}} I^\lambda_{\mathbf{j}}(f)(s,p,x)\mathrm{d}p$$

$$= \int_{C_n} \int_{H_{2n+1}} p^{\lambda_{j'}} \underline{\mathbf{t}}^{\lambda_{\mathbf{j}}} \underline{u}^{\lambda_{j_{m+1}}} \tilde{f}(\mathbf{t}, s \cdot (p,q), u + q \cdot x) \mathrm{d}p \mathrm{d}q \mathrm{d}u \mathrm{d}\mathbf{t}$$
$$= \int_{C_n} \int_{H_{2n+1}} p^{\lambda_{j'}} \underline{\mathbf{t}}^{\lambda_{\mathbf{j}}} \underline{u}^{\lambda_{j_{m+1}}} f(\exp(\mathbf{t}), s \cdot (p,q), u + q \cdot x)$$
$$\mathrm{d}p \mathrm{d}q \mathrm{d}u \mathrm{d}\mathbf{t}$$
$$= \int_{(G = \exp(C_n) \times H_{2n+1})} p^{\lambda_{j'}} \underline{\mathbf{t}}^{\lambda_{\mathbf{j}}} \underline{u}^{\lambda_{j_{m+1}}} f(\exp(\mathbf{t}), s \cdot (p,q), u + q \cdot x) \mathrm{d}p \mathrm{d}q \mathrm{d}u \mathrm{d}\mathbf{t}.$$

As a direct consequence of Theorem 3.6, the following corollary holds.

COROLLARY 3.8

Let $f \in L^2_c(G)$ and let $(\lambda_k)_k$ be a strictly increasing sequence of positive real numbers. Then $(\lambda_j)_{j \geq 0}$ is MS, *if and only if, f is trivial whenever $J^{\lambda}_{\mathbf{j}, j'}(f)(s,x) = 0$ for almost every $s \in K$, $x \in \mathbb{R}^n$ and for any $\mathbf{j}, j' \in \mathbb{N}^{m+1} \times \mathbb{N}$.*

Proof. Let us prove the "only if" part: If for almost every $s \in K$, $x \in \mathbb{R}^n$ and for any $\mathbf{j}, j' \in \mathbb{N}^m \times \mathbb{N}$, $J^{\lambda}_{\mathbf{j}, j'}(f)(s,x) = 0$ then by applying Corollary 2.3, we get that

$$\int_{H_{2n+1}} p^{\lambda_{j'}} \underline{u}^{\lambda_{j_{m+1}}} f(\exp(\mathbf{t}), s \cdot (p,q), u + q \cdot x) \mathrm{d}p \mathrm{d}q \mathrm{d}u = 0,$$

for almost every $\mathbf{t} \in \mathbb{R}^m$, which induces that

$$\int_{H_{2n+1}} p^{\lambda_{j'}} \underline{u}^{\lambda_{j_{m+1}}} f(k, s \cdot (p,q), u + q \cdot x) \mathrm{d}p \mathrm{d}q \mathrm{d}u = 0,$$

for almost every $k \in K$ and for any $j_{m+1} \in \mathbb{N}$. The function

$$F_k : p \mapsto F_k(p) = \int_{\mathbb{R}^{n+1}} \underline{u}^{\lambda_{j_{m+1}}} f(k, s \cdot (p,q), u + q \cdot x) \mathrm{d}q \mathrm{d}u$$

belongs to $L^2_c(\mathbb{R}^n)$. In fact, by applying Cauchy–Schwartz inequality we get

$$|F_k(p)| \leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} |\underline{u}^{\lambda_{j_{m+1}}}|^2 \mathrm{d}q \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |f(k, s \cdot (p,q), u + q \cdot x)|^2 \mathrm{d}q \right)^{\frac{1}{2}} \mathrm{d}u$$
$$\leq M_1 \int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} |f(k, s \cdot (p,q), u + q \cdot x)|^2 \mathrm{d}q \right)^{\frac{1}{2}} \mathrm{d}u$$
$$\leq M_2 \left(\int_{\mathbb{R}} \int_{\mathbb{R}^n} |f(k, s \cdot (p,q), u + q \cdot x)|^2 \mathrm{d}q \mathrm{d}u \right)^{\frac{1}{2}}.$$

