Proc. Indian Acad. Sci. (Math. Sci.) (2019) 129:29
https://doi.org/10.1007/s12044-019-0467-y

# Comparison between two differential graded algebras in noncommutative geometry

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MS received 5 October 2017; revised 11 February 2018; accepted 15 February 2018; published online 11 March 2019

**Abstract.** Starting with a spectral triple, one can associate two canonical differential graded algebras (DGA) defined by Connes (Noncommutative geometry (1994) Academic Press Inc., San Diego) and Fröhlich *et al.* (*Comm. Math. Phys.* **203(1)** (1999) 119–184). For the classical spectral triples associated with compact Riemannian spin manifolds, both these DGAs coincide with the de-Rham DGA. Therefore, both are candidates for the noncommutative space of differential forms. Here we compare these two DGAs in a very precise sense.

**Keywords.** Dirac differential graded algebra; Connes' calculus; FGR differential graded algebra; spectral triple; quantum double suspension.

**2010 Mathematics Subject Classification.** Primary: 58B34; Secondary: 46L87, 16E45.

## 1. Introduction

A differential calculus on a 'space' means the specification of a differential graded algebra (DGA), often interpreted as space of forms. In classical geometry, the 'space' is a manifold and we have the de-Rham DGA, whereas in noncommutative geometry a 'space' is described by a triple called spectral triple. A spectral triple is a tuple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is an associative $\star$-algebra represented on the Hilbert space $\mathcal{H}$ and $D$ is a Dirac-type operator on $\mathcal{H}$. Associated to a spectral triple, there are two canonical DGAs defined by Connes [5] and Fröhlich *et al.* [7]. In literature, these are denoted by $\Omega^\bullet_D(\mathcal{A})$ and $\tilde{\Omega}^\bullet_D(\mathcal{A})$ respectively, and here we call them as the Dirac DGA and the FGR DGA. Note that in [2] we have called the Dirac DGA as the Connes' calculus. It should be noted that for the classical spectral triple associated with compact Riemannian spin manifolds, both these DGAs coincide with the de-Rham DGA ([5,7]). Therefore, both are candidates to be declared as noncommutative space of forms. Moreover, they are same for the noncommutative torus (page 172 in [7]) but not for the $SU_q(2)$ [3]. Hence, it is natural to ask if there is any way to compare these two DGAs so that one can declare one of them as truly

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noncommutative space of forms. This is important because both being generalization of the classical de-Rham forms to the noncommutative set up, any notion in noncommutative geometry involving the noncommutative space of forms, e.g. the Yang-Mills functional [5] can be defined using either the Dirac DGA or the FGR DGA. Hence, a comparison is needed to overcome the difficulty of choice between these two DGAs. This is precisely the goal of our investigation in this article and the comparison is done through explicit computation of these DGAs for a family of spectral triples. In the literature, these DGAs have been computed in very few cases, e.g. noncommutative torus, $SU_q(2)$. This indicates that probably these are difficult to compute and we had no clue on how to compare them. Recently, the authors have identified suitable hypotheses which allow the computation of the Dirac DGA $\Omega_D^\bullet$ for a class of spectral triples. This gives the first systematic computation of $\Omega_D^\bullet$ for a large family of spectral triples [2]. In this article, we compute the FGR DGA $\tilde{\Omega}_D^\bullet$ for the same family of spectral triples, and this leads to a comparison between these two DGAs.

To describe our computation in detail, we recall the concept of the quantum double suspension (QDS) of a $C^*$-algebra $\mathcal{A}$, denoted by $\Sigma^2\mathcal{A}$, introduced by Hong and Szymanski in [8]. Later, QDS of a spectral triple was introduced by Chakraborty and Sundar [4]. We record here a few significance of QDS.

*Significance of QDS*:

(a) Quantum even- and odd-dimensional spheres are produced by iterating QDS to two points and the circle, respectively [8].
(b) Noncommutative analogues of $n$-dimensional balls are obtained by repeated application of the QDS to the classical low-dimensional spaces [9].
(c) If we have one spectral triple $(\mathcal{A}, \mathcal{H}, D)$, then by iterating QDS we produce many spectral triples. Thus, by iterating QDS on the classical cases of manifolds one produces genuine noncommutative spectral triples. Moreover, finite summability and $\Theta$-summability are preserved under the iteration.
(d) All the torus-equivariant spectral triples on the odd-dimensional quantum spheres are obtained by iterating QDS to the spectral triple $(C^\infty(S^1), L^2(S^1), -i\frac{\mathrm{d}}{\mathrm{d}\theta})$.
(e) Most importantly, QDS produces a class of examples of regular spectral triples having simple dimension spectrum [4], essential in the context of local index formula of Connes and Moskovici [6].

This article adds one more significance to the above list namely, QDS provides a comparison between the Dirac DGA and the FGR DGA and establishes the Dirac DGA as a more appropriate generalization of the classical de-Rham DGA to the noncommutative set-up. We work here under the following mild hypotheses on a spectral triple $(\mathcal{A}, \mathcal{H}, D)$:

(1) $[D, a]F - F[D, a]$ is a compact operator for all $a \in \mathcal{A}$, where $F$ is the sign of the operator $D$,
(2) $\mathcal{H}^\infty := \bigcap_{k\geq 1} \mathcal{D}\mathrm{om}(D^k)$ is a left $\mathcal{A}$-module, and $[D, \mathcal{A}] \subseteq \mathcal{A} \otimes \mathcal{E}\mathrm{nd}_{\mathcal{A}}(\mathcal{H}^\infty) \subseteq \mathcal{E}\mathrm{nd}_{\mathbb{C}}(\mathcal{H}^\infty)$.

The notable features of these hypotheses are firstly, the spectral triple associated with a first order differential operator on a manifold will always satisfy them and secondly, they are stable under the quantum double suspension. The authors have computed $\Omega_D^\bullet$ for the quantum double suspended spectral triple $(\Sigma^2\mathcal{A}, \Sigma^2\mathcal{H}, \Sigma^2 D)$ in [2] under these conditions. It turns out that the FGR DGA becomes almost trivial for $(\Sigma^2\mathcal{A}, \Sigma^2\mathcal{H}, \Sigma^2 D)$ in the sense that it does not reflect any information about $(\mathcal{A}, \mathcal{H}, D)$. This phenomenon was observed in [3] for the $SU_q(2)$. Since the torus equivariant spectral triples on the odd-dimensional quantum spheres are obtained through iterated QDS on the spectral triple $(C^\infty(S^1), L^2(S^1), -i\frac{\mathrm{d}}{\mathrm{d}\theta})$, this article also extends the earlier work of Chakraborty and Pal [3]. This helps us to conclude, in view of [2], that the Dirac DGA is more informative than the FGR DGA.

The organization of this paper is as follows. In section 2, we discuss Dirac DGA $\Omega_D^\bullet$, the quantum double suspension and obtain a few results. Section 3 mainly deals with the computation of the FGR DGA $\tilde{\Omega}^\bullet_{\Sigma^2 D}(\Sigma^2\mathcal{A})$ for QDS, which finally leads us to the comparison between Connes DGA and FGR DGA.

## 2. Dirac DGA and the quantum double suspension

In this section, we recall the definition of Dirac DGA $\Omega_D^\bullet$ from [5], and the quantum double suspension from [4,8].

DEFINITION 2.1

A *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ over an involutive associative algebra $\mathcal{A}$ consists of the following:

(1) a $\star$-representation $\pi$ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$,
(2) an unbounded self-adjoint operator $D$ acting on $\mathcal{H}$,
(3) $D$ has compact resolvent and $[D, a]$ extends to a bounded operator on $\mathcal{H}$ for every $a \in \mathcal{A}$.

We will assume that $\mathcal{A}$ is unital and $\pi$ is a unital representation. If $|D|^{-p}$ is in the ideal of Dixmier traceable operators $\mathcal{L}^{(1,\infty)}$, then we say that the spectral triple is $p$-summable. In literature, this is sometimes denoted by $p^+$-summable, $(p, \infty)$-summable, etc. Moreover, if there is a $\mathbb{Z}_2$-grading $\gamma \in \mathcal{B}(\mathcal{H})$ such that $\gamma$ commutes with every element of $\mathcal{A}$ and anticommutes with $D$, then the spectral triple $(\mathcal{A}, \mathcal{H}, D, \gamma)$ is said to be an *even spectral triple*. Associated to every spectral triple, we have the following differential graded algebra (DGA).

DEFINITION 2.2 [2,5]

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple and $\Omega^\bullet(\mathcal{A}) = \bigoplus_{k=0}^\infty \Omega^k(\mathcal{A})$ be the reduced universal differential graded algebra over $\mathcal{A}$. Here, $\Omega^k(\mathcal{A}) := \mathrm{span}\{a_0 da_1 \ldots da_k\colon a_i \in \mathcal{A}, i = 1, \ldots, k\}$, $d$ being the universal differential. With the convention $(da)^* = -da^*$, we get a $\star$-representation $\pi$ of $\Omega^\bullet(\mathcal{A})$ on $\mathcal{Q}(\mathcal{H}) := \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, given by

$$\pi(a_0 da_1 \ldots da_k) := a_0[D, a_1] \ldots [D, a_k] + \mathcal{K}(\mathcal{H});\ a_j \in \mathcal{A}.$$

Let $J_0^{(k)} = \{\omega \in \Omega^k : \pi(\omega) = 0\}$ and $J' = \bigoplus J_0^{(k)}$. Since $J'$ fails to be a differential ideal in $\Omega^\bullet$, consider $J^\bullet = \bigoplus J^{(k)}$, where $J^{(k)} = J_0^{(k)} + dJ_0^{(k-1)}$. Then $J^\bullet$ becomes a differential graded two-sided ideal in $\Omega^\bullet$ and hence, the quotient $\Omega_D^\bullet = \Omega^\bullet/J^\bullet$ becomes a differential graded algebra, called the Connes' calculus or the Dirac DGA.

