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Berger's formulas and their applications in symplectic mean curvature flow

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Abstract. In this paper, we recall some well known Berger's formulas. As their applications, we prove that if the local holomorphic pinching constant is $\lambda < 2$, then there exists a positive constant $\delta > \frac{29(\lambda-1)}{\sqrt{(48-24\lambda)^2+(29\lambda-29)^2}}$ such that $\cos\alpha \geq \delta$ is preserved along the mean curvature flow, improving Li–Yang's main theorem in Li and Yang (*Geom. Dedicata* **170** (2014) 63–69). We also prove that when $\cos\alpha$ is close enough to 1, then the symplectic mean curvature flow exists globally and converges to a holomorphic curve.

Keywords. Symplectic mean curvature flow; holomorphic curve; positive holomorphic sectional curvature.

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1. Introduction

Let $(M, J, \omega, \bar{g})$ be a Kähler surface. For a compact oriented real surface Σ which is smoothly immersed in M, the Kähler angle α of Σ in M was defined by

$$\omega|_\Sigma = \cos\alpha \mathrm{d}\mu_\Sigma,$$

where $\mathrm{d}\mu_\Sigma$ is the area element of Σ in the induced metric from g. We say that Σ is a symplectic surface if $\cos\alpha > 0$; Σ is a holomorphic curve if $\cos\alpha = 1$.

It is important to find the conditions to assure that the symplectic property is preserved along the mean curvature flow. In the case that M is a Kähler–Einstein surface, the symplectic property is preserved. Han and Li [7] proved that the symplectic property is also preserved if the ambient Kähler surface evolves along the Kähler–Ricci flow. In [10], Li and Yang found another condition to assure that the symplectic property is preserved along the mean curvature flow. In this paper, we improve Li and Yang's result in [10].

Here, we only consider the ambient Kähler surface with positive holomorphic sectional curvature. We denote the minimum and maximum of holomorphic sectional curvatures at $p \in M$ by $k_1(p)$ and $k_2(p)$, respectively, and $\lambda(p) = \frac{k_2(p)}{k_1(p)}$. We define

$$k_1 := \min_{p\in M} k_1(p) \quad \text{and} \quad k_2 := \max_{p\in M} k_2(p).$$

We also define the local holomorphic pinching constant by

$$\lambda := \max_{p\in M} \lambda(p).$$

Then we have the first main theorem.

Theorem 1.1. *Suppose M is a Kähler surface with positive holomorphic sectional curvature. If $1 \le \lambda < 2$ and $\cos\alpha(\cdot, 0) \ge \delta > \frac{29(\lambda-1)}{\sqrt{(48-24\lambda)^2+(29\lambda-29)^2}}$, then along the flow*

$$\left(\frac{\partial}{\partial t} - \Delta\right)\cos\alpha \ge |\bar\nabla J_{\Sigma_t}|^2 \cos\alpha + C\sin^2\alpha, \tag{1}$$

where $\bar\nabla$ is the Levi–Civita connection with metric $\bar g$ on M and $|\bar\nabla J_{\Sigma_t}|^2$ is defined by (9) *in subsection* 2.1, *and C is a positive constant depending only on k_1, k_2 and δ. As a corollary*, $\min_{\Sigma_t}\cos\alpha$ *is increasing with respect to t. In particular*, *at each time t*, Σ_t *is symplectic.*

Remark 1.1. Li–Yang's main theorem in [10], i.e., the lower bound of δ is $\frac{53(\lambda-1)}{\sqrt{(53\lambda-53)^2+(48-24\lambda)^2}}$ for $\lambda \in [1, \frac{11}{7})$ and $\frac{8\lambda-5}{\sqrt{(8\lambda-5)^2+(12-6\lambda)^2}}$ for $\lambda \in [\frac{11}{7}, 2)$. It is easy to check that for each $\lambda \in [1, 2)$,

$$\frac{29(\lambda-1)}{\sqrt{(48-24\lambda)^2+(29\lambda-29)^2}} \le \min\left\{\frac{53(\lambda-1)}{\sqrt{(53\lambda-53)^2+(48-24\lambda)^2}}, \frac{8\lambda-5}{\sqrt{(8\lambda-5)^2+(12-6\lambda)^2}}\right\}.$$

Hence we improve Li–Yang's main result in [10].

Similar to Han–Li's main theorem in [6], we also prove the following theorem for a Kähler surface with positive holomorphic sectional curvature and $1 \le \lambda < 2$.