Therefore, we obtain that

$$\int_{\mathbb{R}^n} |F_k(p)|^2 \mathrm{d}p \leq M_2^2 \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n+1}} |f(k, s \cdot (p,q), u + q \cdot x)|^2 \mathrm{d}q \mathrm{d}u \mathrm{d}p$$
$$\leq M_2^2 \int_{\mathbb{R}^{2n+1}} |f(k, (p,q), t|^2 \mathrm{d}p \mathrm{d}q \mathrm{d}t.$$

Hence, we are done using the fact that

$$\int_K \int_{\mathbb{R}^n} |F_k(p)|^2 \mathrm{d}p\mathrm{d}k \leq M_2^2 \|f\|_2^2.$$

Besides, thanks to Corollary 2.3, we get that

$$\int_{\mathbb{R}^{n+1}} \underline{u}^{\lambda_{j_{m+1}}} f(k, s \cdot (p, q), u + q \cdot x)\mathrm{d}q\mathrm{d}u = 0,$$

for any $j_{m+1} \in \mathbb{N}$ and for almost every $(k, s, x) \in K \times \mathbb{R}^{2n}$. Lemma 3.7 applied to the function $(k, (p, q, u)) \mapsto f(k, (p, q, u))$ entails that $f = 0$ almost everywhere on G. On the other hand, the "if" part is a direct consequence of Theorem 3.6. □

3.3 *A second variant of Müntz–Szàsz theorem for* $K \ltimes H_{2n+1}$

We keep our notation and consider a strictly increasing sequence $\{n_k\}_k$ of positive integers and assume that K is arbitrary. Let $\mathscr{V}$ be the vector space generated by the family of monomials $\{x^{n_k}; k \in \mathbb{N}\}$. Let $\mathscr{F}_{I',I}$ be the linear subspace spanned by all the $f_{I',I}$'s for fixed $I', I \in A \times \Gamma$. We shall produce a two-sides discrepant variant of Müntz–Szàsz theorem for G which appears to deeply depend on the fine properties of the group in question and its unitary representations.

Let $\mathscr{A}_{I',I}$ be the operator defined on $L^2_c(G)$ by $\mathscr{A}_{I',I}(f) = f_{I',I}$. So from the last section, $\mathscr{A}_{I',I}$ takes its value on $L^2([-\alpha_\gamma, \alpha_\gamma])$ for some positive real number α_γ. Let $\tau_{I',I} := \mathrm{Ran}(\mathscr{A}_{I',I}) \subset L^2([-\alpha_\gamma, \alpha_\gamma])$ denote the range of the linear map $\mathscr{A}_{I',I}$. The following result is a second variant of Müntz–Szàsz theorem for compact extensions of Heisenberg groups.

Theorem 3.9. *Let $G = K \ltimes H_{2n+1}$. Given a strictly increasing sequence of positive integers $\{n_k\}_k$ the following assertions are equivalent*:

(1) $\{n_k\}_k$ *is* MS.
(2) *For any $I', I \in A \times \Gamma$, $\mathscr{V}$ is dense in $\mathscr{V} + \tau_{I',I}$.*
(3) *For some $I \in A \times \Gamma$, $\mathscr{V}$ is dense in $\mathscr{V} + \tau_{I,I}$.*

Proof. The fact that $\{n_k\}_k$ is MS is equivalent to the fact that $\mathscr{V}$ is dense in $L^2([-\alpha_\gamma, \alpha_\gamma])$ and so in $\mathscr{V} + \tau_{I',I}$. Then only the proof of (3)⇒(1) is required. This amounts to show that $\tau_{I,I}$ is dense in $L^2([-\alpha_\gamma, \alpha_\gamma])$ for some $I \in A \times \Gamma$. Indeed, $\mathscr{V}$ is dense $\mathscr{V} + \tau_{I,I}$ which is dense in $L^2([-\alpha_\gamma, \alpha_\gamma])$ and this entails the fact that $\mathscr{V}$ is dense in $L^2([-\alpha_\gamma, \alpha_\gamma])$.