The representation $\pi$ gives the following isomorphism:

$$\Omega_D^k \cong \pi(\Omega^k)/\pi(dJ_0^{k-1}), \quad \forall k \geq 1. \tag{2.1}$$

The differential $d$ on $\Omega^\bullet(\mathcal{A})$ induces a differential, denoted again by $d$, on the complex $\Omega^\bullet_D(\mathcal{A})$ so that we get a chain complex $(\,\Omega^\bullet_D(\mathcal{A}), d\,)$ and a chain map $\pi_D : \Omega^\bullet(\mathcal{A}) \longrightarrow \Omega^\bullet_D(\mathcal{A})$ such that the following diagram

$$\begin{array}{ccc} \Omega^\bullet(\mathcal{A}) & \xrightarrow{\pi_D} & \Omega^\bullet_D(\mathcal{A}) \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d} \\ \Omega^{\bullet+1}(\mathcal{A}) & \xrightarrow[\pi_D]{} & \Omega^{\bullet+1}_D(\mathcal{A}) \end{array}$$

commutes. Note that $\Omega^\bullet_D(\mathcal{A})$ can be defined for non-unital algebra $\mathcal{A}$ as well as that prescribed in [2], after Remark 2.3.

*Lemma* 2.3. *If there is a decreasing filtration*

$$\mathcal{A} = \mathcal{A}_0 \supseteq \mathcal{A}_{-1} \supseteq \cdots \supseteq \{0\}$$

*of subspaces of* $\mathcal{A}$, *then* $\Omega^\bullet_D(\mathcal{A})$ *becomes a filtered algebra.*

*Proof.* Let $J_0^{k,n} = \ker(\pi^k|_{\Omega^k(\mathcal{A}_n)})$. Then $J_0^{k,n} \subseteq J_0^{k,n+1}$. If we let $J^{k,n} = J_0^{k,n} + dJ_0^{k-1,n}$, then $J^{k,n} \subseteq J^{k,n+1}$. We have

$$\Phi^{k,n} : \frac{\Omega^k(\mathcal{A}_n)}{J^{k,n}} \hookrightarrow \frac{\Omega^k(\mathcal{A}_{n+1})}{J^{k,n}} \twoheadrightarrow \frac{\Omega^k(\mathcal{A}_{n+1})}{J^{k,n+1}}$$

with

$$\begin{aligned} \mathrm{Ker}(\Phi^{k,n}) &= \{\omega \in \Omega^k(\mathcal{A}_n) : \omega \in J^{k,n+1}\} \\ &= J^{k,n+1} \cap \Omega^k(\mathcal{A}_n)/J^{k,n}, \end{aligned}$$

and

$$\mathcal{I}\mathrm{m}(\Phi^{k,n}) = \Omega^k(\mathcal{A}_n)/\,\Omega^k(\mathcal{A}_n) \cap J^{k,n+1}.$$

This gives a filtration on $\Omega^\bullet_D(\mathcal{A})$. □

PROPOSITION 2.4

*The associated graded algebra of the filtered algebra* $\Omega^\bullet_D(\mathcal{A})$ *is given by*

$$\mathcal{G} = \bigoplus_{n\le 0} \bigoplus_{p\ge 0} \frac{\Omega^p(\mathcal{A}_n)}{\Omega^p(\mathcal{A}_{n-1}) + J^{p,n}}.$$

*Proof.* By Lemma 2.3, the filtration on $\Omega_D^\bullet(\mathcal{A})$ is given by $\mathcal{F}_n = \bigoplus_{k\geq 0} \Omega^k(\mathcal{A}_n)/J^{k,n}$. Hence, the associated graded algebra is given by $\mathcal{G} = \bigoplus_{n\leq 0} \mathcal{G}_n$, where

$$\begin{aligned}
\mathcal{G}_n &= \mathcal{F}_n/\mathcal{F}_{n-1} \\
&= \frac{\bigoplus_{p\geq 0} \Omega^p(\mathcal{A}_n)/J^{p,n}}{\bigoplus_{q\geq 0} \Omega^q(\mathcal{A}_{n-1})/J^{q,n-1}} \\
&= \bigoplus_{p\geq 0} \frac{\Omega^p(\mathcal{A}_n)/J^{p,n}}{\Omega^p(\mathcal{A}_{n-1})/J^{p,n-1}} \\
&= \bigoplus_{p\geq 0} \frac{\Omega^p(\mathcal{A}_n)/J^{p,n}}{\mathcal{I}\mathrm{m}(\Phi^{p,n-1})} \\
&= \bigoplus_{p\geq 0} \frac{\Omega^p(\mathcal{A}_n)/J^{p,n}}{\Omega^p(\mathcal{A}_{n-1})/\Omega^p(\mathcal{A}_{n-1}) \cap J^{p,n}} \\
&= \bigoplus_{p\geq 0} \frac{\Omega^p(\mathcal{A}_n)}{\Omega^p(\mathcal{A}_{n-1}) + J^{p,n}}.
\end{aligned}$$

□

Now we define the quantum double suspension (QDS) of $C^*$-algebras and spectral triples.

*Notation.*

(1) We denote by '$l$' the left shift operator on $\ell^2(\mathbb{N})$, defined on the standard orthonormal basis $\{e_n\}$ by $l(e_n) = e_{n-1}$ for $n \geq 1$ and $l(e_0) = 0$.
(2) '$N$' be the number operator on $\ell^2(\mathbb{N})$ defined by $N(e_n) = ne_n$.
(3) '$u$' denotes the rank one projection $|e_0\rangle\langle e_0| := I - l^*l$.
(4) $\mathcal{K}$ denotes the space of compact operators on $\ell^2(\mathbb{N})$.

DEFINITION 2.5 [8]

Let $\mathcal{A}$ be a unital $C^*$-algebra. The quantum double suspension of $\mathcal{A}$, denoted by $\Sigma^2\mathcal{A}$ is the $C^*$-algebra generated by $a \otimes u$ and $1 \otimes l$ in $\mathcal{A} \otimes \mathscr{T}$, where $\mathscr{T}$ is the Toeplitz algebra.

There is a symbol map $\sigma : \mathscr{T} \longrightarrow C(S^1)$ which sends $l$ to the standard unitary generator $z$ of $C(S^1)$ and one gets the following short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathscr{T} \xrightarrow{\sigma} C(S^1) \longrightarrow 0.$$

If $\rho$ denotes the restriction of $1\otimes\sigma$ to $\Sigma^2\mathcal{A}$, then one has the following short exact sequence

$$0 \longrightarrow \mathcal{A}\otimes\mathcal{K} \longrightarrow \Sigma^2\mathcal{A} \xrightarrow{\rho} C(S^1) \longrightarrow 0.$$

There is a $\mathbb{C}$-linear splitting map $\sigma'$ from $C(S^1)$ to $\Sigma^2\mathcal{A}$ which sends the standard unitary generator $z$ of $C(S^1)$ to $1 \otimes l$, and yields the following $\mathbb{C}$-vector spaces (not as algebras) isomorphism:

$$\Sigma^2\mathcal{A} \cong (\mathcal{A}\otimes\mathcal{K}) \bigoplus C(S^1).$$

Notice that $\sigma'$ is injective since it has a left inverse $\rho$ and hence, any $f \in C(S^1)$ can be identified with $1 \otimes \sigma'(f) \in \Sigma^2\mathcal{A}$. For $f = \sum_n \lambda_n z^n \in C(S^1)$, we write $\sigma'(f) := \sum_{n\geq 0} \lambda_n l^n + \sum_{n>0} \lambda_{-n} l^{*n}$. Now let $\mathcal{A}$ be a dense $\star$-subalgebra of a $C^*$-algebra $\mathbb{A}$. Define

$$\Sigma^2_{\text{alg}}\mathcal{A} := \text{span}\{a \otimes T, 1 \otimes l^m, 1 \otimes (l^*)^n : a \in \mathcal{A}, T \in \mathbb{S}(\ell^2(\mathbb{N})), m, n \geq 0\},$$

where $\mathbb{S}(\ell^2(\mathbb{N})) := \{T = (\alpha_{ij}) \colon \sum_{i,j}(1+i+j)^k|\alpha_{ij}| < \infty \ \forall k \geq 0\}$ is the space of Schwartz class operators on $\ell^2(\mathbb{N})$. Clearly, $\Sigma^2_{\text{alg}}\mathcal{A}$ is a dense subalgebra of $\Sigma^2\mathbb{A}$ and we have the following $\mathbb{C}$-vector spaces (not as algebras) isomorphism at the level of subalgebra:

$$\Sigma^2_{\text{alg}}\mathcal{A} \cong (\mathcal{A} \otimes \mathbb{S}(\ell^2(\mathbb{N}))) \bigoplus \mathbb{C}[z, z^{-1}]\,.$$

DEFINITION 2.6 [4]

For any spectral triple $(\mathcal{A}, \mathcal{H}, D)$, $(\Sigma^2_{\text{alg}}\mathcal{A}, \Sigma^2\mathcal{H} := \mathcal{H} \otimes \ell^2(\mathbb{N}), \Sigma^2 D := D \otimes I + F \otimes N)$ becomes a spectral triple, where $F$ is the sign of the operator $D$ and $N$ is the number operator on $\ell^2(\mathbb{N})$. This is called the quantum double suspension of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$.

