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Transversal intersection of monomial ideals

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Abstract. In this paper, we prove conditions for transversal intersection of monomial ideals and derive a simplicial characterization of this phenomenon.

Keywords. Monomial ideals; transversal intersection; Gröbner basis; Taylor complex; minimal free resolution; simplicial complex.

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1. Introduction

The transversal intersection phenomenon for monomial ideals in the polynomial ring happens to be extremely interesting and it turns out that it is equivalent to having disjoint supports for their minimal generating sets; see Theorem 2.2. The Taylor complex which resolves (possibly non-minimally) a monomial ideal has been understood completely for ideals of the form $I + J$, where I and J are monomial ideals intersecting transversally; see Theorem 3.2. As an application of this theorem, we prove in Corollary 3.3 that for two ideals I and J in R intersecting transversally, the ideal $I + J$ is resolved minimally by the complex $\mathbb{M}(I)_{\bullet} \otimes \mathbb{M}(J)_{\bullet}$, if $\mathbb{M}(I)_{\bullet}$ and $\mathbb{M}(J)_{\bullet}$ denote minimal free resolutions of I and J respectively. Minimal free resolutions for ideals of the form $I + J$ have an interesting structure when I and J intersect transversally and are supported simplicially; see 4.3.

2. Monomial ideals

Let $R = K[x_1, x_2, \ldots, x_n]$, where x_i's are indeterminates over the field K. Let $\mathrm{Mon}(R)$ denote the set of all monomials in R. Every nonzero polynomial $f \in R$ is a unique K-linear combination of monomials given by $f = \sum_{v\in \mathrm{Mon}(R)} a_v v$. Let $m(f) := \{v \in \mathrm{Mon}(R) \mid a_v \neq 0\}$. An ideal I in R is said to be a *monomial ideal* if it is generated by monomials of R. We list down some standard facts on monomial ideals.

PROPOSITION 2.1

(1) *I is a monomial ideal if and only if $f \in I \Rightarrow m(f) \subset I$.*
(2) *Let $\{u_1, \ldots, u_m\}$ be a monomial generating set of an ideal I, where u_i's are monomials. A monomial $v \in I$ if and only if $v = u_i w$ for some $1 \le i \le m$.*
(3) *Each monomial ideal I has a unique minimal monomial set of generators $G(I)$.*

Proof. See [1]. □

DEFINITION 1

Let $\emptyset \neq T \subset \mathrm{Mon}(R)$. We define

$$\mathrm{supp}(T) = \{i \mid x_i \text{ divides } m \text{ for some } m \in T\}.$$

If $T = \{m\}$, we simply write $\mathrm{supp}(m)$ instead of $\mathrm{supp}(\{m\})$. If S and T are two nonempty subsets of $\mathrm{Mon}(R)$, then, $\mathrm{supp}(S) \cap \mathrm{supp}(T) = \emptyset$ if and only if $\mathrm{supp}(f) \cap \mathrm{supp}(g) = \emptyset$ for every $f \in S$ and $g \in T$.

DEFINITION 2

We say that the ideals I and J of R intersect transversally if $I \cap J = IJ$.

Theorem 2.2. *Let I and J be two monomial ideals of R. Then, $I \cap J = IJ$ if and only if $\mathrm{supp}(G(I)) \cap \mathrm{supp}(G(J)) = \emptyset$.*

Proof. Let $I \cap J = IJ$. Consider the set

$$S = \{\mathrm{lcm}(f, g) \mid f \in G(I), g \in G(J), \quad \text{and} \quad \mathrm{supp}(f) \cap \mathrm{supp}(g) \neq \emptyset\}.$$

If $S = \emptyset$, then for every $f \in G(I)$ and $g \in G(J)$ we have $\mathrm{supp}(f) \cap \mathrm{supp}(g) = \emptyset$, which proves that $\mathrm{supp}(G(I)) \cap \mathrm{supp}(G(J)) = \emptyset$.

Suppose $S \neq \emptyset$. We define a partial order $\le$ on S in the following way: Given $s, t \in S$, we define $s \le t$ if and only $s \mid t$. Let $\mathbf{m}$ be a minimal element (by Zorn's lemma) of S; then there exist f, g, such that $f \in G(I)$, $g \in G(J)$ and $(f, g) = \mathbf{m}$. Since $\mathrm{lcm}(f, g) \in I \cap J$ and $I \cap J = IJ$, we have $\mathrm{lcm}(f, g) = \mathbf{m} \in IJ$. By Proposition 2.1, there exist $h_1 \in G(I)$ and $h_2 \in G(J)$ such that $h_1 h_2 \mid \mathbf{m}$, since the generating set of IJ is the set $G(I)G(J) = \{uv \mid u \in G(I), v \in G(J)\}$.

We claim that $\mathrm{supp}(h_1) \cap \mathrm{supp}(h_2) = \emptyset$. If this is not the case, then $\mathrm{lcm}(h_1, h_2) \neq h_1 h_2$, in fact $\mathrm{lcm}(h_1, h_2) < h_1 h_2$ and $\mathrm{lcm}(h_1, h_2) \mid \mathbf{m}$; contradicting minimality of $\mathbf{m}$ in S.

