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On solutions of the diophantine equation $F_n - F_m = 3^a$

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Abstract. In this paper, we find non-negative (n, m, a) integer solutions of the diophantine equation $F_n - F_m = 3^a$, where F_n and F_m are Fibonacci numbers. For proving our theorem, we use lower bounds in linear forms.

Keywords. Diophantine equation; lower bounds; logarithmic method.

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1. Introduction

The Fibonacci sequence (F_n) is given by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2$$

with initial conditions $F_0 = 0$, $F_1 = 1$. The Lucas sequence (L_n) is given by the recurrence relation

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2$$

with initial conditions $L_0 = 2$, $L_1 = 1$. F_n and L_n are called the n-th terms of Fibonacci and Lucas sequences, respectively. The Binet formulas for these sequences are given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n,$$

where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$. It is easy to see that $L_n = F_{n-1} + F_{n+1}$ and $5F_n = L_{n-1} + L_{n+1}$. Many properties of these recurrence sequences are given in [6,12].

In [2], Bravo and Luca determined all non-negative solutions (n, m, a) of the diophantine equation $F_n + F_m = 2^a$ with $n \geq m$. Then in [10], Pink and Zeigler considered a more general form $u_n + u_m = wp_1^{z_1} \cdots p_s^{z_s}$ in non-negative integers $n, m, z_1, \ldots, z_s$, where $(u_n)_{n\geq 0}$ is a binary nondegenerate recurrence sequence, $p_1, \ldots, p_s$ are the distinct primes and w is a non-zero integer with $p_i \nmid w$ for all $1 \leq i \leq s$. After noticing Pink and Zeigler's more general diophantine equation for sums of terms of recurrence sequence $(u_n)_{n\geq 0}$, Şiar and Keskin [11] proved that all non-negative integer solutions of the diophantine equation $F_n - F_m = 2^a$ are given by

$$(n, m, a) \in \{(1, 0, 0), (2, 0, 0), (3, 0, 1), (6, 0, 3), (3, 1, 0), (4, 1, 1), (5, 1, 2), (3, 2, 0)\}$$

and

$$(n, m, a) \in \{(4, 3, 0), (4, 2, 1), (5, 2, 2), (9, 3, 5), (5, 4, 1), (7, 5, 3), (8, 5, 4), (8, 7, 3)\}.$$

From this point of [11], we consider the solutions of diophantine equation of the form $F_n - F_m = 3^a$ in non-negative integers. Moreover, Erduvan *et al.* [5] showed that the solutions of the equation $F_n - F_m = 5^a$ is given by

$$F_1 - F_0 = F_2 - F_0 = F_3 - F_2 = F_3 - F_1 = 5^0$$

and

$$F_5 - F_0 = F_4 - F_3 = F_6 - F_4 = F_7 - F_6 = 5,$$

in nonnegative integers m, n and a.

Now we can give the logarithmic height definition from [8].

DEFINITION 1

Let α be an algebraic number of degree d and

$$f(x) = \sum_{i=0}^{d} a_i x^{d-i} \in \mathbb{Z}[x]$$

be the minimal polynomial of α with $a_0 > 0$ and $\gcd(a_0, \ldots, a_d) = 1$. The logarithmic height of α is given by

$$h(\alpha) = \frac{1}{d}\left(\log|a_0| + \sum_{i=1}^{d} \log\max\{|\alpha^{(i)}|, 1\}\right),$$

where $\alpha^{(i)}$'s are the conjugates of α.

Some known properties of the logarithmic heights are as follows:

$$h(\alpha \pm \beta) \leq h(\alpha) + h(\beta) + \log 2, \tag{1.1}$$

$$h(\alpha\beta^{\pm 1}) \leq h(\alpha) + h(\beta), \tag{1.2}$$

$$h(\alpha^k) = |k|\, h(\alpha). \tag{1.3}$$

Now we can give the following lemmas from [7], that will be useful in the proof of Theorem 1.