Theorem 1.2. *Suppose that M is a Kähler surface with positive holomorphic sectional curvature and $1 \le \lambda < 2$. Let α be the Kähler angle of the surface Σ_t which evolves by the mean curvature flow. Suppose that $\cos\alpha(\cdot, 0) > \frac{58(\lambda-1)}{\sqrt{(48-24)^2+(58\lambda-58)^2}}$. r_0 is defined in Remark* 4.2 *and ϵ_0 is the constant in Theorem* 4.1, *and define ϵ_1 as*

$$\epsilon_1 = \frac{\pi^2\epsilon_0^2 r_0^6 (1 - \mathrm{e}^{-\frac{3}{8}(2-\lambda)k_1})^2}{4\mathrm{Area}(\Sigma_0)}.$$

Then if $\int_{\Sigma_0} \frac{\sin^2\alpha(x,0)}{\cos\alpha(x,0)}\mathrm{d}\mu_0 \le \epsilon_1$, the mean curvature flow with initial surface Σ_0 exists globally and it converges to a holomorphic curve.

By Theorem 1.2, it is easy to get the following corollary.

COROLLARY 1.1

Under the same conditions and same notations as in Theorem 1.2, except $\int_{\Sigma_0} \frac{\sin^2\alpha(x,0)}{\cos\alpha(x,0)} \mathrm{d}\mu_0 \leq \epsilon_1$, *there exists a constant* ϵ_2 *depending only on* ϵ_1 *and* $\mathrm{Area}(\Sigma_0)$, *such that if*

$$1 - \cos\alpha(\cdot, 0) \leq \epsilon_2,$$

then the mean curvature flow with initial surface exists globally and it converges to a holomorphic curve.

Remark 1.2. By Corollary 2.3, if $\lambda < \frac{3}{2}$, then the sectional curvature of M is positive, which implies that the bisectional curvature is positive. By Frankel conjecture, which was proved by Siu and Yau [12] and by Mori [11] independently, the Kähler surface is biholomorphic to $\mathbb{CP}^2$. Recently, Yang and Zheng (see Proposition 2.6 in [16]) proved that the Kähler manifold M^n with $\lambda < 2$ must be biholomorphic to $\mathbb{CP}^n$.

2. Preliminaries

In this section, we recall some preliminaries about the curvature and the evolution equations of the mean curvature flow.

2.1 *Evolution equations for the mean curvature flow*

In this subsection, we recall some evolution equations for the mean curvature flow.

Suppose that Σ is a sub manifold in a Riemannian manifold M. We choose an orthonormal basis $\{e_i\}$ for $T\Sigma$ and $\{e_\alpha\}$ for $N\Sigma$. Given an immersed $F_0 : \Sigma \to M$, we consider a one parameter family of smooth maps $F_t = F(\cdot, t) : \Sigma \to M$ with corresponding images $\Sigma_t = F_t(\Sigma)$ immersed in M and F which satisfies the mean curvature flow equation:

$$\begin{cases} \frac{\partial}{\partial t} F(x,t) = H(x,t) \\ F(x,0) = F_0(x). \end{cases} \tag{2}$$

Recall the evolution equation for the second fundamental form h^α_{ij} and $|A|^2$ along the mean curvature flow (see [4,9,13,14]).

Lemma 2.1.

$$\begin{aligned} \frac{\partial}{\partial t} h^\alpha_{ij} &= \Delta h^\alpha_{ij} + (\bar\nabla_k Rm)_{\alpha ijk} + (\bar\nabla_j Rm)_{\alpha kik} - 2R_{lijk} h^\alpha_{lk} \\ &\quad + 2R_{\alpha\beta jk} h^\beta_{ik} + 2R_{\alpha\beta ik} h^\beta_{jk} - R_{lkik} h^\alpha_{lj} - R_{lkjk} h^\alpha_{il} \\ &\quad + R_{\alpha k\beta k} h^\beta_{ij} - H^\beta (h^\beta_{ik} h^\alpha_{jk} + h^\beta_{jk} h^\alpha_{ik}) + h^\alpha_{im} h^\beta_{mk} h^\beta_{kj} \\ &\quad - 2h^\beta_{im} h^\alpha_{mk} h^\beta_{kj} + h^\beta_{ik} h^\beta_{km} h^\alpha_{mj} + h^\alpha_{km} h^\beta_{mk} h^\beta_{ij} + h^\beta_{ij} \langle e_\beta, \bar\nabla_H e_\alpha \rangle, \end{aligned} \tag{3}$$

where R_{ABCD} *is the curvature tensor of* M *and* $\bar\nabla$ *is the covariant derivative of* M. *Therefore,*