We now prove that $\mathrm{supp}(f) \cap \mathrm{supp}(h_2) \neq \emptyset$. We know that $h_1 h_2 \mid \mathrm{lcm}(f, g) = \mathbf{m}$. Therefore, if $\mathrm{supp}(f) \cap \mathrm{supp}(h_2) = \emptyset$, then $h_2 \mid g$; which contradicts minimality of the generating set $G(J)$. Similarly, we can prove that $\mathrm{supp}(g) \cap \mathrm{supp}(h_1) \neq \emptyset$. Now, $h_1 h_2 \mid \mathbf{m} = \mathrm{lcm}(f, g)$ implies that $h_2 \mid \mathbf{m} = \mathrm{lcm}(f, g)$. Moreover, $f \mid \mathrm{lcm}(f, g)$. Therefore, $\mathrm{lcm}(f, h_2) \mid \mathrm{lcm}(f, g)$. Similarly, $\mathrm{lcm}(g, h_1) \mid \mathrm{lcm}(f, g)$. Now if $\mathrm{lcm}(f, h_2) < \mathbf{m}$ or $\mathrm{lcm}(g, h_1) < \mathrm{lcm}(f, g) = \mathbf{m}$, then we have a contradiction, since $\mathrm{lcm}(f, g)$ and $\mathrm{lcm}(g, h_1)$ both are in the set S and $\mathbf{m}$ is a minimal element in S. Therefore, we must have $\mathrm{lcm}(f, h_2) = \mathrm{lcm}(g, h_1) = \mathrm{lcm}(f, g) = \mathbf{m} = h_1 h_2 w$ (say) and $\dfrac{f h_2}{\gcd(f, h_2)} =$

$\frac{gh_1}{\gcd(g, h_1)} = h_1h_2w$, therefore $\frac{f}{\gcd(f, h_2)} = h_1w$. Hence, $h_1 \mid f$ and it contradicts minimality of the generating set $G(I)$. Hence, $S = \emptyset$ and we are done.

Conversely, let us assume that $\mathrm{supp}(G(I)) \cap \mathrm{supp}(G(J)) = \emptyset$. Without loss of generality, we can assume that $\mathrm{supp}(G(I)) = \{1, 2, \ldots, k\}$ and $\mathrm{supp}(G(J)) = \{k+1, k+2, \ldots, n\}$. Let $f \in I \cap J$ such that $f = \sum_{v \in \mathrm{Mon}(R)} a_v v$. We have $v \in I \cap J$ for all $v \in m(f)$. It is therefore enough to show that if m is a monomial and $m \in I \cap J$, then $m \in IJ$. Let $m \in I \cap J$; there exist $m_I \in G(I)$ and $m_J \in G(J)$ such that $m_I \mid m$ and $m_J \mid m$. Let $m_I m_1 = m$; then $m_J \mid m = m_I m_1$. We know that $\mathrm{supp}(m_I) \cap \mathrm{supp}(m_J) = \emptyset$, therefore $m_J \mid m_1$ and it follows that $m_1 = m_J m_2$ and $m = m_I m_J m_2$. Therefore, $m \in IJ$ since $m_I m_J \in IJ$. □

Theorem 2.3. *Let I and J be two ideals of R such that with respect to some monomial order $\preceq$, $\mathrm{supp}(\mathrm{Lt}(I)) \cap \mathrm{supp}(\mathrm{Lt}(J)) = \emptyset$. Then I and J intersect transversally.*

Proof. Let $\mathcal{G}_I$ and $\mathcal{G}_J$ be Gröbner bases with respect to the monomial order $\preceq$ of the ideals I and J respectively. We assume that $f \in I \cap J$. There exist polynomials $p \in \mathcal{G}_I$ and $q \in \mathcal{G}_J$ such that $\mathrm{Lt}(p) \mid \mathrm{Lt}(f)$ and $\mathrm{Lt}(q) \mid \mathrm{Lt}(f)$. Since $\mathrm{supp}(\mathrm{Lt}(I)) \cap \mathrm{supp}(\mathrm{Lt}(J)) = \emptyset$, we have $\mathrm{Lt}(p) \cdot \mathrm{Lt}(q) \mid \mathrm{Lt}(f)$. After division we write $f = pq + r$. Then $r \in I \cap J$ and $\mathrm{Lt}(r) \preceq \mathrm{Lt}(f)$. We can apply the same process on r, and after a finite stage, we get $f \in IJ$. □

3. The Taylor complex

We define the *multidegree* of a monomial $x^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ to be the n-tuple $(a_1, \ldots, a_n)$, denoted by $\mathrm{mdeg}(\mathbf{x}^{\mathbf{a}})$. Hence $R = \oplus_{m \in \mathrm{Mon}(R)} Km$ has a direct sum decomposition as K-vector spaces Km, where $Km = \{cm \mid c \in K\}$.