Lemma 1. *If* $n \equiv m \pmod 2$, *then*

$$F_n - F_m = \begin{cases} F_{(n-m)/2} L_{(n+m)/2}, & n \equiv m \pmod 4 \\ F_{(n+m)/2} L_{(n-m)/2}, & n \equiv m + 2 \pmod 4). \end{cases}$$

Lemma 2. *Let* $L_n = 3^s \cdot y^b$ *for some integers* $n \geq 1, y \geq 1, b \geq 2$ *and* $s \geq 0$. *The solutions of this equation are given by* $n \in \{1, 2, 3\}$.

The following lemma was an interesting problem and was completely proved by Bugeaud *et al.* [3]. This lemma will be used in the proof of Theorem 1.

Lemma 3. *The only perfect powers in the Fibonacci sequence are* $F_0 = 0,\ F_1 = F_2 = 1,\ F_6 = 8$ *and* $F_{12} = 144$.

In [1], Baker gave an effective lower bound for a non-zero expression of the form $c_1 \log \alpha_1 + \cdots + c_n \log \alpha_n$, where α_i are the algebraic numbers and c_i are the integers for all $1 \leq i \leq n$. We will use the reduction method of Baker–Davenport in the proof of Theorem 1.

2. Preliminaries

Before proving the main theorem, we shall state a useful inequality associated with Fibonacci sequence.

Lemma 4. *Let* $n \geq 1$. *Then* $\alpha^{n-2} \leq F_n \leq \alpha^{n-1}$.

Since the proof of the following lemma is given firstly in [9] and then in [3], we will omit its proof.

Lemma 5 [3,9]. *Let* $\mathbb{L}$ *be a number field of degree* D *and* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *be non-zero elements of* $\mathbb{L}$, *and let* $b_1, b_2, \ldots, b_n$ *be rational integers such that*

$$\Lambda = a_1^{b_1} \cdots a_n^{b_n} - 1$$

and

$$B = \max\{|b_1|, \ldots, |b_n|\}.$$

Let h *denote the absolute logarithmic height and* $A_1, \ldots, A_n$ *be real numbers with*

$$A_j \geq \max\{Dh(\alpha_j), |\log \alpha_j|, 0.16\} \quad \textit{for all } 1 \leq j \leq n.$$

If $\Lambda \neq 0$, *then*

$$\log |\Lambda| > -3 \cdot 30^{n+4} (n+1)^{5.5} D^2 (1 + \log D)(1 + \log nB) A_1 \cdots A_n.$$

Furthermore, if $\mathbb{L}$ *is real, then*

$$\log|\Lambda| > -1.4 \cdot 30^{n+3} n^{4.5} D^2 (1+\log D)(1+\log B) A_1 \cdots A_n.$$

We give the following lemma from [4].

Lemma 6. *Let* M *be a positive integer and* p/q *be a convergent of the continued fraction of the irrational number* γ *such that* $q > 6M$. *Let* A, B, μ *be some real numbers with* $A > 0$ *and* $B > 1$. *Let* $\varepsilon := \|\mu q\| - M\|\gamma q\|$, *where* $\|\cdot\|$ *denotes the distance from the nearest integer. If* $\varepsilon > 0$, *then there exist no solutions of the inequality*

$$0 < |u\gamma - v + \mu| < AB^{-w},$$

in positive integers u, v *and* w *with*

$$u \le M \quad \text{and} \quad w \ge \frac{\log(Aq/\varepsilon)}{\log B}.$$

3. Main theorem

Theorem 1. *Let* n, m *and* a *be non-negative integers with* $n > m$. *Then all solutions of the equation*

$$F_n - F_m = 3^a \tag{3.1}$$

are given by

$$(n, m, a) \in \{(1,0,0), (2,0,0), (4,0,1), (3,1,0), (3,2,0), (4,3,0), (5,3,1), (6,5,1), (11,6,4)\}.$$

Proof. If $m = 0$, then from Lemma 3, we get $(n, m, a) = (1,0,0), (2,0,0), (4,0,1)$. Let $1 \le m < n \le 100$. Then the solutions of (3.1) are $(n, m, a) \in \{(3,1,0), (3,2,0), (4,3,0), (5,3,1), (6,5,1), (11,6,4)\}$. Thus, from now on, we will assume that $n > 100$. If $n - m = 1$, then we get $3^a = F_n - F_m = F_{m-1}$. This implies that $m - 1 = 1, 2, 4$, by Lemma 3. So $m = 2, 3, 5$. If $n - m = 2$, then $3^a = F_n - F_m = F_{m+1}$. This implies that $m + 1 = 2, 4$, by Lemma 3. So $m = 1, 3$. But these solutions are given in the theorem for $1 \le m < n \le 100$. Therefore, we may suppose $n - m \ge 3$. Since $3^a = F_n - F_m < F_n \le \alpha^{n-1} < 3^{n-1}$ by Lemma 4, we get $a < n$.