$$
\begin{aligned}
\frac{\partial}{\partial t}|A|^2 = \Delta|A|^2 - 2|\nabla A|^2 \\
&+ [(\bar{\nabla}_k Rm)_{\alpha ijk} + (\bar{\nabla}_j Rm)_{\alpha kik}]h_{ij}^{\alpha} - 4R_{lijk}h_{lk}^{\alpha}h_{ij}^{\alpha} \\
&+ 8R_{\alpha\beta jk}h_{ik}^{\beta}h_{ij}^{\alpha} - 4R_{lkik}h_{lj}^{\alpha}h_{ij}^{\alpha} + 2R_{\alpha k\beta k}h_{ij}^{\beta}h_{ij}^{\alpha} + 2P_1 + 2P_2,
\end{aligned}
\tag{4}
$$

where

$$
\begin{aligned}
P_1 &= \Sigma_{\alpha,\beta,i,j}\left(\Sigma_k\left(h_{ik}^{\alpha}h_{jk}^{\beta} - h_{jk}^{\alpha}h_{ik}^{\beta}\right)\right)^2, \\
P_2 &= \Sigma_{\alpha,\beta}(\Sigma_{i,j}h_{ij}^{\alpha}h_{ij}^{\beta})^2.
\end{aligned}
$$

$$
\frac{\partial}{\partial t}|H|^2 = \Delta|H|^2 - 2|\nabla H|^2 + 2R_{\alpha k\beta k}H^{\alpha}H^{\beta} + 2P_3, \tag{5}
$$

where

$$
P_3 = \Sigma_{i,j}(\Sigma_{\alpha}H^{\alpha}h_{ij}^{\alpha})^2.
$$

Choose an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ on $(M, \bar{g})$ along Σ_t such that $\{e_1, e_2\}$ is the frame of the tangent bundle $T\Sigma_t$ and $\{e_3, e_4\}$ is the frame of the normal bundle $N\Sigma_t$. Then along the surface Σ_t, we can take the complex structure on M as the form (cf. [9])

$$
J = \begin{pmatrix} 0 & \cos\alpha & y & z \\ -\cos\alpha & 0 & -z & y \\ -y & z & 0 & -\cos\alpha \\ -z & -y & \cos\alpha & 0 \end{pmatrix} \tag{6}
$$

or

$$
J = \begin{pmatrix} 0 & \cos\alpha & y & z \\ -\cos\alpha & 0 & z & -y \\ -y & -z & 0 & \cos\alpha \\ -z & y & -\cos\alpha & 0 \end{pmatrix}. \tag{7}
$$

Since Kähler form is self-dual, then J must be of the form (7).

Remark 2.1. In fact, the above argument also shows that the Kähler form is self-dual. If J is of the form (6), then the Kähler form is anti-self-dual, i.e., $*\omega = -\omega$, and hence it is impossible to obtain Kähler form. Hence J must be of the form (7), then the Kähler form ω must be self-dual.

Recall the evolution equation of the Kähler angle $\cos\alpha$ (cf. [4,8]).

Lemma 2.2. *The evolution equation for* $\cos\alpha$ *along* Σ_t *is*

$$
\left(\frac{\partial}{\partial t} - \Delta\right)\cos\alpha = |\bar{\nabla}J_{\Sigma_t}|^2\cos\alpha + \sin^2\alpha\,\mathrm{Ric}(Je_1, e_2). \tag{8}
$$

Here

$$
|\bar{\nabla}J_{\Sigma_t}|^2 = |h_{1k}^4 + h_{2k}^3|^2 + |h_{2k}^4 - h_{1k}^3|^2. \tag{9}
$$

Then $|\bar{\nabla} J_{\Sigma_t}|^2$ is independent of the choice of the frame and depend only on the orientation of the frame. It is proved in [4,7] that

$$|\bar{\nabla} J_{\Sigma_t}|^2 \geq \frac{1}{2}|H|^2 \tag{10}$$

and

$$|\nabla \cos\alpha|^2 \leq \sin^2\alpha |\bar{\nabla} J_{\Sigma_t}|^2. \tag{11}$$

2.2 *Berger's formulas*

In this subsection, we recall some well known identities, which are called Berger's formulas. We first recall the definitions of the Riemannian curvature and the holomorphic sectional curvature; secondly, we recall some Berger's formulas, which are the relations between Riemannian curvatures and the holomorphic sectional curvature.

The Riemann curvature tensor R of (M, g) is defined by

$$R(X, Y, Z, W) = -g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W)$$

for any vector fields X, Y, Z, W.

Set $R(X, Y) = R(X, Y, X, Y)$ and $R(X) = R(X, JX)$. Fix a point $p \in M$ and a two-dimensional plane $\Pi \subset T_p M$. The sectional curvature of Π is defined by

$$K(\Pi) = \frac{R(X, Y)}{g(X, X)g(Y, Y) - g(X, Y)^2},$$

where $\{X, Y\}$ is a basis of Π. We also denote it by $K(X, Y)$. For a Kähler manifold (M, g, J), if the two-dimensional plane Π is spanned by $\{X, JX\}$, i.e., Π is a holomorphic plane, then the sectional curvature of Π is called a holomorphic sectional curvature of Π. We denote it by $H(X)$, where $\{X, JX\}$ is a basis of Π. Then

$$H(X) = \frac{R(X)}{g(X, X)^2}.$$

For any orthogonal four-frames $\{e_1, e_2, e_3, e_4\} \subset T_p M$, we have for any index $A, B, C, D \in \{1, 2, 3, 4\}$ the below lemma.