DEFINITION 3 (The Taylor complex)

Let M be a monomial ideal of R minimally generated by the monomials $m_1, m_2, \ldots, m_p$. The Taylor complex for M is given by $\mathbb{T}(M)_{\cdot}$ as follows: $\mathbb{T}(M)_i$ is the free R-module generated by $e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_i}$ for all $1 \le j_1 < \cdots < j_i \le p$, where $\{e_1, \ldots, e_p\}$ is the standard basis for the free R-module R^p. The differential $\delta(M)_{\cdot}$ is given by the following:

$$\begin{aligned}&\delta(M)_i(e_{j_1} \wedge \cdots \wedge e_{j_i})\\ &\quad = \sum_{1 \le k \le i} (-1)^{k-1} \frac{\mathrm{lcm}(m_{j_1}, \ldots, m_{j_i})}{\mathrm{lcm}(m_{j_1}, \ldots, \widehat{m_{j_k}}, \ldots, m_{j_i})} (e_{j_1} \wedge \cdots \wedge \widehat{e_{j_k}} \wedge \cdots \wedge e_{j_i}).\end{aligned}$$

We define $\mathrm{mdeg}(e_{j_1} \wedge, \ldots \wedge e_{j_i}) = \mathrm{mdeg}(\mathrm{lcm}(m_{j_1}, \ldots, m_{j_i}))$. The Taylor complex $(\mathbb{T}(M)_{\cdot}, \delta(M)_{\cdot})$ defined above is indeed a chain complex of free R modules and gives a free resolution for the R-module R/M; see section 26 in [2].

Theorem 3.1. *The Taylor complex $(\mathbb{T}(M)_{\cdot}, \delta(M)_{\cdot})$ defined above is a free resolution (though not minimal) of the monomial ideal M called the Taylor resolution of M.*

Proof. See Theorem 26.7 in [2]. □

Theorem 3.2. *Let I and J be two monomial ideals of R intersecting transversally. Let $\mathbb{T}(I)_{\textbf{.}}$ and $\mathbb{T}(J)_{\textbf{.}}$ be the Taylor resolutions of I and J respectively. Then the Taylor's resolution of $I+J$ is isomorphic to the complex $\mathbb{T}(I)_{\textbf{.}} \otimes \mathbb{T}(J)_{\textbf{.}}$ and hence $\mathbb{T}(I)_{\textbf{.}} \otimes \mathbb{T}(J)_{\textbf{.}}$ is acyclic.*

Proof. We know that $\mathbb{T}(I)_i = R^{\binom{p}{i}}$ and $\mathbb{T}(J)_j = R^{\binom{q}{j}}$, where p and q denote the minimal number of generators of I and J respectively. Then

$$\begin{aligned}(\mathbb{T}(I)_{\textbf{.}} \otimes \mathbb{T}(J)_{\textbf{.}})_r &= \oplus_{i+j=r}(\mathbb{T}(I)_i \otimes \mathbb{T}(J)_j)\\ &= \oplus_{i+j=r} R^{\binom{p}{i}} \otimes R^{\binom{q}{j}}\\ &= R^{\sum_{i+j=r}\binom{p}{i}\binom{q}{j}}\\ &= R^{\binom{p+q}{r}} = \mathbb{T}(I+J)_r.\end{aligned}$$

A basis of $(\mathbb{T}(I)_{\textbf{.}} \otimes \mathbb{T}(J)_{\textbf{.}})_r$ is given by $(e_{k_1} \wedge \cdots \wedge e_{k_i}) \otimes (e'_{k'_1} \wedge \cdots \wedge e'_{k'_j})$ such that $i+j=r$, $1 \le k_1 < k_2 < \cdots < k_i \le p$ and $1 \le k'_1 < k'_2 < \cdots < k'_j \le q$. Now for each r, the free module $(\mathbb{T}(I)_{\textbf{.}} \otimes \mathbb{T}(J)_{\textbf{.}})_r$ can be graded with the help of mdeg as follows. We define

$$\begin{aligned}&\mathrm{mdeg}((e_{k_1} \wedge \cdots \wedge e_{k_i}) \otimes (e'_{k'_1} \wedge \cdots \wedge e'_{k'_j}))\\ &\quad = \mathrm{mdeg}(\mathrm{lcm}(m_{k_1}, \ldots, m_{k_i}) \cdot \mathrm{lcm}(m'_{k'_1}, \ldots, m'_{k'_j})).\end{aligned}$$

This defines a multi-graded structure for the complex $\mathbb{T}(I)_{\textbf{.}} \otimes \mathbb{T}(J)_{\textbf{.}}$.

Let $G(I) = \{m_1, m_2, \ldots, m_p\}$ and $G(J) = \{m'_1, m'_2, \ldots, m'_q\}$ denote the minimal sets of generators for I and J respectively. Then $I+J$ is minimally generated by $G(I) \cup G(J)$, by Theorem 2.2, since $G(I)$ and $G(J)$ are of disjoint support. We will now show that the tensor product complex $\mathbb{T}(I)_{\textbf{.}} \otimes \mathbb{T}(J)_{\textbf{.}}$ is isomorphic to the Taylor resolution $\mathbb{T}(I+J)_{\textbf{.}}$. We define $\psi_r : (\mathbb{T}(I)_{\textbf{.}} \otimes \mathbb{T}(J)_{\textbf{.}})_r \longrightarrow (\mathbb{T}(I+J)_{\textbf{.}})_r$ as

$$(e_{k_1} \wedge \cdots \wedge e_{k_i}) \otimes (e'_{k'_1} \wedge \cdots \wedge e'_{k'_j}) \mapsto (e_{k_1} \wedge \cdots \wedge e_{k_i} \wedge e'_{k'_1} \wedge \cdots \wedge e'_{k'_j}).$$