It is easy to see that if $(\mathcal{A}, \mathcal{H}, D)$ is $p$-summable, then $(\Sigma^2_{\text{alg}}\mathcal{A}, \Sigma^2\mathcal{H}, \Sigma^2 D)$ is a $(p+1)$-summable spectral triple. Notice that for any $f \in \mathbb{C}[z, z^{-1}]$, we have $[\Sigma^2 D, 1 \otimes \sigma'(f)] = F \otimes [N, f]$. The finite subalgebra $(\Sigma^2_{\text{alg}}\mathcal{A})_{\text{fin}}$ is generated by $a \otimes T$ and $\sum_{0\leq n<\infty} \lambda_n l^n + \sum_{0<n<\infty} \lambda_{-n} l^{*n}$, where $a \in \mathcal{A}$ and $T \in \mathcal{B}\left(\ell^2(\mathbb{N})\right)$ is a finitely supported matrix.

*Remark* 2.7. In [5], Connes represented $\Omega^\bullet(\mathcal{A})$ on $\mathcal{B}(\mathcal{H})$ instead of on $\mathcal{Q}(\mathcal{H})$. But the explicit computation of $\Omega^\bullet_{\Sigma^2 D}((\Sigma^2_{\text{alg}}\mathcal{A})_{\text{fin}})$ is very difficult, even in the particular cases. In [2], the authors have computed $\Omega^\bullet_{\Sigma^2 D}((\Sigma^2_{\text{alg}}\mathcal{A})_{\text{fin}})$ following the prescription given in Definition 2.2. The justification for this is also discussed in [2].

The computation of $\Omega^\bullet_{\Sigma^2 D}((\Sigma^2_{\text{alg}}\mathcal{A})_{\text{fin}})$ has been done in [2] under the following conditions on spectral triples $(\mathcal{A}, \mathcal{H}, D)$.

*Conditions:*

(A) $[D, a]F - F[D, a]$ is a compact operator for all $a \in \mathcal{A}$, where $F = \text{sign}(D)$.
(B) $\mathcal{H}^\infty := \bigcap_{k\geq 1} \mathcal{D}\text{om}(D^k)$ is a left $\mathcal{A}$-module and $[D, \mathcal{A}] \subseteq \mathcal{A} \otimes \mathcal{E}\text{nd}_{\mathcal{A}}(\mathcal{H}^\infty) \subseteq \mathcal{E}\text{nd}_{\mathbb{C}}(\mathcal{H}^\infty)$.

The notable features of these conditions are given by the following proposition.

PROPOSITION 2.8 [2]

*These conditions are valid for the classical case, where $\mathcal{A} = C^\infty(\mathbb{M})$ and $D$ is a first-order differential operator. Moreover, if a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ satisfies these conditions, then the quantum double suspended spectral triple $(\Sigma^2_{\text{alg}}\mathcal{A}, \Sigma^2\mathcal{H}, \Sigma^2 D)$ also satisfies them.*

*Notation.*

(1) In this article, we will work with $(\Sigma^2_{\text{alg}}\mathcal{A})_{\text{fin}}$ and denote it by $\Sigma^2\mathcal{A}$ for notational brevity.

(2) For all $f \in \mathbb{C}[z, z^{-1}]$, we denote $[N, f]$ by $f'$ for notational brevity.
(3) '$\mathcal{S}$' denotes the space of finitely supported matrices in $\mathcal{B}(\ell^2(\mathbb{N}))$.
(4) $(e_{ij})$ will denote infinite matrix with 1 at the $ij$-th place and zero elsewhere. We call it an elementary matrix.

The notion of unitary equivalence of spectral triples forms a category of spectral triples. That is, we have the following definition.

DEFINITION 2.9

The objects of the category $\mathcal{S}$pec are spectral triples $(\mathcal{A}, \mathcal{H}, D)$. A morphism between two such objects $(\mathcal{A}_i, \mathcal{H}_i, D_i)$, $i = 1, 2$ is a tuple $(\phi, \Phi)$, where $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ is a unital algebra morphism and $\Phi : \mathcal{H}_1 \to \mathcal{H}_2$ is a unitary which intertwines the algebra representations and the Dirac operators $D_1, D_2$.

PROPOSITION 2.10

*The association* $\mathcal{F} : (\mathcal{A}, \mathcal{H}, D) \longmapsto \Omega^\bullet_D(\mathcal{A})$ *gives a covariant functor from* $\mathcal{S}$pec *to* DGA, *the category of differential graded algebras over* $\mathbb{C}$.

*Proof.* Consider two objects $(\mathcal{A}_1, \mathcal{H}_1, D_1), (\mathcal{A}_2, \mathcal{H}_2, D_2) \in \mathcal{O}b(\mathcal{S}pec)$ and suppose there is a morphism $(\phi, \Phi) : (\mathcal{A}_1, \mathcal{H}_1, D_1) \longrightarrow (\mathcal{A}_2, \mathcal{H}_2, D_2)$. Define

$$\Psi : \Omega^\bullet_{D_1}(\mathcal{A}_1) \longrightarrow \Omega^\bullet_{D_2}(\mathcal{A}_2)$$
$$\left[\sum a_0 \prod_{i=1}^{n} [D_1, a_i]\right] \longmapsto \left[\sum \phi(a_0) \prod_{i=1}^{n} [D_2, \phi(a_i)]\right]$$

for all $a_j \in \mathcal{A}_1$, $n \geq 0$. To show $\Psi$ is well-defined, we must show that $\Psi(\pi(d_1 J_0^m)) \subseteq \pi(d_2 J_0^m)$ for all $m \geq 1$, where $d_1, d_2$ are the universal differentials for $\Omega^\bullet(\mathcal{A}_1), \Omega^\bullet(\mathcal{A}_2)$ respectively. Observe that

$$\Phi \circ \left(\sum a_0 \prod_{i=1}^{n} [D_1, a_i]\right) = \left(\sum \phi(a_0) \prod_{i=1}^{n} [D_2, \phi(a_i)]\right) \circ \Phi. \tag{2.2}$$

Consider an arbitrary element $\xi \in \pi(d_1 J_0^n)$. By definition, $\xi = \sum \prod_{i=0}^{n} [D_1, a_i] \in \pi(d_1 J_0^n)$ such that $\sum a_0 \prod_{i=1}^{n} [D_1, a_i] = 0$. Now, using equation (2.2) and $\Phi$ is a unitary (surjectivity is enough), we have

$$\sum \phi(a_0) \prod_{i=1}^{n} [D_2, \phi(a_i)] = 0.$$

This shows well-definedness of $\Psi$. Now it is easy to check that $\Psi$ is a DGA morphism. □

*Remark* 2.11. One can weaken the definition of morphism of spectral triples by demanding the map $\Phi$ to be only linear. This was defined in [1]. But for Proposition 2.10 to hold, one requires surjectivity of $\Phi$. However, the reason why we have assumed $\Phi$ to be unitary will be justified in the next section.

*Lemma* 2.12. *The quantum double suspension of a spectral triple is a covariant functor* $\Sigma^2$ *on the category* $\mathcal{S}$pec.