Lemma 2.3 [1].

$$\begin{aligned}
&24 R_{ABCD} \\
&\quad = R(e_A + e_C, e_B + e_D) - R(eS_A + e_C, e_B - e_D) \\
&\qquad - R(e_A - e_C, e_B + e_D) \\
&\qquad + R(e_A - e_C, e_B - e_D) - R(e_A + e_D, e_B + e_C) \\
&\qquad + R(e_A + e_D, e_B - e_C) \\
&\qquad + R(e_A - e_D, e_B + e_C) - R(e_A - e_D, e_B - e_C)
\end{aligned} \tag{12}$$

For the proof, see the proof of Proposition 1.9 in [3]. Then we get the following property.

COROLLARY 2.1 [1]

Let (M, g) be a Riemannian manifold, and let p be an arbitrary point in M. Suppose that $\underline{\kappa} \leq K(\pi) \leq \bar{\kappa}$ for all two-dimensional planes $\pi \subset T_pM$. Then

$$R(e_1, e_2, e_3, e_4) \leq \frac{2}{3}(\bar{\kappa} - \underline{\kappa}) \tag{13}$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset T_pM$.

If we take $e_A = e_C = X, e_B = Y, e_D = Z$ in the equality (12), then we obtain the following lemma.

Lemma 2.4. *For any vector fields X, Y and Z on M,*

$$R(X, Y, X, Z) = \frac{1}{4}(R(X, Y + Z) - R(X, Y - Z)). \tag{14}$$

It is well known that we can express the sectional curvatures by the holomorphic sectional curvatures (see Proposition 2.1 in [2]).

Lemma 2.5. *Let (M, ω, J) be a Kähler manifold. Then*

$$\begin{aligned} R(X, Y) = \frac{1}{32}[&3R(X + JY) + 3R(X - JY) - R(X + Y) - R(X - Y) \\ &- 4R(X) - 4R(Y)]. \end{aligned} \tag{15}$$

Then we have the following corollary, also see Corollary 2.1 in [2].

COROLLARY 2.2

For any two orthonormal vectors X and Y, if $\langle X, JY\rangle = x$, then

$$\begin{aligned} K(X, Y) = \frac{1}{8}[&3(1 + x)^2 H(X + JY) \\ &+ 3(1 - x)^2 H(X - JY) - H(X + Y) - H(X - Y) \\ &- H(X) - H(Y)]. \end{aligned}$$

Thus we have the following Corollary.

COROLLARY 2.3

For any two orthonormal vectors X and Y, if $\langle X, JY\rangle = x$, then

$$\frac{1}{4}[(3(1 + x^2)k_1 - 2k_2] \leq K(X, Y) \leq \frac{1}{4}[(3(1 + x^2)k_2 - 2k_1] \tag{16}$$

Remark 2.2. In fact, Bishop and Goldberg (see Proposition 3.1 in [2]) also obtained the following interesting formula.

Lemma 2.6. *Let X, Y be the orthonormal vectors, if $\langle X, JY\rangle = \cos\alpha$. Denote*

$$H(X,Y) = \frac{1}{\pi}\int_0^{\pi} H(X\cos\alpha + Y\sin\alpha)\mathrm{d}\alpha,$$
$$A(X,Y) = \frac{1}{\pi}\int_0^{\pi} K(X\cos\alpha + Y\sin\alpha, -JX\sin\alpha + JY\cos\alpha)\mathrm{d}\alpha.$$

Then

$$K(X,Y) = H(X,Y) - 3A(X,Y)\sin^2\alpha.$$

If $\cos\alpha = 0$, by the Corollary 2.3, we have

$$\frac{1}{4}(3k_1 - 2k_2) \leq K(X,Y) \leq \frac{1}{4}(3k_2 - 2k_1).$$

On the other hand, for any orthonormal vectors X, Y with $\langle X, JY\rangle \neq 0$ and $|\langle X, JY\rangle| \neq 1$, let $\tilde{Y} = \langle X, JY\rangle X - JY$. Then $\langle X, \tilde{Y}\rangle = \langle X, J\tilde{Y}\rangle = 0$ and $\mathrm{Span}\{X, JY\} = \mathrm{Span}\{X, \tilde{Y}\}$. Hence we obtain

$$\frac{3k_1 - 2k_2}{4} \leq K(X\cos\theta + Y\sin\theta, -JX\sin\theta + JY\cos\theta) \leq \frac{3k_2 - 2k_1}{4}$$

for every $\theta \in [0, \pi]$. Hence for any orthonormal vectors X, Y, we have

$$\frac{3k_1 - 2k_2}{4} \leq A(X,Y) \leq \frac{3k_2 - 2k_1}{4}.$$

On the other hand, $K(X,Y)$ also can be expressed as follows:

$$K(X,Y) = \frac{1}{4}[(1+\cos\alpha)^2 H(X+JY) + (1-\cos\alpha)^2 H(X-JY)]$$
$$-A(X,Y)\sin^2\alpha.$$