Moreover,

$$\begin{aligned}&\mathrm{mdeg}(e_{k_1} \wedge \cdots \wedge e_{k_i} \wedge e_{k'_1} \wedge \cdots \wedge e_{k'_j})\\ &\quad = \mathrm{mdeg}(\mathrm{lcm}(m_{k_1}, \ldots, m_{k_s}, m'_{k'_1}, \ldots, m'_{k'_{s'}}))\\ &\quad = (\mathrm{mdeg}(\mathrm{lcm}(m_{k_1}, \ldots, m_{k_s})), \mathrm{mdeg}(\mathrm{lcm}(m'_{k'_1}, \ldots, m'_{k'_{s'}}))),\end{aligned}$$

since $G(I)$ and $G(J)$ are with disjoint supports. Hence, the map is a graded isomorphism between the free modules $(\mathbb{T}(I)_{\textbf{.}} \otimes \mathbb{T}(J)_{\textbf{.}})_r$ and $(\mathbb{T}(I+J)_{\textbf{.}})_r$. We therefore have the following diagram of complexes:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & (\mathbb{T}(I)_{\bullet} \otimes \mathbb{T}(J)_{\bullet})_r & \xrightarrow{d_r} & (\mathbb{T}(I)_{\bullet} \otimes \mathbb{T}(J)_{\bullet})_{r-1} & \longrightarrow \cdots \\
& \downarrow \psi_r & & \downarrow \psi_{r-1} & \\
\cdots \longrightarrow & (\mathbb{T}(I+J)_{\bullet})_r & \xrightarrow{\delta(I+J)_r} & (\mathbb{T}(I+J)_{\bullet})_{r-1} & \longrightarrow \cdots
\end{array}$$

We now show that $\delta(I+J)_r \circ \psi_r = \psi_{r-1} \circ d_r$. Let us take an arbitrary basis element of $(\mathbb{T}(I)_{\bullet} \otimes \mathbb{T}(J)_{\bullet})_r$, which is of the form

$$(e_{k_1} \wedge e_{k_2} \wedge \cdots \wedge e_{k_i}) \otimes (e_{k'_1} \wedge e_{k'_2} \wedge \cdots \wedge e_{k'_j}) \quad \text{with} \quad i + j = r.$$

Then

$$\begin{aligned}
&(\psi_{r-1} \circ d_r)((e_{k_1} \wedge \cdots \wedge e_{k_i}) \otimes (e'_{k'_1} \wedge \cdots \wedge e'_{k'_j})) \\
&= \psi_{r-1}(\delta(I)_i (e_{k_1} \wedge \cdots \wedge e_{k_i}) \otimes (e'_{k'_1} \wedge \cdots \wedge e'_{k'_j}) \\
&\quad + (-1)^i (e_{k_1} \wedge \cdots \wedge e_{k_i}) \otimes \delta(J)_j (e'_{k'_1} \wedge \cdots \wedge e'_{k'_j})).
\end{aligned}$$

Let us write

$$\begin{aligned}
Q_t &= \frac{\operatorname{lcm}(m_{k_1}, \ldots, m_{k_i})}{\operatorname{lcm}(m_{k_1}, \ldots, \widehat{m_{k_t}}, \ldots, m_{k_i})}, \ 1 \le t \le i; \\
V_t &= \frac{\operatorname{lcm}(m_{k_1}, \ldots, m_{k_i}, m'_{k'_1}, \ldots, m'_{k'_j})}{\operatorname{lcm}(m_{k_1}, \ldots, \widehat{m_{k_t}}, \ldots, m_{k_i}, m'_{k'_1}, \ldots, m'_{k'_j})}, \ 1 \le t \le i.
\end{aligned}$$

We know that $G(I)$ and $G(J)$ have disjoint supports. Therefore,

$$V_t = \frac{\operatorname{lcm}(m_{k_1}, \ldots, m_{k_i}) \cdot \operatorname{lcm}(m'_{k'_1}, \ldots, m'_{k'_j})}{\operatorname{lcm}(m_{k_1}, \ldots, \widehat{m_{k_t}}, \ldots, m_{k_i}) \cdot \operatorname{lcm}(m'_{k'_1}, \ldots, m'_{k'_j})} = Q_t, \quad \forall\, 1 \le t \le i.$$

Then

$$\begin{aligned}
&\big(\delta(I)_i (e_{k_1} \wedge \cdots \wedge e_{k_i})\big) \otimes (e'_{k'_1} \wedge \cdots \wedge e'_{k'_j}) \\
&= \sum_{t=1}^{i} (-1)^t Q_t \cdot (e_{k_1} \wedge \cdots \wedge \widehat{e_{k_t}} \cdots \wedge e_{k_i}) \otimes (e'_{k'_1} \wedge \cdots \wedge e'_{k'_j})
\end{aligned}$$

and

$$\begin{aligned}
&\psi_{r-1}(\delta(I)_i (e_{k_1} \wedge \cdots \wedge e_{k_i}) \otimes (e'_{k'_1} \wedge \cdots \wedge e'_{k'_j})) \\
&= \sum_{t=1}^{i} (-1)^t Q_t \cdot (e_{k_1} \wedge \cdots \wedge \widehat{e_{k_t}} \cdots \wedge e_{k_i}) \otimes (e'_{k'_1} \wedge \cdots \wedge e'_{k'_j}) \\
&= \sum_{t=1}^{i} (-1)^t V_t \cdot (e_{k_1} \wedge \cdots \wedge \widehat{e_{k_t}} \wedge e_{k_i} \wedge e'_{k'_1} \wedge \cdots \wedge e'_{k'_j}).
\end{aligned}$$