*Proof.* Easy to verify. □

PROPOSITION 2.13

*The functor* $\Sigma^2$ *gives an equivalence* $\Sigma^2(\mathcal{S}\text{pec}) \cong \mathcal{S}\text{pec}$ *of categories, and hence* $\Sigma^2$ *is not a constant functor.*

*Proof.* Recall that as a linear space $\Sigma^2\mathcal{A} = \mathcal{A}\otimes\mathcal{S} \bigoplus \mathbb{C}[z, z^{-1}]$ and $\Sigma^2 D = D\otimes 1 + F\otimes N$. Suppose $(\phi, \Phi) : (\Sigma^2\mathcal{A}_1, \Sigma^2\mathcal{H}_1, \Sigma^2 D_1) \to (\Sigma^2\mathcal{A}_2, \Sigma^2\mathcal{H}_2, \Sigma^2 D_2)$ is an isomorphism in the sense of Definition 2.9. One can replace $N$ by $N + g(N)$ for a suitable function $g$ such that $(\Sigma^2\mathcal{A}, \Sigma^2\mathcal{H}, D \otimes I + F \otimes (N + g(N)))$ remains an honest spectral triple, and computations done in [2] does not get affected. It is possible to choose such a function $g$ so that the following map

$$\begin{aligned} \sigma(|D_1|) \times \sigma(N + g(N)) &\longrightarrow \mathbb{N}_+ \\ (\lambda_n, n + g(n)) &\longmapsto \lambda_n + n + g(n) \end{aligned}$$

becomes one-to-one. This is possible since $D_1$ has discrete spectrum. This will imply that any unitary $\tilde{\Phi} : \mathcal{H}_1 \otimes \ell^2(\mathbb{N}) \to \mathcal{H}_2 \otimes \ell^2(\mathbb{N})$ is of the form $\Phi \otimes 1$, where $\Phi : \mathcal{H}_1 \to \mathcal{H}_2$ is a unitary. This will assure that algebra isomorphism $\tilde{\phi} : \Sigma^2\mathcal{A}_1 \to \Sigma^2\mathcal{A}_2$ is of the form $\phi \otimes 1 \bigoplus 1$, where $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ is an algebra isomorphism. This shows that

$$\begin{aligned} &(\Sigma^2\mathcal{A}_1, \Sigma^2\mathcal{H}_1, \Sigma^2 D_1) \cong (\Sigma^2\mathcal{A}_2, \Sigma^2\mathcal{H}_2, \Sigma^2 D_2) \\ &\Longrightarrow (\mathcal{A}_1, \mathcal{H}_1, D_1) \cong (\mathcal{A}_2, \mathcal{H}_2, D_2). \end{aligned}$$

The other implication '$\Leftarrow$' is obvious. □

Recall Theorem (3.22) from [2].

**Theorem 2.14 [2].**

*For the spectral triple* $(\Sigma^2\mathcal{A}, \Sigma^2\mathcal{H}, \Sigma^2 D)$, *we have*

(1) $\Omega^1_{\Sigma^2 D}(\Sigma^2\mathcal{A}) \cong \Omega^1_D(\mathcal{A}) \otimes \mathcal{S} \bigoplus \Sigma^2\mathcal{A}$.
(2) $\Omega^n_{\Sigma^2 D}(\Sigma^2\mathcal{A}) \cong \Omega^n_D(\mathcal{A}) \otimes \mathcal{S}$, *for all* $n \geq 2$.
(3) *The differential* $\delta^0 : \Sigma^2\mathcal{A} \longrightarrow \Omega^1_{\Sigma^2 D}(\Sigma^2\mathcal{A})$ *is given by*

$$a \otimes T + f \longmapsto [D,a] \otimes T \bigoplus \big(a \otimes [N,T] + f'\big).$$

(4) *The differential* $\delta^1 : \Omega^1_{\Sigma^2 D}(\Sigma^2\mathcal{A}) \longrightarrow \Omega^2_{\Sigma^2 D}(\Sigma^2\mathcal{A})$ *is given by*

$$\delta^1|_{\Omega^1_D(\mathcal{A})\otimes\mathcal{S}} = d^1 \otimes 1 \quad \textit{and} \quad \delta^1|_{\Sigma^2\mathcal{A}} = 0.$$

(5) *The differential* $\delta^n : \Omega^n_{\Sigma^2 D}(\Sigma^2\mathcal{A}) \longrightarrow \Omega^{n+1}_{\Sigma^2 D}(\Sigma^2\mathcal{A})$ *is given by* $\delta^n = d^n \otimes 1$ *for all* $n \geq 2$.

*Here,* $d : \Omega^\bullet_D(\mathcal{A}) \longrightarrow \Omega^{\bullet+1}_D(\mathcal{A})$ *is the differential of the Dirac DGA.*

*Remark* 2.15. The DGA $\Omega^{\bullet}_{\Sigma^2 D}(\Sigma^2\mathcal{A})$ can be described alternatively as follows. Notice that for the (graded) algebra $\Omega^{\bullet}_{D}(\mathcal{A})$, one can consider $\Sigma^2(\Omega^{\bullet}_{D}(\mathcal{A})) = \Omega^{\bullet}_{D}(\mathcal{A}) \otimes \mathcal{S} \bigoplus \mathbb{C}[z, z^{-1}]$. This is a graded algebra whose degree zero term is $\mathcal{A} \otimes \mathcal{S} \bigoplus \mathbb{C}[z, z^{-1}] = \Sigma^2\mathcal{A}$ and the degree $n$ term is $\Omega^{n}_{D}(\mathcal{A}) \otimes \mathcal{S}$ for $n \geq 1$. That is,

$$\Sigma^2(\Omega^{\bullet}_{D}(\mathcal{A}) = \Sigma^2\mathcal{A} \bigoplus \Omega^{1}_{D}(\mathcal{A}) \otimes \mathcal{S} \bigoplus \Omega^{2}_{D}(\mathcal{A}) \otimes \mathcal{S} \bigoplus \cdots$$

as a graded algebra. Then, as a graded algebra $\Omega^{\bullet}_{\Sigma^2 D}(\Sigma^2\mathcal{A}) = \Sigma^2(\Omega^{\bullet}_{D}(\mathcal{A})) \bigoplus \Sigma^2\mathcal{A}$, where $\Sigma^2\mathcal{A}$ sits in the degree 1 term.

COROLLARY 2.16

*The cohomology of* $(\Omega^{\bullet}_{\Sigma^2 D}(\Sigma^2\mathcal{A}), \delta^{\bullet}$ *is given by*

(1) $H^0(\Sigma^2\mathcal{A}) = H^0(\mathcal{A}) \otimes \mathcal{S}_{\text{diag}} \bigoplus \mathbb{C}$,
(2) $H^1(\Sigma^2\mathcal{A}) = H^1(\mathcal{A}) \otimes \mathcal{S}_{\text{diag}} \bigoplus \mathcal{A} \otimes \mathcal{S}_{\text{diag}} \bigoplus \text{Ker}(d^1) \otimes \mathcal{S}_{\text{off}} \bigoplus \mathbb{C}$,
(3) $H^n(\Sigma^2\mathcal{A}) = H^n(\mathcal{A}) \otimes \mathcal{S}$, *for all* $n \geq 2$,

*where* $H^{\bullet}(\mathcal{A})$ *denotes the cohomology of* $(\Omega^{\bullet}_{D}(\mathcal{A}), d^{\bullet})$ *and* $\mathcal{S}_{\text{diag}}$ ,$\mathcal{S}_{\text{off}}$ *denote the spaces of finitely supported diagonal and off-diagonal matrices respectively.*

*Proof.* We have $H^0(\Sigma^2\mathcal{A}) = \text{Ker}(\delta^0)$. A general element of $\mathcal{A} \otimes \mathcal{S}$ can be written in terms of elementary matrices of the form $a \otimes T = \sum_{i,j} a_{ij} \otimes e_{ij}$. We now have the following:

$$\begin{aligned}\text{Ker}(\delta^0) &= \{a_{ij} = 0 \ \ for\ i \neq j\ ,\ [D, a_{ii}] = 0\ \forall\, i\ ,\ f = \{\text{constant}\} \\ &= \text{Ker}(d^0) \otimes \mathcal{S}_{\text{diag}} \bigoplus \mathbb{C} \\ &= H^0(\mathcal{A}) \otimes \mathcal{S}_{\text{diag}} \bigoplus \mathbb{C}\,.\end{aligned}$$

This proves part (1). For part (2), observe that

$$\text{Ker}(\delta^1) = \text{Ker}(d^1) \otimes \mathcal{S} \bigoplus \Sigma^2\mathcal{A}$$

and $\text{Im}(\delta^0) = \text{Im}(\delta^0|_{\mathcal{A}\otimes\mathcal{S}}) \bigoplus \mathbb{C}[z, z^{-1}]/\mathbb{C}$. Hence, we need to determine $\frac{\text{Ker}(d^1)\otimes\mathcal{S} \bigoplus \mathcal{A}\otimes\mathcal{S}}{\text{Im}(\delta^0|_{\mathcal{A}\otimes\mathcal{S}})}$. Now

$$\begin{aligned}&\delta^0 :\ \mathcal{A} \otimes \mathcal{S}_{\text{off}} \bigoplus \mathcal{A} \otimes \mathcal{S}_{\text{diag}} \\ &\quad \longrightarrow (\text{Ker}(d^1) \oplus \mathcal{A}) \otimes \mathcal{S}_{\text{off}} \bigoplus (\text{Ker}(d^1) \oplus \mathcal{A}) \otimes \mathcal{S}_{\text{diag}} \\ &\left(\sum_{i\neq j} a_{ij} \otimes e_{ij}\, , \sum_{i} b_i \otimes e_{ii}\right) \\ &\quad \longmapsto \left(\sum_{i\neq j}(d^0 a_{ij}, (i-j)a_{ij}) \otimes e_{ij}\, , \sum_{i}(d^0 b_i, 0) \otimes e_{ii}\right).\end{aligned}$$