Then Bishop and Goldberg (Proposition 4.2 in [2]) established the following estimate.

PROPOSITION 2.1 [2]

Let X, Y be the orthonormal vectors with $\langle X, JY\rangle = \cos\alpha$. Then

$$k_1 - \frac{3k_2}{4}\sin^2\alpha \leq K(X,Y) \leq k_2 - \frac{3k_1}{4}\sin^2\alpha.$$

It is easy to check that

$$\frac{1}{4}[(3+3\cos^2\alpha)k_2 - 2k_1] \geq (or \leq) k_2 - \frac{3k_1}{4}\sin^2\alpha, \quad if\ \cos^2\alpha \geq (or \leq) 1/3.$$

Lemma 2.7. *For the orthonormal basis $\{e_1, e_2, e_3, e_4\}$ on (M, g) along Σ_t, it takes the form J as (7). Hence $\cos\alpha, y, z$ are defined by (7). Then we have the following estimates*:

(1) $\frac{1}{4}(3+3\cos^2\alpha)k_1 - \frac{1}{2}k_2 \leq R_{1212} \leq \frac{1}{4}(3+3\cos^2\alpha)k_2 - \frac{1}{2}k_1$;
(2) $\frac{1}{4}(3+3y^2)k_1 - \frac{1}{2}k_2 \leq R_{2424} \leq \frac{1}{4}(3+3y^2)k_2 - \frac{1}{2}k_1$;

(3) $\frac{1}{4}(3+3z^2)k_1 - \frac{1}{2}k_2 \le R_{2323} \le \frac{1}{4}(3+3z^2)k_2 - \frac{1}{2}k_1$;
(4) $\frac{1}{32}[(23+6(\cos\alpha+y)^2)k_1 - (23+6(\cos\alpha-y)^2)k_2] \le R_{2131} \le \frac{1}{32}[(23+6(\cos\alpha+y)^2)k_2 - (23+6(\cos\alpha-y)^2)k_1]$;
(5) $\frac{1}{32}[(23+6(\cos\alpha-y)^2)k_1 - (23+6(\cos\alpha+y)^2)k_2] \le R_{2434} \le \frac{1}{32}[(23+6(\cos\alpha-y)^2)k_2 - (23+6(\cos\alpha+y)^2)k_1]$.

Proof. By (7), we have

- $Je_1 = \cos\alpha e_2 + ye_3 + ze_4$,
- $Je_2 = -\cos\alpha e_1 + ze_3 - ye_4$,
- $Je_3 = -ye_1 - ze_2 + \cos\alpha e_4$,
- $Je_4 = -ze_1 + ye_2 - \cos\alpha e_3$.

Hence $\langle Je_1, e_2\rangle = \cos\alpha$, $\langle Je_4, e_2\rangle = y$, $\langle Je_2, e_3\rangle = z$. Then by Corollary 2.3, we get (1)–(3).

By Lemma 2.4,

$$R_{1213} = \frac{1}{4}(R(e_1, e_2+e_3) - R(e_1, e_2-e_3)). \tag{17}$$

Hence $Je_1 = \cos\alpha e_2 + ye_3 + ze_4$, $\langle Je_1, e_2+e_3\rangle = \cos\alpha + y$ and $\langle Je_1, e_2-e_3\rangle = \cos\alpha - y$. Then by Corollary 2.3,

$$\begin{aligned} &\frac{1}{16}[(27+12(\cos\alpha+y)^2)k_1 - 19k_2] \\ &\quad \le R(e_1, e_2+e_3) \le \frac{1}{16}[(27+12(\cos\alpha+y)^2)k_2 - 19k_1] \end{aligned} \tag{18}$$

and

$$\begin{aligned} &\frac{1}{16}[(27+12(\cos\alpha-y)^2)k_1 - 19k_2] \le R(e_1, e_2-e_3) \\ &\quad \le \frac{1}{16}[(27+12(\cos\alpha-y)^2)k_2 - 19k_1]. \end{aligned} \tag{19}$$

Hence

$$\begin{aligned} R_{1213} &\le \frac{1}{64}[(46+12(\cos\alpha+y)^2)k_2 - (46+12(\cos\alpha-y)^2)k_1] \\ &= \frac{1}{32}[(23+6(\cos\alpha+y)^2)k_2 - (23+6(\cos\alpha-y)^2)k_1] \end{aligned} \tag{20}$$

and

$$R_{1213} \ge \frac{1}{32}[(23+6(\cos\alpha+y)^2)k_1 - (23+6(\cos\alpha-y)^2)k_2]. \tag{21}$$

Hence we obtain (4).