Let $g \in L^2([-\alpha_\gamma, \alpha_\gamma])$, then for any $f = f_1 \otimes f_2 \otimes f_3$ where $f_1 \in C^\infty(K)$, $f_2 \in C_c^\infty(\mathbb{R}^{2n})$ and $f_3 \in C_c^\infty(\mathbb{R})$, we have

$$\begin{aligned} 0 &= \int_{-\alpha_\gamma}^{\alpha_\gamma} g(u) f_{I,I}(u)\mathrm{d}u \\ &= \int_{-\alpha_\gamma}^{\alpha_\gamma} g(u) \int_{\mathbb{R}^n} \int_{K^2} \int_{\mathbb{R}^{2n}} f(s \cdot k^{-1}, s \cdot (p, q), u + q \cdot x) e_i(k) \\ &\quad \Psi_\gamma(x - p)\bar{e}_i(s)\bar{\Psi}_\gamma(x)\mathrm{d}p\mathrm{d}q\mathrm{d}s\mathrm{d}k\mathrm{d}x\mathrm{d}u \end{aligned}$$

$$= \int_{-\alpha_\gamma}^{\alpha_\gamma} g(u) \int_{\mathbb{R}^n} \int_{K^2} \int_{\mathbb{R}^{2n}} f_1(s \cdot k^{-1}) f_2(s \cdot (p,q)) f_3(u + q \cdot x) e_i(k)$$
$$\Psi_\gamma(x-p) \bar{e}_i(s) \bar{\Psi}_\gamma(x) \mathrm{d}p \mathrm{d}q \mathrm{d}s \mathrm{d}k \mathrm{d}x \mathrm{d}u.$$

As this holds for any $f_1 \in C^\infty(K)$, we obtain that for almost all $k \in K$,

$$0 = \int_{-\alpha_\gamma}^{\alpha_\gamma} g(u) \int_{\mathbb{R}^n} \int_{K} \int_{\mathbb{R}^{2n}} f_2(s \cdot (p,q)) f_3(u + q \cdot x) e_i$$
$$(k^{-1} \cdot s) \Psi_\gamma(x-p) \bar{e}_i(s)$$
$$\bar{\Psi}_\gamma(x) \mathrm{d}p \mathrm{d}q \mathrm{d}s \mathrm{d}x \mathrm{d}u$$
$$= \int_{-\alpha_\gamma}^{\alpha_\gamma} g(u) \int_{\mathbb{R}^n} \int_{K} \int_{\mathbb{R}^{2n}} f_2(p,q) f_3(u + \pi_2(s^{-1} \cdot (p,q)) \cdot x)$$
$$e_i(k^{-1} \cdot s) \Psi_\gamma(x - \pi_1(s^{-1} \cdot (p,q))) \bar{e}_i(s) \bar{\Psi}_\gamma(x) \mathrm{d}p \mathrm{d}q \mathrm{d}s \mathrm{d}x \mathrm{d}u,$$

where π_1, π_2 designate the first and the second projections of $\mathbb{R}^{2n}$. Since again this holds for any $f_2 \in C_c^\infty(\mathbb{R}^{2n})$, we get for almost all $(k, p, q) \in K \times \mathbb{R}^{2n}$,

$$0 = \int_{-\alpha_\gamma}^{\alpha_\gamma} g(u) \int_{\mathbb{R}^n} \int_{K} f_3(u + \pi_2(s^{-1} \cdot (p,q)) \cdot x) e_i(k^{-1} \cdot s)$$
$$\Psi_\gamma(x - \pi_1(s^{-1} \cdot (p,q)))) \bar{e}_i(s) \bar{\Psi}_\gamma(x) \mathrm{d}s \mathrm{d}x \mathrm{d}u.$$

As the function

$$\nu : k \mapsto \int_{-\alpha_\gamma}^{\alpha_\gamma} g(u) \int_K e_i(k^{-1} \cdot s) \overline{e_i}(s)$$
$$\int_{\mathbb{R}^n} f_3(u + \pi_2(s^{-1} \cdot (p,q)) \cdot \Psi_\gamma(x - \pi_1(s^{-1} \cdot (p,q))) \bar{\Psi}_\gamma(x) \mathrm{d}x \mathrm{d}s \mathrm{d}u,$$

is continuous using the theorem of continuity under the integral sign, one gets that $\nu(k) = 0$ for any $k \in K$. Hence

$$\mu : (p,q) \mapsto \int_{-\alpha_\gamma}^{\alpha_\gamma} g(u) \int_K |e_i(s)|^2$$
$$\int_{\mathbb{R}^n} f_3(u + \pi_2(s^{-1} \cdot (p,q)) \cdot x) \Psi_\gamma(x - \pi_1(s^{-1} \cdot (p,q))) \bar{\Psi}_\gamma(x) \mathrm{d}x \mathrm{d}s \mathrm{d}u$$

is null for almost all $(p,q) \in \mathbb{R}^{2n}$. Likewise, we get that μ also vanishes on $\mathbb{R}^{2n}$. We finally deduce that $0 = \mu(0,0) := \int_{-\alpha_\gamma}^{\alpha_\gamma} g(u) f_3(u) \mathrm{d}u \int_K |e_i(s)|^2 \mathrm{d}s \int_{\mathbb{R}^n} |\Psi_\gamma(x)|^2 \mathrm{d}x$ which induces that $\int_{-\alpha_\gamma}^{\alpha_\gamma} g(u) f_3(u) \mathrm{d}u = 0$. As this holds for any $f_3 \in C_c^\infty(\mathbb{R})$, we find that g is null on $[-\alpha_\gamma, \alpha_\gamma]$. This completes the proof of the theorem. □