Now recalling $F_n - F_m = 3^a$, we get

$$\frac{\alpha^n}{\sqrt{5}} - 3^a = \frac{\beta^n}{\sqrt{5}} + F_m. \tag{3.2}$$

Taking absolute value of (3.2), we have

$$\left|\frac{\alpha^n}{\sqrt{5}} - 3^a\right| \le \frac{|\beta|^n}{\sqrt{5}} + F_m < \frac{|\beta|^n}{\sqrt{5}} + \alpha^m < \frac{1}{2} + \alpha^m, \tag{3.3}$$

where we take into account $|\beta| \cong 0.6 < 1$ and $\dfrac{|\beta|^n}{\sqrt{5}} < \dfrac{1}{2}$. Dividing both sides of (3.3) by $\dfrac{\alpha^n}{\sqrt{5}}$, we obtain

$$|1 - 3^a \cdot \alpha^{-n} \cdot \sqrt{5}| < \sqrt{5} \cdot \alpha^{m-n} \cdot \left(\frac{1}{2}\alpha^{-m} + 1\right),$$

which implies that

$$|1 - 3^a \cdot \alpha^{-n} \cdot \sqrt{5}| < \sqrt{5} \cdot \alpha^{m-n} \cdot \left(\frac{1}{2} + 1\right) = \frac{3}{2}\sqrt{5} \cdot \alpha^{m-n} < \frac{4}{\alpha^{n-m}}. \tag{3.4}$$

Take the parameters $t := 3$, $\gamma_1 = 3$, $\gamma_2 = \alpha$, $\gamma_3 = \sqrt{5}$, $b_1 = a$, $b_2 = -n$ and $b_3 = 1$ in Lemma 5. We also notice that $D = 2$ and $\gamma_1, \gamma_2, \gamma_3$ are positive real numbers and belong to $\mathbb{Q}(\sqrt{5})$.

Now it is necessary to show that $\Lambda = 1 - 3^a \cdot \alpha^{-n} \cdot \sqrt{5}$ is non-zero. Assume that $\Lambda = 1 - 3^a \cdot \alpha^{-n} \cdot \sqrt{5} = 0$. Then we get $\alpha^{2n} = 5 \cdot 3^a$, which is impossible since $\alpha^{2n} \notin \mathbb{Z}$.

Thus we have $h(\gamma_1) = \log 3$, $h(\gamma_2) = h(\alpha) = \dfrac{\log \alpha}{2}$ and $h(\gamma_3) = h(\sqrt{5}) = \log \sqrt{5}$. Therefore, we can choose

$$A_1 := 2.2 \geq \max\{2 \log 3, |\log \gamma_1|, 0.16\} = 2.1972, \ A_2 := 0.5\ , \ A_3 := 1.7.$$

Now by considering $a < n$, we can take

$$B := \max\{|a|, |-n|, 1\} = n.$$

According to Lemma 5, we get

$$|1 - 3^a \cdot \alpha^{-n} \cdot \sqrt{5}| > \exp\{-1.4 \times 30^6 \times 3^{4.5} \times 4 \times (1 + \log 2)(1 + \log n) \times 2.2 \times 0.5 \times 1.7\}. \tag{3.5}$$

If we take logarithms in equality (3.4) and combine the calculation of the right-hand side of (3.5), we get

$$(n - m) \log \alpha < 18.139 \times 10^{11}(1 + \log n) + \log 4. \tag{3.6}$$

Using the fact that $1 + \log n < 2 \log n$ for all $n \geq 3$, we have

$$(n - m) \log \alpha < 3.63 \times 10^{12} \log n. \tag{3.7}$$

Moreover, we can obtain a second linear form by using equation (3.1) as follows:

$$\left|\frac{\alpha^n}{\sqrt{5}}(1 - \alpha^{m-n}) - 3^a\right| = \left|\frac{\beta^n}{\sqrt{5}} - \frac{\beta^m}{\sqrt{5}}\right| \leq \frac{|\beta|^n + |\beta|^m}{\sqrt{5}} < 0.445. \tag{3.8}$$