Using Lemma 2.4, Corollary 2.3 and the same argument as in the proof of (4) , we can obtain (5). $\square$

3. Lower bound along the mean curvature flow

In this section, we follow the argument in [10] to prove the first main theorem of this paper, which improves the main theorem in [10].

Theorem 3.1. *Suppose M is a Kähler surface with positive holomorphic sectional curvatures. If $1 \le \lambda < 2$ and $\cos\alpha(\cdot, 0) \ge \delta > \frac{29(\lambda-1)}{\sqrt{(48-24\lambda)^2+(29\lambda-29)^2}}$, then along the flow*

$$\left(\frac{\partial}{\partial t} - \Delta\right)\cos\alpha \ge |\bar{\nabla} J_{\Sigma_t}|^2 \cos\alpha + C\sin^2\alpha, \tag{22}$$

where C is a positive constant depending only on k_1, k_2 and δ. As a corollary, $\min_{\Sigma_t}\cos\alpha$ is increasing with respect to t. In particular, at each time t, Σ_t is symplectic.

Proof. For simplicity, we can take $y = \sin\alpha$, $z = 0$ in the form of J. Due to the evolution of $\cos\alpha$ (see Lemma 2.2),

$$\left(\frac{\partial}{\partial t} - \Delta\right)\cos\alpha = |\bar{\nabla} J_{\Sigma_t}|^2 \cos\alpha + \mathrm{Ric}(Je_1, e_2)\sin^2\alpha. \tag{23}$$

In order to prove this theorem, we need to estimate $\mathrm{Ric}(Je_1, e_2)$. Then

$$\begin{aligned}\mathrm{Ric}(Je_1, e_2) &= \sum_{i=1}^{4} R(Je_1, e_i, e_2, e_i)\\ &= \sum_{i=1}^{4} R(\cos\alpha e_2 + \sin\alpha e_3, e_i, e_2, e_i)\\ &= \cos\alpha R_{22} + \sin\alpha(R_{1213} + R_{4243})\\ &= \cos\alpha R_{22} + \sin\alpha R_{23}.\end{aligned} \tag{24}$$

By Lemma 2.7, we have

$$R_{22} = R_{1212} + R_{3232} + R_{4242} \ge 3k_1 - \frac{3}{2}k_2 \tag{25}$$

and

$$|R_{23}| = |R_{1213} + R_{4243}| \le \frac{29}{16}(k_2 - k_1). \tag{26}$$

Hence we have

$$\begin{aligned}&\mathrm{Ric}(Je_1, e_2)\\ &\quad\ge \cos\alpha\left(3k_1 - \frac{3}{2}k_2\right) - \sqrt{1-\cos^2\alpha}\,\frac{29}{16}(k_2 - k_1)\\ &\quad= \left(3\cos\alpha + \frac{29}{16}\sqrt{1-\cos^2\alpha}\right)k_1 - \left(\frac{3}{2}\cos\alpha + \frac{29}{16}\sqrt{1-\cos^2\alpha}\right)k_2\\ &\quad= k_1\left\{\frac{3}{2}\cos\alpha(2-\lambda) + \frac{29}{16}\sqrt{1-\cos^2\alpha}(1-\lambda)\right\}.\end{aligned} \tag{27}$$

We set

$$f_\lambda(x) = \frac{3}{2}(2-\lambda)x + \frac{29}{16}(1-\lambda)\sqrt{1-x^2}. \tag{28}$$

When $1 \leq \lambda < 2$, $f_\lambda(x) > 0$ is equivalent to

$$\frac{3}{2}(2-\lambda)x > \frac{29}{16}(\lambda-1)\sqrt{1-x^2}.$$

Furthermore, if $x > 0$, $f_\lambda(x) > 0$ is equivalent to

$$\left(\frac{3}{2}(2-\lambda)x\right)^2 > \left(\frac{29}{16}(\lambda-1)(1-x^2)\right),$$

it is equivalent to

$$\left\{\left(\frac{3}{2}(2-\lambda)\right)^2 + \left(\frac{29}{16}(\lambda-1)\right)^2\right\} x^2 > \left(\frac{29}{16}(\lambda-1)\right)^2,$$