3.4 *The setting of Heisenberg groups*

The Heisenberg group can be regarded as a trivial case of G_K where $K = \{e\}$. We consider $\{\tilde{\Psi}_\gamma\}_{\gamma \in \Gamma} \subset L_c^2(\mathbb{R}^n)$ an orthonormal basis of $L^2(\mathbb{R}^n, \lambda)$. Let $f \in L^2(H_{2n+1})$ such that $\mathrm{supp}(f) \subset \mathscr{B}(0, \alpha) \times [-\alpha, \alpha]$ for some positive number α. We define in a similar way as in formula (3.1), the function $f_{\gamma', \gamma}$ as follows:

$$f_{\gamma',\gamma}(u) = \int_{\mathbb{R}^{3n}} f(p, q, u + q \cdot x)\Psi_{\gamma'}(x - p)\bar{\Psi}_{\gamma}(x)\mathrm{d}p\mathrm{d}q\mathrm{d}x.$$

Then we have the following.

Theorem 3.10. *Let $f \in L^2_c(H_{2n+1})$ and let $\{\tilde{\Psi}_\gamma\}_{\gamma\in\Gamma} \subset L^2_c(\mathbb{R}^n)$ be an orthonormal basis of $L^2(\mathbb{R}^n, \lambda)$ defined as in Section 3.2. Consider $(\lambda_k)_k$ a strictly increasing sequence of positive real numbers. Let $\mathbf{k} = (k_1, \ldots, k_n, k_{n+1}) \in \mathbb{N}^{n+1}$, $\mathbf{p} = (p_1, \ldots, p_n)$ and $\underline{\mathbf{p}}^{\lambda_{\boldsymbol{k}}} = \underline{p_1}^{\lambda_{k_1}} \cdots \underline{p_n}^{\lambda_{k_n}}$ and define*

$$I^\lambda_{\mathbf{k}}(f)(x) = \int_{H_{2n+1}} \underline{\mathbf{p}}^{\lambda_{\boldsymbol{k}}} \underline{u}^{\lambda_{k_{n+1}}} f(p, q, u + q \cdot x)\mathrm{d}u\mathrm{d}p\mathrm{d}q.$$

Then

(1) *For $\lambda \neq 0$, the operator-valued Fourier transform expands as*

$$\pi_\lambda(f)\tilde{\Psi}_\gamma = \sum_{\gamma'\in\Gamma} \hat{f}_{\gamma',\gamma}(\lambda)\tilde{\Psi}_{\gamma'},$$

and $\hat{f}_{\gamma',\gamma}$ admits an analytic continuation to the whole complex plane with the property that for some positive real number α_γ, $\hat{f}_{\gamma',\gamma}(z) = O(e^{\alpha_\gamma |z|})$ for any $z \in \mathbb{C}$.

(2) $(\lambda_k)_{k\geq 0}$ *is* MS, *if and only if the trivial function is the unique function satisfying $I^\lambda_{\boldsymbol{k}}(f)(x) = 0$ for almost every $x \in \mathbb{R}^n$, and for any $\boldsymbol{k} \in \mathbb{N}^{n+1}$.*

3.5 *A concluding remark*

We find back the result by Cook in [5]. Let $G = \exp(\mathfrak{g})$ be a nilpotent Lie group which admits an ideal $\mathfrak{b} \subset \mathfrak{g}$, polarizing all representations for which the orbits of the related linear forms are of maximal dimension, which is the case of Heisenberg groups. The author proved a Müntz–Szàsz theorem for the matrix coefficients of the operator valued Fourier transform on G. As a part of the proof, he constructed a so called an almost strong Malcev basis, which is a Malcev basis of $\mathfrak{g}$ with respect to $\mathfrak{b}$. Then the author uses such a basis to only prove partial similar result as in Theorem 3.6.

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