Now dividing both sides of (3.8) by $\frac{\alpha^n}{\sqrt{5}}(1-\alpha^{m-n})$, we get

$$|1 - 3^a \cdot \alpha^{-n} \cdot \sqrt{5} \cdot (1-\alpha^{m-n})^{-1}| \leq \frac{0.445 \times \sqrt{5}}{\alpha^n(1-\alpha^{m-n})}.$$

Since $\alpha^{m-n} = \frac{1}{\alpha^{n-m}} < \frac{1}{\alpha} < 0.67$, it is obvious that $\frac{1}{1-\alpha^{m-n}} < 2.71$. Thus it follows that

$$|1 - 3^a \cdot \alpha^{-n} \cdot \sqrt{5} \cdot (1-\alpha^{m-n})^{-1}| < \frac{2.71}{\alpha^n}. \tag{3.9}$$

Let $\Lambda = 3^a \cdot \alpha^{-n} \cdot \sqrt{5} \cdot (1-\alpha^{m-n})^{-1} - 1$. Now, in order to apply Lemma 5 again, we take parameters $t := 3$ and

$$\gamma_1 = 3,\ \gamma_2 = \alpha,\ \gamma_3 = \sqrt{5} \cdot (1-\alpha^{m-n})^{-1},\ b_1 = a, b_2 = -n,\ b_3 = 1.$$

As before, we have $D := [\mathbb{Q}(\sqrt{5}) : \mathbb{Q}] = 2$. It is obvious that $\Lambda \neq 0$. Because, if $\Lambda = 0$, we get $\alpha^n - \alpha^m = 3^a\sqrt{5}$. Taking the conjugate of this equation, we have $\beta^n - \beta^m = -3^a\sqrt{5}$. Addition of these two conjugate equations gives $L_n - L_m = 0$, which contradicts the fact that $n > m$.

Hence the left-hand side of (3.9) is non-zero. As before, $A_1 = 2.2$, $A_2 = 0.5$ and $B = n$ in Lemma 5.

Now we calculate A_3. For all $n - m > 3$, we have $|\log \gamma_3| < 1$. Taking the identities (1.1), (1.2) and (1.3) into account, we get

$$\begin{aligned} h(\gamma_3) &= h(\sqrt{5} \cdot (1-\alpha^{m-n})^{-1}) \\ &\leq h(\sqrt{5}) + h(1) + h(\alpha^{m-n}) + \log 2 \\ &\leq \log\sqrt{5} + |m-n|\, h(\alpha) + \log 2 \\ &= \log\sqrt{5} + (n-m)\frac{\log\alpha}{2} + \log 2 \\ &= (n-m)\frac{\log\alpha}{2} + \log 2\sqrt{5}. \end{aligned}$$

Thus, it follows that

$$A_3 \geq \max\{2h(\gamma_3), |\log\gamma_3|, 0.16\} \geq \log 20 + (n-m)\log\alpha.$$

Hence, we can take $A_3 = \log 20 + (n-m)\log\alpha$.

By Lemma 5, we obtain a lower bound for the left-hand side of (3.9) as

$$\begin{aligned} \frac{3.03}{\alpha^n} > |\Lambda| > \exp\{&-1.4 \times 30^6 \times 3^{4.5} \times 4 \times (1+\log 2)(1+\log n) \\ &\times 2.2 \times 0.5 \times [\log 20 + (n-m)\log\alpha]\}. \end{aligned}$$

Taking logarithms on both sides of this inequality and considering the fact that $1+\log n < 2\log n$ for all $n > 1$, we get

$$n\log\alpha < 2.134 \times 10^{12}\log n \times \left[\log 20 + (n-m)\log\alpha\right]. \tag{3.10}$$

By a fast calculation with Mathematica, we obtain $n < 7.09616 \times 10^{28}$.