which is equivalent to

$$x^2 > \frac{29^2(\lambda-1)^2}{24^2(2-\lambda)^2 + 29^2(\lambda-1)^2}.$$

Hence if $1 \leq \lambda < 2$ and $\cos\alpha > \frac{29(\lambda-1)}{\sqrt{(48-24\lambda)^2+(29\lambda-29)^2}}$, we have $f_\lambda(\cos\alpha) > 0$, that is, $\mathrm{Ric}(Je_1, e_2) > 0$. Furthermore, if $\cos\alpha \geq \delta > \frac{29(\lambda-1)}{\sqrt{(48-24\lambda)^2+(29\lambda-29)^2}}$, then $f_\lambda(\cos\alpha) \geq f_\lambda(\delta) > 0$. Then by the maximum principle, the condition $\cos\alpha \geq \delta > \frac{29(\lambda-1)}{\sqrt{(48-24\lambda)^2+(29\lambda-29)^2}}$ is preserved by the mean curvature flow. Hence we obtain the theorem. □

Remark 3.1. For the estimate of the term $R_{1213} + R_{4243}$, we use more better estimate of R_{1213} and R_{4243} than the Li–Yang's estimate [10], which is the key point to improve the Li–Yang's main result.

We also have the following corollary and theorem as Corollary 1.2 and Theorem 1.3 in [10]. Using the same argument as in [5], we have the following.

COROLLARY 3.1

Suppose M is a Kähler surface with positive holomorphic sectional curvatures and $1 \leq \lambda < 2$. Then every symplectic minimal surface satisfying

$$\cos\alpha > \frac{29(\lambda-1)}{\sqrt{(48-24\lambda)^2 + (29\lambda-29)^2}}$$

in M is a holomorphic curve.

Using the same argument as in [4] or [14], we have as follows.

Theorem 3.2. *Under the same condition of Theorem 3.1, the symplectic mean curvature flow has no type I singularity at any $T > 0$.*

4. When $\cos\alpha$ is close to 1

In this section, we use the same argument of Han and Li [6]. We prove Kähler manifold M with positive holomorphic sectional curvature and $1 \le \lambda < 2$, when $\cos\alpha$ is close enough to 1. Then the mean curvature flow exists globally and converges to a holomorphic curve.

PROPOSITION 4.1

Suppose that M is a Kähler surface with positive holomorphic sectional curvature and $1 \le \lambda < 2$. Let α be the Kähler angle of the surface Σ_t which evolves by the mean curvature flow. Suppose that $\cos\alpha(\cdot, 0) > \frac{58(\lambda-1)}{\sqrt{(48-24)^2+(58\lambda-58)^2}}$. Then

$$\int_{\Sigma_t} \frac{\sin^2\alpha}{\cos\alpha} \mathrm{d}\mu_t \le C_0 \mathrm{e}^{-\frac{3}{4}(2-\lambda)k_1 t},$$
$$\int_t^{t+1} \int_{\Sigma_t} |H|^2 \mathrm{d}\mu_t \mathrm{d}t \le C_0 \mathrm{e}^{-\frac{3}{4}(2-\lambda)k_1 t}, \tag{29}$$

where C_0 is defined by $C_0 = \int_{\Sigma_0} \frac{\sin^2\alpha(x,0)}{\cos\alpha(x,0)} \mathrm{d}\mu_0$.

Proof. By Theorem 3.1, we know $\cos\alpha(\cdot, t) > \frac{58(\lambda-1)}{\sqrt{(48-24)^2+(58\lambda-58)^2}}$ is preserved along the mean curvature flow. Since $\cos\alpha > \frac{58(\lambda-1)}{\sqrt{(48-24)^2+(58\lambda-58)^2}}$, then by (27), we have

$$\mathrm{Ric}(Je_1, e_2) > \frac{3}{4}(2-\lambda)k_1 \cos\alpha.$$

Hence

$$\left(\frac{\partial}{\partial t} - \Delta\right)\cos\alpha > |\bar{\nabla} J_{\Sigma_t}|^2 \cos\alpha + \frac{3}{4}(2-\lambda)k_1 \cos\alpha \sin^2\alpha$$
$$\ge \frac{3}{4}(2-\lambda)k_1 \cos\alpha \sin^2\alpha.$$

Then using the same argument as in the proof of Proposition 2.1 in [6], we get the proposition. □

Using the same argument as in the proof of Proposition 2.2 in [6], we also get the following.