The last equality in the above expression follows from the fact that $G(I)$ and $G(J)$ have disjoint supports. Similarly, it can be proved that

$$
\begin{aligned}
&\psi_{r-1}((e_{k_1} \wedge \cdots \wedge e_{k_i}) \otimes (\delta(J)_j(e'_{k'_1} \wedge \cdots \wedge e'_{k'_j})))\\
&= \sum_{l=1}^{j} (-1)^l W_l \cdot (e_{k_1} \wedge \cdots \wedge e_{k_i} \wedge e'_{k'_1} \wedge \cdots \widehat{e'_{k'_l}} \cdots \wedge e'_{k'_j});
\end{aligned}
$$

where

$$
W_l = \frac{\mathrm{lcm}(m_{k_1}, \ldots, m_{k_i}, m'_{k'_1}, \ldots, m'_{k'_j})}{\mathrm{lcm}(m_{k_1}, \ldots, m_{k_i}, m'_{k'_1}, \ldots, \widehat{m'_{k'_l}}, \ldots, m'_{k'_j})}, \quad 1 \le l \le j.
$$

Hence

$$
\begin{aligned}
&(*)\ (\psi_{r-1} \circ d_r)((e_{k_1} \wedge \cdots \wedge e_{k_i}) \otimes (e'_{k'_1} \wedge \cdots \wedge e'_{k'_j}))\\
&= \sum_{t=1}^{i} (-1)^t V_t \cdot (e_{k_1} \wedge \cdots \wedge \widehat{e_{k_t}} \wedge e_{k_i} \wedge e'_{k'_1} \wedge \cdots \wedge e'_{k'_j})\\
&\quad + \sum_{l=1}^{j} (-1)^{i+l} W_l \cdot (e_{k_1} \wedge \cdots \wedge e_{k_i} \wedge e'_{k'_1} \wedge \cdots \widehat{e'_{k'_l}} \cdots \wedge e'_{k'_j}).
\end{aligned}
$$

We now introduce $i + j$ new symbols w_s such that $w_s = m_{k_s}$ for $s \le i$ and $w_s = m'_{k'_{s-i}}$ for $s > i$. We also introduce another set of $i + j$ symbols E_s such that $E_s = e_{k_s}$ for $s \le i$ and $E_s = e'_{k'_{s-i}}$ for $s > i$. Hence, the expression after the last equality in (*) can be written in a compact form as

$$
\sum_{k=1}^{i+j} (-1)^k \frac{\mathrm{lcm}(w_1, \ldots, w_{i+j})}{\mathrm{lcm}(w_1, \ldots, \widehat{w_k}, \ldots, w_{i+j})} (E_1 \wedge \cdots \wedge \widehat{E_k} \cdots \wedge E_{i+j}).
$$

Now

$$
\begin{aligned}
&(\delta(I + J)_r \circ \psi_r)((e_{k_1} \wedge \cdots \wedge e_{k_i}) \otimes (e'_{k'_1} \wedge \cdots \wedge e'_{k'_j}))\\
&= \delta(I + J)_r (e_{k_1} \wedge \cdots \wedge e_{k_i} \wedge e'_{k'_1} \wedge \cdots \wedge e'_{k'_j})\\
&= \sum_{k=1}^{i+j} (-1)^k \frac{\mathrm{lcm}(w_1, \ldots, w_{i+j})}{\mathrm{lcm}(w_1, \ldots, \widehat{w_k}, \ldots, w_{i+j})} (E_1 \wedge \cdots \wedge \widehat{E_k} \cdots \wedge E_{i+j}).
\end{aligned}
$$

Hence the diagram is commutative. □

COROLLARY 3.3

Let I and J be ideals in R such that $IJ = I \cap J$. Let $\mathbb{M}(I)_{\bullet}$ and $\mathbb{M}(J)_{\bullet}$ denote minimal free resolutions of I and J respectively. Then $\mathbb{M}(I)_{\bullet} \otimes \mathbb{M}(J)_{\bullet}$ is a minimal free resolution of the ideal $I + J$.

Proof. The minimal free resolutions $\mathbb{M}(I)_{\bullet}$ and $\mathbb{M}(J)_{\bullet}$ are direct summands of the Taylor resolutions $\mathbb{T}(I)_{\bullet}$ and $\mathbb{T}(J)_{\bullet}$ respectively. It follows that $\mathbb{M}(I)_{\bullet} \otimes \mathbb{M}(J)_{\bullet}$ is a direct summand of $\mathbb{T}(I)_{\bullet} \otimes \mathbb{T}(J)_{\bullet} = \mathbb{T}(I + J)_{\bullet}$. Hence $\mathbb{M}(I)_{\bullet} \otimes \mathbb{M}(J)_{\bullet}$ is a free resolution of $I + J$ and it is minimal since $\mathbb{M}(I)_{\bullet}$ and $\mathbb{M}(J)_{\bullet}$ are both minimal. □

4. A simplicial characterization of transversal intersection of monomial ideals

We first introduce the basic definitions of a simplicial complex; see [1].