If (n, m, a) is a positive integer solution of equation (3.1) with $n > m$, then we have $a < n < 7.09616 \times 10^{28}$. We have obtained an upper bound for n and now we will reduce this bound to a size that can be easily dealt with. For doing this, we will use Lemma 4 again.

Let $z_1 := a\log 3 - n\log\alpha + \log\sqrt{5}$. By considering equation (3.4), we get

$$\left|1 - \mathrm{e}^{z_1}\right| < \frac{4}{\alpha^{n-m}}.$$

Thus, by using equation (3.1) and the Binet formula, we obtain

$$\frac{\alpha^n}{\sqrt{5}} = F_n + \frac{\beta^n}{\sqrt{5}} > F_n - 1 > F_n - F_m = 3^a.$$

Hence $z_1 = \log(3^a\sqrt{5}/\alpha^n) < 0$. It is obvious that $\dfrac{4}{\alpha^{n-m}} < 0.945$ for all $n-m \geq 3$. Therefore we get $\mathrm{e}^{|z_1|} < 18.2$ and therefore, it follows that

$$0 < |z_1| < \mathrm{e}^{|z_1|} - 1 \leq \mathrm{e}^{|z_1|}|1 - \mathrm{e}^{z_1}| < \frac{73}{\alpha^{n-m}}.$$

Thus it can be seen that

$$0 < |a\log 3 - n\log\alpha + \log\sqrt{5}| < \frac{73}{\alpha^{n-m}}. \tag{3.11}$$

Dividing both sides of the inequality (3.11) by $\log\alpha$, we get

$$0 < \left|a\frac{\log 3}{\log\alpha} - n + \frac{\log\sqrt{5}}{\log\alpha}\right| < \frac{73}{\log\alpha}\cdot\alpha^{-(n-m)}. \tag{3.12}$$

Now considering Lemma 6, we have the irrational $\gamma = \dfrac{\log 3}{\log\alpha}$ with

$$\mu = \frac{\log\sqrt{5}}{\log\alpha},\ A = \frac{73}{\log\alpha},\ B = \alpha,\ w = n - m.$$

Also, we know that $a < n < 7.09616 \times 10^{28}$. So it follows that $M := 7.09616 \times 10^{28}$, according to Lemma 6 and $q > 6M$ is the denominator of a convergent of the continued fraction of γ such that $\varepsilon = \|\mu q\| - M\,\|\gamma q\| > 0$. Considering the denominator of the 61-st convergence of $\frac{\log 3}{\log \alpha}$, we have $q = 10.52 \times 10^{29}$. By some calculations with Mathematica, we obtain $\varepsilon = 0.154453$.

According to Lemma 6, we know that there is no solution of the inequality (3.12) for the values $n - m$ with $n - m \geq \frac{\log(Aq/\varepsilon)}{\log B}$. Therefore, it follows that inequality (3.12) has no solutions for $n - m \geq 157.972$. This means that a bound for $n - m$ is $n - m \leq 157$. Considering this fact in inequality (3.10), we get $n < 1.29184 \times 10^{16}$.

Let us work on (3.9) for finding an upper bound on n. Now take

$$z_2 := a \log 3 - n \log \alpha + \log(\sqrt{5}(1 - \alpha^{m-n})^{-1}).$$

Thus (3.9) implies that

$$|1 - \mathrm{e}^{z_2}| < \frac{2.71}{\alpha^n}.$$

It is obvious that $\frac{2.71}{\alpha^n} < \frac{1}{2}$. If $z_2 > 0$, then $0 < z_2 < \mathrm{e}^{z_2} - 1 < \frac{2.71}{\alpha^n}$. If $z_2 < 0$, then $|1 - \mathrm{e}^{z_2}| = 1 - \mathrm{e}^{z_2} < \frac{2.71}{\alpha^n} < \frac{1}{2}$. Thus, we get $1 - \frac{1}{2} < \mathrm{e}^{z_2}$, so that $\mathrm{e}^{|z_2|} < 2$. Therefore, we have

$$0 < |z_2| < \mathrm{e}^{|z_2|} - 1 \leq \mathrm{e}^{|z_2|} \cdot |1 - \mathrm{e}^{z_2}| < 2 \times \frac{2.71}{\alpha^n}.$$