Hence $\delta^0 = \delta_1^0 \oplus \delta_2^0$. Observe that $\mathrm{Im}(\delta_2^0) = \mathrm{Im}(d^0) \otimes \mathcal{S}_{\mathrm{diag}}$. Now

$$\Psi : \frac{(\mathrm{Ker}(d^1) \oplus \mathcal{A}) \otimes \mathcal{S}_{\mathrm{off}}}{\mathrm{Im}(\delta_1^0)} \longrightarrow \mathrm{Ker}(d^1) \otimes \mathcal{S}_{\mathrm{off}}$$
$$\sum_{i \neq j} [(\omega_{ij}, a_{ij}) \otimes e_{ij}] \longmapsto \sum_{i \neq j} (\omega_{ij} - (i-j)^{-1} d^0 a_{ij}) \otimes e_{ij}$$

is a well-defined linear isomorphism. Hence,

$$\begin{aligned} H^1(\Sigma^2\mathcal{A}) &= \frac{(\mathrm{Ker}(d^1) \oplus \mathcal{A}) \otimes \mathcal{S}_{\mathrm{off}}}{\mathrm{Im}(\delta_1^0)} \bigoplus \frac{\mathrm{Ker}(d^1)}{\mathrm{Im}(d^0)} \otimes \mathcal{S}_{\mathrm{diag}} \bigoplus \mathcal{A} \otimes \mathcal{S}_{\mathrm{diag}} \bigoplus \mathbb{C} \\ &= \mathrm{Ker}(d^1) \otimes \mathcal{S}_{\mathrm{off}} \bigoplus H^1(\mathcal{A}) \otimes \mathcal{S}_{\mathrm{diag}} \bigoplus \mathcal{A} \otimes \mathcal{S}_{\mathrm{diag}} \bigoplus \mathbb{C}. \end{aligned}$$

This proves part (2), and part (3) is easy to verify. □

If $\mathcal{A}$ comes with a decreasing filtration

$$\mathcal{A} = \mathcal{A}_0 \supseteq \mathcal{A}_{-1} \supseteq \cdots \supseteq \{0\},$$

then the algebra $\Sigma^2\mathcal{A}$ has the induced filtration. By Lemma (2.3), $\Omega^{\bullet}_{\Sigma^2 D}(\Sigma^2\mathcal{A})$ then becomes a filtered algebra.

PROPOSITION 2.17

*The associated graded algebra of the filtered algebra* $\Omega^{\bullet}_{\Sigma^2 D}(\Sigma^2\mathcal{A})$ *is*

$$\begin{aligned} \mathcal{G}(\Sigma^2\mathcal{A}) = \bigoplus_{n \leq 0} \Bigg( & \frac{\mathcal{A}_n}{\mathcal{A}_{n-1}} \bigoplus \left( \frac{\Omega^1(\mathcal{A}_n)}{\Omega^1(\mathcal{A}_{n-1})} \oplus \frac{\mathcal{A}_n}{\mathcal{A}_{n-1}} \right) \\ & \bigoplus \left( \bigoplus_{p \geq 2} \frac{\Omega^p(\mathcal{A}_n)}{\Omega^p(\mathcal{A}_{n-1}) + dJ_0^{p-1}(\mathcal{A}_n)} \right) \Bigg) \otimes \mathcal{S}. \end{aligned}$$

*Hence*, $\mathcal{G}(\Sigma^2\mathcal{A})$ *depends only on the filtration of* $\mathcal{A}$.

*Proof.* By Lemma 2.4, the associated graded algebra is

$$\mathcal{G}(\Sigma^2\mathcal{A}) = \bigoplus_{n \leq 0} \bigoplus_{p \geq 0} \frac{\Omega^p(\Sigma^2\mathcal{A}_n)}{\Omega^p(\Sigma^2\mathcal{A}_{n-1}) + J^{p,n}(\Sigma^2\mathcal{A})}.$$

For $p = 0$,

$$\begin{aligned} \frac{\Omega^p(\Sigma^2\mathcal{A}_n)}{\Omega^p(\Sigma^2\mathcal{A}_{n-1}) + J^{p,n}(\Sigma^2\mathcal{A})} &\cong \frac{\Sigma^2\mathcal{A}_n}{\Sigma^2\mathcal{A}_{n-1}} \\ &\cong \frac{\mathcal{A}_n}{\mathcal{A}_{n-1}} \otimes \mathcal{S}, \end{aligned}$$

and for $p \geq 2$,

$$\begin{aligned}
&\frac{\Omega^p(\Sigma^2\mathcal{A}_n)}{\Omega^p(\Sigma^2\mathcal{A}_{n-1}) + J^{p,n}(\Sigma^2\mathcal{A})}\\
&\cong \frac{\pi(\Omega^p(\Sigma^2\mathcal{A}_n))}{\pi(\Omega^p(\Sigma^2\mathcal{A}_{n-1})) + \pi(dJ_0^{p-1,n}(\Sigma^2\mathcal{A}))}\\
&\cong \frac{\pi(\Omega^p(\mathcal{A}_n \otimes \mathcal{S})) \bigoplus \pi(\Omega^p(\mathbb{C}[z, z^{-1}]))}{(\pi(\Omega^p(\mathcal{A}_{n-1} \otimes \mathcal{S})) + \pi(dJ_0^{p-1}(\mathcal{A}_n \otimes \mathcal{S}))) \bigoplus \pi(\Omega^p(\mathbb{C}[z, z^{-1}]))}\\
&\cong \frac{\pi(\Omega^p(\mathcal{A}_n \otimes \mathcal{S}))}{\pi(\Omega^p(\mathcal{A}_{n-1} \otimes \mathcal{S}) + dJ_0^{p-1}(\mathcal{A}_n \otimes \mathcal{S}))}\\
&\cong \frac{\pi(\Omega^p(\mathcal{A}_n)) \otimes \mathcal{S}}{\pi(\Omega^p(\mathcal{A}_{n-1}) + dJ_0^{p-1}(\mathcal{A}_n)) \otimes \mathcal{S}}\\
&\cong \frac{\Omega^p(\mathcal{A}_n)}{\Omega^p(\mathcal{A}_{n-1}) + dJ_0^{p-1}(\mathcal{A}_n)} \otimes \mathcal{S}
\end{aligned}$$

by Propositions (3.8) and (3.10) in [2]. Finally, for $p = 1$,

$$\frac{\Omega^1(\Sigma^2\mathcal{A}_n)}{\Omega^1(\Sigma^2\mathcal{A}_{n-1}) + J^{1,n}(\Sigma^2\mathcal{A})} \cong \frac{\Omega^1(\mathcal{A}_n)}{\Omega^1(\mathcal{A}_{n-1})} \otimes \mathcal{S} \bigoplus \frac{\mathcal{A}_n}{\mathcal{A}_{n-1}} \otimes \mathcal{S}$$

by part (1) of Theorem 3.20 in [2]. Hence, our claim follows. □

## 3. FGR DGA for the quantum double suspension

In this section, our objective is to compute the DGA of Fröhlich *et al.* for the quantum double suspension. We first recall its definition from [7].

DEFINITION 3.1

For any $p$-summable spectral triple $(\mathcal{A}, \mathcal{H}, D)$, consider the following functional:

$$\begin{aligned}
\int : \pi(\Omega^\bullet(\mathcal{A})) &\longrightarrow \mathbb{C}\\
[v] &\longmapsto \lim_{\varepsilon \to 0^+} \frac{\mathrm{Tr}_{\mathcal{H}}(v\mathrm{e}^{-\varepsilon D^2})}{\mathrm{Tr}_{\mathcal{H}}(\mathrm{e}^{-\varepsilon D^2})}.
\end{aligned} \tag{3.3}$$

Let

$$K(\mathcal{A}) := \bigoplus_{n=0}^{\infty} K^n(\mathcal{A}), \quad K^n(\mathcal{A}) := \{\omega \in \Omega^n(\mathcal{A}) : \int \pi(\omega)^*\pi(\omega) = 0\}.$$

Then,

$$\tilde{\Omega}^{\bullet}_{D}(\mathcal{A}) := \bigoplus_{n=0}^{\infty} \tilde{\Omega}^{n}_{D}(\mathcal{A}), \quad \tilde{\Omega}^{n}_{D}(\mathcal{A}) := \Omega^{n}(\mathcal{A})/(K^{n} + dK^{n-1}) \cong \pi(\Omega^{n}(\mathcal{A}))/\pi(K^{n} + dK^{n-1})$$

is a differential graded algebra called the FGR DGA.

*Remark* 3.2.

(1) Note that for a $p$-summable spectral triple $(\mathcal{A}, \mathcal{H}, D)$,

$$\lim_{\lambda\to\infty}\left(\frac{1}{\lambda}\mathrm{Tr}(T\mathrm{e}^{-\lambda^{-2/p}D^{2}})\right) = \Gamma\left(\frac{p}{2}+1\right)\mathrm{Tr}_{\omega}(T|D|^{-p})$$

for all $T \in \mathcal{B}(\mathcal{H})$ ([5], page 563). Hence, the functional considered in equation (3.3) is nothing but the Dixmier trace $\mathrm{Tr}_{\omega}$ up to a positive constant.

(2) Since, for any compact operator $K \in \mathcal{K}(\mathcal{H})$, $\mathrm{Tr}_{\omega}(K|D|^{-p}) = 0$, the functional in (3.3) is well-defined on $\pi(\Omega^{\bullet}(\mathcal{A})) \subseteq \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.

(3) For the classical case of manifolds and the noncommutative torus, $K^{n} = J_{0}^{n}$ (Definition 2.2). Hence, the FGR DGA coincides with the Dirac DGA in these cases [7].