DEFINITION 4

A *simplicial complex* Δ on the vertex set $\{1, \ldots, m\}$ is a collection of subsets called faces or simplices satisfying the following condition: $\sigma \in \Delta$ and $\tau \subset \sigma$ imply that $\tau \in \Delta$. A simplex $\sigma \in \Delta$ of cardinality $|\sigma| = r + 1$ has dimension r and it is called an r *face* of Δ.

DEFINITION 5

A *facet* is a maximal simplex of a simplicial complex Δ. Let $\Gamma(\Delta)$ denote the set of all facets of Δ. Then, the vertex set of Δ and the set $\Gamma(\Delta)$ determine Δ completely.

DEFINITION 6

A *standard simplicial complex* of dimension $m - 1$ on the vertex set $\{1, \ldots, m\}$ is the simplicial complex whose $\Gamma(\Delta) = \{\{1, \ldots, m\}\}$.

DEFINITION 7

Let Δ_1 and Δ_2 be simplicial complexes on disjoint vertex sets. Let the vertext set of Δ_1 be $\{1, \ldots, m\}$ and that of Δ_2 be $\{m + 1, \ldots, m + p\}$. The *join of simplicial complexes* Δ_1 and Δ_2 is the simplicial complex $\Delta_1 * \Delta_2$, whose vertex set is $\{1, \ldots, m + p\}$ and

$$\Gamma(\Delta_1 * \Delta_2) = \{\sigma_1 \cup \sigma_2 \mid \sigma_1 \in \Gamma(\Delta_1),\ \sigma_2 \in \Gamma(\Delta_2)\}.$$

We now define a *frame* as a complex of K-vector spaces with a fixed basis, which encodes the minimal free resolution of a monomial ideal; see [2].

DEFINITION 8

An r-frame $\mathbb{U}_{\bullet}$ is a complex of finite K vector spaces with differential ∂ and a fixed basis satisfying the following:

(i) $\mathbb{U}_i = 0$ for $i < 0$;
(ii) $\mathbb{U}_0 = K$;
(iii) $\mathbb{U}_1 = K^r$, with basis $\{w_1, \ldots, w_r\}$;
(iv) $\partial(w_j) = 1$, for all $j = 1, \ldots, r$.

Construction of a frame of a simplicial complex. Let Δ be a simplicial complex on the vertex set $\{1, \ldots, m\}$. For each integer i, let $\Gamma_i(\Delta)$ be the set of all i dimensional faces of Δ and let $K^{\Gamma_i(\Delta)}$ denote the K-vector space generated by the basis elements e_σ for $\sigma \in \Gamma_i(\Delta)$. The *chain complex* of Δ over K is the complex $\mathcal{C}(\Delta)_{\bullet}$,

$$0 \longrightarrow K^{\Gamma_{m-1}(\Delta)} \xrightarrow{\partial_{m-1}} \cdots K^{\Gamma_i(\Delta)} \xrightarrow{\partial_i} K^{\Gamma_{i-1}(\Delta)} \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_0} K^{\Gamma_{-1}(\Delta)} \longrightarrow 0.$$

The boundary maps ∂_i are defined as

$$\partial_i(e_\sigma) = \sum_{j\in\sigma} \text{sign}(j,\sigma)e_{\sigma-j}, \quad 0 \le i \le m-1$$

such that $\text{sign}(j,\sigma) = (-1)^{t-1}$, where j is the t-th element of the set $\sigma \subset \{1,\dots,m\}$ written in increasing order. If $i < -1$ or $i > m-1$, then $K^{\Gamma_i(\Delta)} = 0$ and $\partial_i = 0$. We see that the chain complex of a simplicial complex is an m-frame.

If Δ is a standard simplicial complex of dimension $m-1$ on the vertex set $\{1,\dots,m\}$, then $\mathcal{C}(\Delta)_{\bullet}$ is acyclic and $K^{\Gamma_i(\Delta)} = K^{\binom{m}{i}}$. Therefore, the m-frame of a standard simplicial complex of dimension $m-1$ is nothing but the Koszul complex.

DEFINITION 9

Let $M = \{m_1, m_2, \dots, m_r\}$ be a set of monomials in R. An M-complex $\mathbb{G}_{\bullet}$ is a multigraded complex of finitely generated free multigraded R modules with differential Δ and a fixed multihomogeneous basis with multidegrees that satisfy

(i) $\mathbb{G}_i = 0$ for $i < 0$;
(ii) $\mathbb{G}_0 = R$;
(iii) $\mathbb{G}_1 = R(m_1) \oplus \cdots \oplus R(m_r)$;
(iv) $\Delta(w_j) = m_j$ for each basis vector $w_j \in \mathbb{G}_1$.