PROPOSITION 4.2

Suppose that M is a Kähler surface with positive holomorphic sectional curvature and $1 \leq \lambda < 2$. Let α be the Kähler angle of the surface Σ_t which evolves by the mean curvature flow. Suppose that $\cos\alpha(\cdot, 0) > \frac{29(\lambda-1)}{\sqrt{(48-24)^2+(29\lambda-29)^2}}$. *Then*

$$\int_0^T \int_{\Sigma_t} |H| \mathrm{d}\mu_t \mathrm{d}t \leq (C_0)^{1/2} \frac{\mathrm{Area}(\Sigma_0)^{1/2}}{1 - \mathrm{e}^{-\frac{3}{8}(2-\lambda)k_1}}, \tag{30}$$

where the constant C_0 is defined in Proposition 4.1.

Remark 4.1. Han and Li [6] proved the above propositions in the case of Kähler–Einstein manifold M with positive scalar curvature R. They obtained

$$\int_{\Sigma_t} \frac{\sin^2\alpha}{\cos\alpha} \mathrm{d}\mu_t \leq C_0 \mathrm{e}^{-Rt},$$
$$\int_t^{t+1} \int_{\Sigma_t} |H|^2 \mathrm{d}\mu_t \mathrm{d}t \leq C_0 \mathrm{e}^{-Rt},$$
$$\int_0^T \int_{\Sigma_t} |H| \mathrm{d}\mu_t \mathrm{d}t \leq (C_0)^{1/2} \frac{\mathrm{Area}(\Sigma_0)^{1/2}}{1 - \mathrm{e}^{-R/2}}.$$

We recall White's local regularity theorem. Let $H(X, X_0, t)$ be the backward heat kernel on $\mathbb{R}^4$. Define

$$\rho(X, t) = 4\pi(t_0 - t) H(X, X_0, t) = \frac{1}{4\pi(t_0 - t)} \exp\left(-\frac{|X - X_0|^2}{4(t_0 - t)}\right) \tag{31}$$

for $t < t_0$. Let i_M be the injective radius of M^4. We choose a cutoff function $\phi \in C_0^\infty(B_{2r}(X_0))$ with $\phi \equiv 1$ in $B_r(X_0)$, where $X_0 \in M, 0 < 2r < i_M$. Choose normal coordinates in $B_{2r}(X_0)$ and express F using the coordinates (F^1, F^2, F^3, F^4) as a surface in $\mathbb{R}^4$. The parabolic density of the mean curvature flow is defined by

$$\Phi(X_0, t_0, t) = \int_{\Sigma_t} \phi(F) \rho(F, t) \mathrm{d}\mu_t. \tag{32}$$

The following local regularity theorem was proved by White (see Theorems 3.1 and 4.1 in [15]).

Theorem 4.1. *There is a positive constant $\epsilon_0 > 0$ such that if*

$$\Phi(X_0, t_0, t_0 - r^2) \leq 1 + \epsilon_0, \tag{33}$$

then the second fundamental form $A(t)$ of Σ_t in M is bounded in $B_{r/2}(X_0)$, that is,

$$\sup_{B_{r/2} \times (t_0 - r^2/4, t_0]} |A| \leq C, \tag{34}$$

where C is a positive constant depending only on M.

Remark 4.2. Since Σ_0 is smooth, it is well known that

$$\lim_{r\to 0}\int_{\Sigma_0}\phi(F)\frac{\mathrm{e}^{-(|F-X_0|^2/4r^2)}}{4\pi r^2}\mathrm{d}\mu_0 = 1$$

for any $X_0 \in \Sigma_0$. So we can find a sufficiently small r_0 such that

$$\int_{\Sigma_0}\phi(F)\frac{\mathrm{e}^{-(|F-X_0|^2/4r_0^2)}}{4\pi r_0^2}\mathrm{d}\mu_0 \le 1+\frac{\epsilon_0}{2},$$

i.e.,

$$\Phi(X_0, r_0^2, 0) \le 1+\frac{\epsilon_0}{2}$$

for all $X_0 \in M$, where ϵ_0 is the constant in White's theorem.

Using the same argument as in the proof of Theorem 2.5 in [6], we get the following theorem.

Theorem 4.2. *Suppose that M is a Kähler surface with positive holomorphic sectional curvature and $1 \le \lambda < 2$. Let α be the Kähler angle of the surface Σ_t which evolves by the mean curvature flow. Suppose that $\cos\alpha(\cdot, 0) > \frac{29(\lambda-1)}{\sqrt{(48-24)^2+(29\lambda-29)^2}}$. C_0, r_0, ϵ_0 are defined as in Proposition 4.1, Remark 4.2 and Theorem 4.1, respectively. We denote*

$$\epsilon_1 = \frac{\pi^2\epsilon_0^2 r_0^6 (1-\mathrm{e}^{-\frac{3}{8}(2-\lambda)k_1})^2}{4\,\mathrm{Area}(\Sigma_0)}.$$

Then if $C_0 \le \epsilon_1$, the mean curvature flow with initial surface Σ_0 exists globally and it converges to a holomorphic curve.

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