*Lemma* 3.3. *The association* $\mathcal{G} : (\mathcal{A}, \mathcal{H}, D) \longmapsto \tilde{\Omega}^{\bullet}_{D}(\mathcal{A})$ *gives a covariant functor from* $\mathcal{S}$pec *to* $DGA$, *the category of differential graded algebras over* $\mathbb{C}$.

*Proof.* Consider two objects $(\mathcal{A}_1, \mathcal{H}_1, D_1), (\mathcal{A}_2, \mathcal{H}_2, D_2) \in \mathcal{O}\mathrm{b}(\mathcal{S}\mathrm{pec})$ and suppose there is a morphism $(\phi\,, \Phi) : (\mathcal{A}_1, \mathcal{H}_1, D_1) \longrightarrow (\mathcal{A}_2, \mathcal{H}_2, D_2)$. Define

$$\Psi : \tilde{\Omega}^{\bullet}_{D_1}(\mathcal{A}_1) \longrightarrow \tilde{\Omega}^{\bullet}_{D_2}(\mathcal{A}_2)$$
$$\left[\sum a_0 \prod_{i=1}^{n}[D_1, a_i]\right] \longmapsto \left[\sum \phi(a_0) \prod_{i=1}^{n}[D_2, \phi(a_i)]\right]$$

for all $a_j \in \mathcal{A}_1$, $n \geq 0$. To show $\Psi$ is well-defined, we must show that $\Psi(\pi_1(K_1^{m})) \subseteq \pi_2(K_2^{m})$ for all $m \geq 0$. Observe that

$$\Phi \circ \left(\sum a_0 \prod_{i=1}^{n}[D_1, a_i]\right) = \left(\sum \phi(a_0) \prod_{i=1}^{n}[D_2, \phi(a_i)]\right) \circ \Phi\,. \tag{3.4}$$

Now, $\Phi D_1 = D_2 \Phi$ will imply that $\Phi \mathrm{e}^{-tD_1^2}\Phi^{*} = \mathrm{e}^{-tD_2^2}$. Let us denote

$$\pi_1(\omega) := \sum a_0 \prod_{i=1}^{n}[D_1, a_i],$$
$$\pi_2(\tilde{\omega}) := \sum \phi(a_0) \prod_{i=1}^{n}[D_2, \phi(a_i)].$$

Now,

$$\begin{aligned}\mathrm{Tr}(\pi_2(\tilde{\omega})^*\pi_2(\tilde{\omega})\mathrm{e}^{-tD_2^2}) &= \mathrm{Tr}(\pi_2(\tilde{\omega})^*\pi_2(\tilde{\omega})\Phi\mathrm{e}^{-tD_1^2}\Phi^*)\\ &= \mathrm{Tr}(\pi_1(\omega)^*\Phi^*\Phi\pi_1(\omega)\mathrm{e}^{-tD_1^2})\\ &= \mathrm{Tr}(\pi_1(\omega)^*\pi_1(\omega)\mathrm{e}^{-tD_1^2})\end{aligned}$$

and $\mathrm{Tr}(\mathrm{e}^{-tD_1^2}) = \mathrm{Tr}(\mathrm{e}^{-tD_2^2})$. This proves that $\Psi(\pi_1(K_1^m)) = \pi_2(K_2^m)$, i.e. $\Psi$ is well-defined, and one can check that it is a DGA morphism. □

*Remark* 3.4.

(1) Although, surjectivity of $\Phi$ is enough to ensure that Dirac DGA is a functor, it fails in this case of FGR DGA. This is the reason we have chosen $\Phi$ to be unitary. Unless $\Phi$ is both one-to-one and onto, it is not guaranteed that $\Psi(\pi_1(K_1^m)) \subseteq \pi_2(K_2^m)$.

(2) One may come up with a different definition of the category $\mathcal{S}$pec of spectral triples which allows larger set of morphisms than ours; such that both the Dirac DGA and FGR DGA become functors. Here we stress the point that *it will not contradict* our main result in this article as we shall see. Because of this reason, we have chosen the simplest possible definition for the category $\mathcal{S}$pec.

To make the computation possible, we need to use the functional in (3.3) in a different disguise, namely

$$\begin{aligned}\oint : \pi(\Omega^\bullet(\mathcal{A})) &\longrightarrow \mathbb{C}\\ [\tilde{v}] &\longmapsto \lim_{t\to 0}(t^p\mathrm{Tr}(\tilde{v}\mathrm{e}^{-t|D|})).\end{aligned} \tag{3.5}$$

The well-definedness of this functional follows from the next lemma.

*Lemma* 3.5. *Let* $(\mathcal{A}, \mathcal{H}, D)$ *be a* $p$*-summable spectral triple. Then the functional* $\oint$ *is equal to the Dixmier trace up to a positive constant* (*which depends only on* $p$).

*Proof.* Recall the following equality

$$\omega\left(\frac{1}{t}\mathrm{Tr}\left(\exp\left(-(tA)^{-q}\right)B\right)\right) = \Gamma\left(1+\frac{1}{q}\right)\mathrm{Tr}_\omega(AB)$$

proved in [10] for any $B \in \mathcal{B}(\mathcal{H})$. Now take $q = 1/p$ and $A = |D|^{-p}$. □

COROLLARY 3.6

*For any* $T_1 \otimes T_2 \in \mathcal{B}(\mathcal{H} \otimes \ell^2(\mathbb{N}))$,

$$t^{p+1}\mathrm{Tr}((T_1\otimes T_2)\mathrm{e}^{-t|\Sigma^2 D|}) = t^p\mathrm{Tr}(T_1\mathrm{e}^{-t|D|})\,t\mathrm{Tr}(T_2\mathrm{e}^{-tN}).$$

*Remark* 3.7. It is this corollary which makes the computation in this section possible. Moreover, since both the functionals $\int$ and $\oint$ become equal up to a constant, and we are interested in the spaces $K^n$ in Definition 3.3, it is absolutely permissible to choose $\oint$ over $\int$.

*Lemma* 3.8. $K^0(\Sigma^2\mathcal{A}) = \mathcal{A} \otimes \mathcal{S}$.

*Proof.* Choose arbitrary element $\sum_k a_k \otimes T_k$ of $\mathcal{A} \otimes \mathcal{S}$. In terms of elementary matrices, we can write $T_k = \sum_{i,j} \alpha_{ij}^{(k)} e_{ij}$ . Then

$$\begin{aligned}
&\oint \left(\sum_k a_k \otimes T_k\right)\left(\sum_k a_k \otimes T_k\right)^* \\
&= \oint \left(\sum_{k,i,j} a_{kij} \otimes e_{ij}\right)\left(\sum_{k,i,j} a_{kij}^* \otimes e_{ji}\right) \\
&= \oint \sum_{k,k',i,j,i'} a_{kij} a_{k'i'j}^* \otimes e_{ii'} \\
&= \lim_{t\to 0} t^{p+1}\mathrm{Tr}\left(\left(\sum_{k,k',i,j,i'} a_{kij} a_{k'i'j}^* \otimes e_{ii'}\right)\mathrm{e}^{-t|\Sigma^2 D|}\right) \\
&= \sum_{k,k',i,j,i'} \lim_{t\to 0}(t^p \mathrm{Tr}(a_{kij} a_{k'i'j}^* \mathrm{e}^{-t|D|}))(t\mathrm{Tr}(e_{ii'}\mathrm{e}^{-tN})) \\
&= \sum_{k,k',i,j} \lim_{t\to 0}(t^p \mathrm{Tr}(a_{kij} a_{k'ij}^* \mathrm{e}^{-t|D|}))(t\mathrm{e}^{-ti}) \\
&= 0.
\end{aligned}$$

Hence $\mathcal{A} \otimes \mathcal{S} \subseteq K^0(\Sigma^2\mathcal{A})$. Now, for arbitrary $\sum_k a_k \otimes T_k + f \in K^0(\Sigma^2\mathcal{A})$,

$$\begin{aligned}
0 &= \oint \left(\sum_k a_k \otimes T_k + f\right)\left(\sum_k a_k \otimes T_k + f\right)^* \\
&= \oint \left(\sum_k a_k \otimes T_k\right)\left(\sum_k a_k \otimes T_k\right)^* + \oint ff^* + \oint f\left(\sum_k a_k \otimes T_k\right)^* \\
&\quad + \oint \left(\sum_k a_k \otimes T_k\right) f^* \\
&= \oint ff^*
\end{aligned}$$

because the same calculation as above proves that both $\oint f(\sum_k a_k \otimes T_k)^*$ and $\oint(\sum_k a_k \otimes T_k) f^*$ are zero. For any $f \in \mathbb{C}[z, z^{-1}]$, $\oint ff^*$ is just the integration of the function $ff^* \equiv |f|^2$ against the Haar measure on $S^1$. This shows that $f = 0$, i.e. $K^0(\Sigma^2\mathcal{A}) \subseteq \mathcal{A} \otimes \mathcal{S}$. □

*Remark* 3.9. In Assumption 2.13, page 131 in [7], the authors have assumed that $K^0 = \{0\}$. Lemma 3.8 shows that this is never true in the case of quantum double suspension.