Construction. Let $\mathbb{U}_{\bullet}$ be an r-frame and $M = \{m_1, m_2, \dots, m_r\}$ be a set of monomials in R. The *M-homogenization* of $\mathbb{U}_{\bullet}$ is defined to be the complex $\mathbb{G}_{\bullet}$ with $\mathbb{G}_0 = R$ and $\mathbb{G}_1 = R(m_1) \oplus \cdots \oplus R(m_r)$. Let $\bar{v_1}, \dots, \bar{v_p}$ and $\bar{u_1}, \dots, \bar{u_q}$ be the given bases of $\mathbb{U}_i$ and $\mathbb{U}_{i-1}$ respectively. Let $u_1, u_2, \dots, u_q$ be the basis of $\mathbb{G}_{i-1}$ chosen in the previous step of induction. We take $v_1, \dots, v_p$ to be a basis of $\mathbb{G}_i$. If

$$\partial(\bar{v_j}) = \sum_{1\le s\le q} \alpha_{sj}\bar{u_s}$$

with coefficients $\alpha_{sj} \in k$, then set $\text{mdeg}(v_j) = \text{lcm}\left(mdeg(u_s)|\alpha_{sj} \neq 0\right)$, where $\text{lcm}(\emptyset) = 1$. We define

$$\mathbb{G}_i = \oplus_{1\le i\le p} R\left(\text{mdeg}(v_j)\right), \quad \Delta(v_j) = \Sigma_{1\le s\le q}\alpha_{sj}\frac{\text{mdeg}(v_j)}{\text{mdeg}(u_s)}u_s.$$

DEFINITION 10

Let $I \subset R$ be a monomial ideal with $G(I) = \{m_1, \dots, m_r\}$. We say that a minimal free resolution of I is supported by a simplicial complex Δ on the vertex set $\{1, \dots, r\}$ if it is isomorphic to the $G(I)$ homogenization of the chain complex $\mathcal{C}(\Delta)_{\bullet}$ of Δ.

Lemma 4.1. *Let I and J be two monomial ideals such that $I \cap J = IJ$, $|G(I)| = r$ and $|G(J)| = s$, and whose minimal free resolutions are supported by standard simplicial* *complexes Δ_I and Δ_J on the vertex sets $\{1,\ldots,r\}$ and $\{r+1,\ldots,r+s\}$ respectively. Then minimal free resolution of $I+J$ is supported by the standard simplicial complex $\Delta_I * \Delta_J$.*

Proof. We have seen that if Δ is a standard simplicial complex of dimension $m-1$ on the vertex set $\{1,\ldots,m\}$, then the m-frame of Δ is nothing but the Koszul complex. Therefore, it follows that if a minimal free resolution $\mathbb{M}(I)_{\bullet}$ of I is supported by the standard simplicial complex Δ_I, then it is actually isomorphic to the Taylor complex $\mathbb{T}(I)_{\bullet}$. Since $I\cap J = IJ$, by Theorem 2.2, we have $G(I)\cap G(J)=\emptyset$, then $|G(I)\cup G(J)| = r+s$. Again $\Delta_I * \Delta_J$ is also the standard simplicial complex of dimension $r+s-1$. Therefore, $G(I)\cup G(J)$-homogenization of $\mathcal{C}(\Delta_I * \Delta_J)_{\bullet}$ is isomorphic to the Taylor complex $\mathbb{T}(I+J)_{\bullet}$. Thus by Theorem 3.2, we have

$$\mathbb{M}(I)_{\bullet} \otimes \mathbb{M}(J)_{\bullet} \cong \mathbb{T}(I)_{\bullet} \otimes \mathbb{T}(J)_{\bullet} \cong \mathbb{T}(I+J)_{\bullet}$$

Again by Corollary 3.3, $\mathbb{M}(I)_{\bullet} \otimes \mathbb{M}(J)_{\bullet}$ is a minimal free resolution of $I+J$. Therefore, minimal free resolution of $I+J$ is supported by the standard simplicial complex $\Delta_I * \Delta_J$. □

Lemma 4.2. *Let Δ_I be a simplicial complex on vertex $\{1,\ldots,r\}$ and Δ_J be another simplicial complex on vertex $\{r+1,\ldots,r+s\}$, i.e., the vertex sets of Δ_I and Δ_J are disjoint. Then*

$$\mathcal{C}(\Delta_I)_{\bullet} \otimes \mathcal{C}(\Delta_J)_{\bullet} \cong \mathcal{C}(\Delta_I * \Delta_J)_{\bullet}$$

Proof. Let $\Gamma(\Delta_I) = \{\gamma_1,\ldots,\gamma_{l_1}\}$ and $\Gamma(\Delta_J) = \{\sigma_1,\ldots,\sigma_{l_2}\}$. We have $\Gamma(\Delta_I * \Delta_J) = \{\gamma_i \cup \sigma_j \mid 1 \le i \le l_1, 1 \le j \le l_2\}$. Now assuming $\binom{s}{t} = 0$ for $s < t$, we have $|\Gamma_p(\Delta_I)| = \sum_{i=1}^{l_1} \binom{|\gamma_i|}{p}$ for all $1 \le p \le l_1$ and $|\Gamma_q(\Delta_J)| = \sum_{j=1}^{l_2} \binom{|\sigma_j|}{q}$ for all $1 \le q \le l_2$. On the other hand, $|\Gamma_t(\Delta_I * \Delta_J)| = \sum_{j=1}^{l_2}\sum_{i=1}^{l_1} \binom{|\gamma_i \cup \sigma_j|}{t} = \sum_{j=1}^{l_2}\sum_{i=1}^{l_1} \binom{|\gamma_i| + |\sigma_j|}{t} = \sum_{p+q=t} \sum_{j=1}^{l_2}\sum_{i=1}^{l_1} \binom{|\gamma_i|}{p}\binom{|\sigma_j|}{q}$. Therefore, the map