Thus it follows that

$$0 < \left| a \frac{\log 3}{\log \alpha} - n + \frac{\log \sqrt{5} \cdot (1 - \alpha^{m-n})^{-1}}{\log \alpha} \right| < \frac{5.42}{\log \alpha} \cdot \alpha^{-n}. \tag{3.13}$$

Now considering Lemma 6, we obtain

$$\gamma = \frac{\log 3}{\log \alpha}, \ \mu = \frac{\log \sqrt{5} \cdot (1 - \alpha^{m-n})^{-1}}{\log \alpha}, \ A = \frac{5.42}{\log \alpha}, \ B = \alpha, \ w = n.$$

It is obvious that γ is irrational. Also, $3 \leq n - m \leq 157$. Firstly, calculate the denominator q of continued fraction of γ. Since $M = 1.29184 \times 10^{16}$, we must choose the 39-th denominator $q = 48, 9 \times 10^{16}$ such that $q > 6M = 7.75107 \times 10^{16}$. Hence by applying Lemma 6 to (3.13) with $3 \leq n - m \leq 157$ except for $n - m = 4$ or 8, we obtain

$$\varepsilon = \|\mu q_{40}\| - M\|\gamma q_{40}\| \geq 0.492868$$

by a fast computation with Mathematica. Furthermore, according to Lemma 6, we know that there is no solution of the inequality (3.13) for values n with $n \geq \frac{\log(Aq/\varepsilon)}{\log B} = 91.1453$. Thus, an upper bound for n must be $n \leq 91$. This contradicts our assumption that $n > 100$.

Finally, we consider the cases $n - m = 4$ or 8. According to Lemma 1, when $n \equiv m \pmod 4$, we have $F_n - F_m = F_{(n-m)/2}L_{(n+m)/2}$. Thus, it follows that $F_n - F_m = F_2L_{m+2} = L_{m+2}$ for $n - m = 4$. This gives $L_{m+2} = 3^a$. According to Lemma 2, the possible values for $m + 2$ are $1, 2, 3$. Since $4 \nmid 3^a$ and m is a non-negative integer, we have that $m + 2 \neq 3$ and $m + 2 \neq 1$. If $m + 2 = 2$, then we get $L_2 = 3^a$. Thus, it follows that $(n, m, a) = (4, 0, 1)$ is a solution to (3.1). Moreover, the case $n - m = 8$ gives $F_n - F_m = F_4L_{m+4}$ by Lemma 1 . Hence we get $L_{m+4} = 3^{a-1}$, which implies that $m + 4 = 1, 2, 3$ by Lemma 2. This is impossible since m is a non-negative integer. □

4. Conclusion

In [7], it is shown that if $n \equiv m \pmod 2$, then all the solutions of the equation

$$F_n - F_m = y^p,\ p \geq 2,\ y \geq 1 \tag{4.1}$$

satisfy $\max\{n, m\} \leq 36$. Then the authors conjectured that all the solutions of equation (4.1) are

$$\begin{aligned}
&F_1 - F_0 = 1,\ F_2 - F_0 = 1,\ F_3 - F_1 = 1,\ F_3 - F_2 = 1,\ F_4 - F_3 = 1,\\
&F_5 - F_1 = 2^2,\ F_5 - F_2 = 2^2,\ F_6 - F_4 = 5,\ F_7 - F_5 = 2^3,\\
&F_7 - F_6 = 5,\ F_8 - F_5 = 2^4, F_8 - F_7 = 2^3,\ F_9 - F_3 = 2^5,\\
&F_{11} - F_6 = 9^2,\ F_{13} - F_6 = 15^2,\ F_{13} - F_{11} = 12^2,\ F_{14} - F_9 = 7^3,\\
&F_{14} - F_{13} = 12^2,\ F_{15} - F_9 = 24^2.
\end{aligned}$$

Consequently, it is true that the above conjecture is valid for $y = 2, 3$ by our result and the results in [5,11]. It is reasonable to conjecture that if $F_n - F_m = p^a$ for some prime p and positive integer a, then $p = 2, 3, 5, 7$.

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