*Lemma* 3.10. $\oint (F \otimes 1)\pi(\omega) = 0$ *for any* $\omega \in \Omega^1(\mathcal{A} \otimes \mathcal{S})$.

*Proof.* Let

$$\begin{aligned}\pi(\omega) &= \sum (a_0 \otimes T_0)[\Sigma^2 D, a_1 \otimes T_1] \\ &= \sum a_0[D, a_1] \otimes T_0 T_1 + F a_0 a_1 \otimes T_0[N, T_1].\end{aligned}$$

Then, using elementary matrices $(e_{ij})$, we have

$$\begin{aligned}
&\oint (F \otimes 1)\pi(\omega) \\
&= \lim_{t\to 0} (t^{p+1}\mathrm{Tr}(\pi(\omega)\mathrm{e}^{-t|\Sigma^2 D|})) \\
&= \sum \lim_{t\to 0} (t^{p+1}\mathrm{Tr}((a_0[D, a_1] \otimes T_0 T_1)\mathrm{e}^{-t|\Sigma^2 D|})) \\
&\quad + \lim_{t\to 0} (t^{p+1}\mathrm{Tr}((F a_0 a_1 \otimes T_0[N, T_1])\mathrm{e}^{-t|\Sigma^2 D|})) \\
&= \sum \lim_{t\to 0} \left( t^{p+1}\mathrm{Tr}\left( \sum_{i,j,q} (a_{0ij}[D, a_{1jq}] \otimes e_{iq})\mathrm{e}^{-t|\Sigma^2 D|} \right)\right) \\
&\quad + \lim_{t\to 0} \left( t^{p+1}\mathrm{Tr}\left( \left( \sum_{i,j,q} F a_{0ij} a_{1jq}(j-q) \otimes e_{iq} \right) \mathrm{e}^{-t|\Sigma^2 D|} \right)\right) \\
&= \sum \sum_{i,j,q} \lim_{t\to 0} \left( t^p \mathrm{Tr}\left( a_{0ij}[D, a_{1jq}]\mathrm{e}^{-t|D|}\right) t\mathrm{Tr}\left( e_{iq}\mathrm{e}^{-tN}\right)\right) \\
&\qquad + \lim_{t\to 0} \left( t^p \mathrm{Tr}\left( F a_{0ij} a_{1jq}(j-q)\mathrm{e}^{-t|D|}\right) t\mathrm{Tr}\left( e_{iq}\mathrm{e}^{-tN}\right)\right) \\
&= \sum \sum_{i,j} \lim_{t\to 0} \left( t^p \mathrm{Tr}\left( a_{0ij}[D, a_{1ji}]\mathrm{e}^{-t|D|}\right) (t\mathrm{e}^{-ti})\right) \\
&\qquad + \lim_{t\to 0} (t^p \mathrm{Tr}(F a_{0ij} a_{1ji}(j-i)\mathrm{e}^{-t|D|})(t\mathrm{e}^{-ti})) \\
&= 0,
\end{aligned}$$

and this concludes the proof. □

*Lemma* 3.11. $\pi(K^1(\Sigma^2\mathcal{A})) = \pi(\Omega^1(\mathcal{A} \otimes \mathcal{S})) \bigoplus \pi(K^1(\mathbb{C}[z, z^{-1}]))$.

*Proof.* We first prove that $\pi(\Omega^1(\mathcal{A} \otimes \mathcal{S}) \subseteq \pi(K^1(\Sigma^2\mathcal{A}))$. The arbitrary element of $\pi(\Omega^1(\mathcal{A} \otimes \mathcal{S}))$ looks like $\pi(\omega) = \sum_k (a_{0k} \otimes T_{0k})[\Sigma^2 D, a_{1k} \otimes T_{1k}]$. Then, using elementary matrices $(e_{ij})$, we get

$$\begin{aligned}\pi(\omega) &= \sum_k \left( \sum_{i,j} a_{0kij} \otimes e_{ij} \right) \left[ \Sigma^2 D, \sum_{p,q} a_{1kpq} \otimes e_{pq} \right] \\ &= \sum_{k,i,j,q} (a_{0kij}[D, a_{1kjq}] + F a_{0kij} a_{1kjq}(j-q)) \otimes e_{iq}\ ,\end{aligned}$$

and

$$\pi(\omega)^* = \sum_{k,i,j,q} (a_{0kij}[D, a_{1kjq}] + Fa_{0kij}a_{1kjq}(j-q))^* \otimes e_{qi}\,.$$

Let $T_{kjiq} = a_{0kij}[D, a_{1kjq}] + Fa_{0kij}a_{1kjq}(j-q)$. Now

$$\begin{aligned}
&\oint \pi(\omega)\pi(\omega)^* \\
&= \lim_{t\to 0} t^{p+1}\mathrm{Tr}(\pi(\omega)\pi(\omega)^*\mathrm{e}^{-t|\Sigma^2 D|}) \\
&= \lim_{t\to 0} t^{p+1}\mathrm{Tr}\left(\left(\sum_{i,q}\sum_{k,j} T_{kjiq}\otimes e_{iq}\right)\left(\sum_{i',q'}\sum_{k,j} T^*_{kji'q'}\otimes e_{q'i'}\right)\mathrm{e}^{-t|\Sigma^2 D|}\right) \\
&= \lim_{t\to 0} t^{p+1}\mathrm{Tr}\left(\left(\sum_{i,q,i'}\left(\sum_{k,j} T_{kjiq}\right)\left(\sum_{k,j} T^*_{kji'q}\right)\otimes e_{ii'}\right)\mathrm{e}^{-t|\Sigma^2 D|}\right) \\
&= \lim_{t\to 0}\sum_{i,q,i'} t^{p}\mathrm{Tr}\left(\left(\sum_{k,j} T_{kjiq}\right)\left(\sum_{k,j} T_{kji'q}\right)^*\mathrm{e}^{-t|D|}\right) t\,\mathrm{Tr}\left(e_{ii'}\mathrm{e}^{-tN}\right) \\
&= \lim_{t\to 0}\sum_{i,q} t^{p}\mathrm{Tr}\left(\left(\sum_{k,j} T_{kjiq}\right)\left(\sum_{k,j} T_{kjiq}\right)^*\mathrm{e}^{-t|D|}\right)(t\mathrm{e}^{-ti}) \\
&= 0\,.
\end{aligned}$$

Hence, $\pi(\Omega^1(\mathcal{A}\otimes\mathcal{S}))\bigoplus\pi(K^1(\mathbb{C}[z,z^{-1}]))\subseteq\pi(K^1(\Sigma^2\mathcal{A}))$.

To show the converse, choose an arbitrary element $\pi(\omega) = \sum_k (a_{0k}\otimes T_{0k} + f_{0k})[\Sigma^2 D, a_{1k}\otimes T_{1k} + f_{1k}]$ in $\pi\left(K^1(\Sigma^2\mathcal{A})\right)$. Then

$$\pi(\omega) \;=\; \sum_k F\otimes f_{0k}f'_{1k} + \pi(\tilde{\omega})$$

where $\pi(\tilde{\omega}) \in \pi(\Omega^1(\mathcal{A}\otimes\mathcal{S}))$. Hence $\pi(\omega)^* = \pi(\tilde{\omega})^* + \sum_k F\otimes(f_{0k}f'_{1k})^*$. Since $\pi(\omega)\in\pi\left(K^1(\Sigma^2\mathcal{A})\right)$, we have

$$\begin{aligned}
0 &= \oint \pi(\omega)^*\pi(\omega) \\
&= \oint \pi(\tilde{\omega})^*\pi(\tilde{\omega}) + \oint\left(\sum_k F\otimes(f_{0k}f'_{1k})^*\right)\pi(\tilde{\omega}) \\
&\quad + \oint \pi(\tilde{\omega})^*\left(\sum_k F\otimes(f_{0k}f'_{1k})\right) \\
&\quad + \oint\left(\sum_k f_{0k}f'_{1k}\right)^*\left(\sum_k f_{0k}f'_{1k}\right).
\end{aligned}$$

This shows that $\oint(\sum_k f_{0k} f'_{1k})^*(\sum_k f_{0k} f'_{1k}) = 0$ (using Lemma 3.10). That is, $\sum_k F \otimes f_{0k} f'_{1k} \in \pi(K^1(\mathbb{C}[z, z^{-1}]))$. Hence $\pi(K^1(\Sigma^2\mathcal{A})) \subseteq \pi(\Omega^1(\mathcal{A}\otimes\mathcal{S})) \bigoplus \pi(K^1(\mathbb{C}[z, z^{-1}]))$. □

PROPOSITION 3.12

*$\tilde{\Omega}^1_{\Sigma^2 D}(\Sigma^2\mathcal{A}) \cong \mathbb{C}[z, z^{-1}]$ as $\Sigma^2\mathcal{A}$-bimodule.*