$$\theta_t : (\mathcal{C}(\Delta_I)_{\bullet} \otimes \mathcal{C}(\Delta_J)_{\bullet})_t \longrightarrow (\mathcal{C}(\Delta_I * \Delta_J)_{\bullet})_t$$

defined by

$$\theta_t(e_\gamma \otimes e_\sigma) = e_{\gamma\cup\sigma}$$

is an isomorphism, where $\gamma \in \Delta_I, |\gamma| = p$ and $\sigma \in \Delta_J, |\gamma| = q$ and $p+q=t$. We, therefore, have to show that the following diagram commutes:

$$\begin{array}{ccc}
(\mathcal{C}(\Delta_I)_{\cdot} \otimes \mathcal{C}(\Delta_J)_{\cdot})_t & \xrightarrow{(\partial(\Delta_I)\otimes\partial(\Delta_J))_t} & (\mathcal{C}(\Delta_I)_{\cdot} \otimes \mathcal{C}(\Delta_J)_{\cdot})_{t-1} \\
\downarrow{\theta_t} & & \downarrow{\theta_{t-1}} \\
\mathcal{C}(\Delta_I * \Delta_J)_t & \xrightarrow{(\partial(\Delta_I * \Delta_J))_t} & \mathcal{C}(\Delta_I * \Delta_J)_{t-1}.
\end{array}$$

Let $\gamma \in \Delta_I, |\gamma| = p$ and $\sigma \in \Delta_J, |\gamma| = q$ be such that $p+q=t$. Then

$$\theta_{t-1} \circ (\partial(\Delta_I) \otimes \partial(\Delta_J))_t\, (e_\gamma \otimes e_\sigma)$$

$$
\begin{aligned}
&= \theta_{t-1}((\partial(\Delta_I)_p e_\gamma) \otimes e_\sigma + (-1)^p e_\gamma \otimes (\partial(\Delta_J)_p e_\sigma)) \\
&= \theta_{t-1}\left(\sum_{j\in\gamma} \mathrm{sign}(j,\gamma) e_{\gamma - j}\right) \otimes e_\sigma + (-1)^p e_\gamma \otimes \left(\sum_{s\in\sigma} \mathrm{sign}(s,\sigma) e_{\sigma - s}\right) \\
&= \sum_{j\in\gamma} \mathrm{sign}(j,\gamma) e_{\gamma\cup\sigma - j} + (-1)^p \left(\sum_{s\in\sigma} \mathrm{sign}(s,\sigma) e_{\gamma\cup\sigma - s}\right) \\
&= \sum_{j\in\gamma\cup\sigma} \mathrm{sign}(j,\gamma\cup\sigma) e_{\gamma\cup\sigma - j} = (\partial(\Delta_I * \Delta_J)_t \circ \theta_t)(e_\gamma \otimes e_\sigma).
\end{aligned}
$$

□

Theorem 4.3. *Let I and J be two monomial ideals such that $I \cap J = IJ$, $|G(I)| = r$ and $|G(J)| = s$, and their minimal free resolutions are supported by the simplicial complexes Δ_I and Δ_J on the vertex sets $\{1,\ldots,r\}$ and $\{r+1,\ldots,r+s\}$ respectively. Then minimal free resolution of $I + J$ is supported by the simplicial complex $\Delta_I * \Delta_J$.*

Proof. We have $I \cap J = IJ$. Therefore, $\mathrm{supp}(G(I)) \cap \mathrm{supp}(G(J)) = \emptyset$, by Theorem 2.2. Again, by Lemma 4.2,

$$\mathcal{C}(\Delta_I)_{\bullet} \otimes \mathcal{C}(\Delta_J)_{\bullet} \cong \mathcal{C}(\Delta_I * \Delta_J)_{\bullet}$$

Since $\mathbb{M}(I)_{\bullet}$ is the $G(I)$-homogenization of the complex $\mathcal{C}(\Delta_I)_{\bullet}$ and $\mathbb{M}(J)_{\bullet}$ is the $G(J)$-homogenization of the complex $\mathcal{C}(\Delta_J)_{\bullet}$, we can proceed in the same way as Lemma 4.2 to prove that $\mathbb{M}(I)_{\bullet} \otimes \mathbb{M}(J)_{\bullet}$ is the $G(I)\cup G(J)$–homogenization of the complex $\mathcal{C}(\Delta_I * \Delta_J)_{\bullet}$. Again, by Corollary 3.3, $\mathbb{M}(I)_{\bullet} \otimes \mathbb{M}(J)_{\bullet}$ is a minimal free resolution of $I + J$. Therefore, minimal free resolution of $I + J$ is supported by the simplicial complex $\Delta_I * \Delta_J$. □

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