*Proof.* We have $\tilde{\Omega}^1_{\Sigma^2 D}(\Sigma^2\mathcal{A}) \cong \pi(\Omega^1(\Sigma^2\mathcal{A}))/(\pi(K^1(\Sigma^2\mathcal{A})) + \pi(\mathrm{d}K^0(\Sigma^2\mathcal{A})))$. But $\pi(\mathrm{d}K^0(\Sigma^2\mathcal{A})) \subseteq \pi(\Omega^1(\mathcal{A}\otimes\mathcal{S}))$ and $K^0(\mathbb{C}[z, z^{-1}]) = \{0\}$. This says that

$$\begin{aligned}
&\tilde{\Omega}^1_{\Sigma^2 D}(\Sigma^2\mathcal{A}) \\
&\quad\cong (\pi(\Omega^1(\mathcal{A}\otimes\mathcal{S})) \oplus \pi(\Omega^1(\mathbb{C}[z, z^{-1}]))/(\pi(\Omega^1(\mathcal{A}\otimes\mathcal{S})) \\
&\qquad\oplus\pi(K^1(\mathbb{C}[z, z^{-1}]))) \\
&\quad\cong \pi(\Omega^1(\mathbb{C}[z, z^{-1}]))/\pi(K^1(\mathbb{C}[z, z^{-1}]))) \\
&\quad\cong \tilde{\Omega}^1_N(\mathbb{C}[z, z^{-1}])) \\
&\quad\cong \mathbb{C}[z, z^{-1}].
\end{aligned}$$

Here, the first isomorphism follows from the fact that (see Proposition 3.8 in [2])

$$\pi(\Omega^1(\Sigma^2\mathcal{A})) = \pi(\Omega^1(\mathcal{A}\otimes\mathcal{S})) \bigoplus \pi(\Omega^1(\mathbb{C}[z, z^{-1}]),$$

and we refer [3] for the following fact:

$$\tilde{\Omega}^n_N(\mathbb{C}[z, z^{-1}]) = \begin{cases} \mathbb{C}[z, z^{-1}]; & n = 0, 1 \\ \{0\}; & \text{otherwise.} \end{cases}$$

□

*Remark* 3.13. Recall that $\Sigma^2\mathcal{A} \cong \mathcal{A}\otimes\mathcal{S} \bigoplus \mathbb{C}[z, z^{-1}]$ as $\mathbb{C}$-vector spaces, where $\mathbb{C}[z, z^{-1}]$ is identified with the quotient $\Sigma^2\mathcal{A}/\mathcal{A}\otimes\mathcal{S}$. These direct sum and isomorphism are also as $\Sigma^2\mathcal{A}$-bimodule. Hence, $\tilde{\Omega}^1_{\Sigma^2 D}\left(\Sigma^2\mathcal{A}\right)$ is always a finitely generated projective $\Sigma^2\mathcal{A}$-bimodule (compare with Assumption 2.13 in [7], page 131).

*Lemma* 3.14. *$\oint \pi(\omega) = 0$ for any $\omega \in \Omega^n(\mathcal{A}\otimes\mathcal{S})$ and for all $n \geq 2$.*

*Proof.* Recall Lemma 3.15 from [2] which says that

$$\pi(\Omega^n(\mathcal{A}\otimes\mathcal{S})) = \sum_{r=0}^{n} F^r\pi(\Omega^{n-r}(\mathcal{A}))\otimes\mathcal{S}.$$

Hence, for $\omega \in \Omega^n(\mathcal{A}\otimes\mathcal{S})$, we have $\pi(\omega) = \sum_{r=0}^n \sum_k F^r\pi(v_{r,k})\otimes T_{r,k}$ with $v_{r,k} \in \Omega^{n-r}(\mathcal{A})$. Writing each $T_{r,k}$ in terms of elementary matrices $(e_{ij})$, we get that

$$\pi(\omega) = \sum_{r=0}^{n}\sum_{k,i,j} F^r\pi(v^{ij}_{r,k})\otimes e_{ij}\,.$$

Then

$$\begin{aligned}\oint \pi(\omega) &= \lim_{t\to 0} t^{p+1}\mathrm{Tr}(\pi(\omega)\mathrm{e}^{-t|\Sigma^2 D|}))\\ &= \sum_{r=0}^{n}\sum_{k,i}\lim_{t\to 0}(t^p\mathrm{Tr}(F^r\pi(v^{ii}_{r,k})\mathrm{e}^{-t|D|}))(t\mathrm{e}^{-ti})\\ &= 0\end{aligned}$$

and we are done. □

*Lemma* 3.15. $\pi(K^n(\Sigma^2\mathcal{A})) = \pi(\Omega^n(\mathcal{A}\otimes\mathcal{S}))\bigoplus\pi(K^n(\mathbb{C}[z,z^{-1}]))$, *for all* $n\geq 2$.

*Proof.* Note that for any algebra $\mathcal{A}$, we have

$$\Omega^n(\mathcal{A}) = \underbrace{\Omega^1(\mathcal{A})\otimes_{\mathcal{A}}\cdots\otimes_{\mathcal{A}}\Omega^1(\mathcal{A})}_{n\text{ times}}.$$

Lemma 3.11 proves that $\pi(\Omega^1(\mathcal{A}\otimes\mathcal{S}))\subseteq\pi(K^1(\Sigma^2\mathcal{A}))$. Since

$$\Omega^n(\mathcal{A}\otimes\mathcal{S}) = \Omega^{n-1}(\mathcal{A}\otimes\mathcal{S})\bigotimes_{\Sigma^2\mathcal{A}}\Omega^1(\mathcal{A}\otimes\mathcal{S}),$$

we get that

$$\pi(\Omega^n(\mathcal{A}\otimes\mathcal{S}))\subseteq\pi(K^n(\Sigma^2\mathcal{A}))$$

because $K^\bullet$ is a graded ideal in $\Omega^\bullet$. Hence, we have the inclusion '$\supseteq$'. Now, recall Proposition 3.8 from [2], which says that

$$\pi(\Omega^n(\Sigma^2\mathcal{A})) = \pi(\Omega^n(\mathcal{A}\otimes\mathcal{S}))\bigoplus\pi(\Omega^n(\mathbb{C}[z,z^{-1}]));\ \ \forall n\geq 0.$$

Since $K^n\subseteq\Omega^n$, using Lemma 3.14, we get the inclusion '$\subseteq$' and this completes the proof. □

**Theorem 3.16.** *For* $(\Sigma^2\mathcal{A},\Sigma^2\mathcal{H},\Sigma^2 D)$,
(1) $\tilde{\Omega}^n_{\Sigma^2 D}(\Sigma^2\mathcal{A}) = \mathbb{C}[z,z^{-1}]$ *for* $n=0,1$;
(2) $\tilde{\Omega}^n_{\Sigma^2 D}(\Sigma^2\mathcal{A}) = 0$ *for all* $n\geq 2$.

*Proof.* Part (1) follows from Lemma 3.8 and Proposition 3.12. Now, Lemmas 3.11 and 3.15 shows that for all $n\geq 1$,

$$K^n(\Sigma^2\mathcal{A}) + J^n_0(\Sigma^2\mathcal{A}) = \Omega^n(\mathcal{A}\otimes\mathcal{S}) + K^n(\mathbb{C}[z,z^{-1}]) + J^n_0(\Sigma^2\mathcal{A}). \quad (3.6)$$

But $J^n_0(\Sigma^2\mathcal{A})\subseteq K^n(\Sigma^2\mathcal{A})$. Hence, equation (3.6) reduces to

$$K^n(\Sigma^2\mathcal{A}) \ = \ \Omega^n(\mathcal{A}\otimes\mathcal{S}) + K^n(\mathbb{C}[z,z^{-1}]) + J^n_0(\Sigma^2\mathcal{A}). \quad (3.7)$$

So, for all $n\geq 1$,

$$\mathrm{d}K^n(\Sigma^2\mathcal{A}) = \mathrm{d}\Omega^n(\mathcal{A}\otimes\mathcal{S}) + \mathrm{d}K^n(\mathbb{C}[z,z^{-1}]) + \mathrm{d}J^n_0(\Sigma^2\mathcal{A});$$

and consequently for all $n\geq 1$,

$$\pi(\mathrm{d}K^n(\Sigma^2\mathcal{A})) = \pi(\mathrm{d}\Omega^n(\mathcal{A}\otimes\mathcal{S})) + \pi(\mathrm{d}K^n(\mathbb{C}[z,z^{-1}]) + \pi(\mathrm{d}J_0^n(\Sigma^2\mathcal{A})). \tag{3.9}$$

Recall Propositions 3.8 and 3.10 from [2], which say that

$$\pi(\Omega^n(\Sigma^2\mathcal{A})) = \pi(\Omega^n(\mathcal{A}\otimes\mathcal{S}))\bigoplus\pi(\Omega^n(\mathbb{C}[z,z^{-1}])); \ \forall n\geq 0 \tag{3.10}$$

and

$$\pi(\mathrm{d}J_0^n(\Sigma^2\mathcal{A})) = \pi(\mathrm{d}J_0^n(\mathcal{A}\otimes\mathcal{S}))\bigoplus\pi(\mathrm{d}J_0^n(\mathbb{C}[z,z^{-1}])); \ \forall n\geq 1. \tag{3.11}$$

Hence, equation (3.9) turns out to be

$$\pi(\mathrm{d}K^n(\Sigma^2\mathcal{A})) = \pi(\mathrm{d}\Omega^n(\mathcal{A}\otimes\mathcal{S}))\bigoplus\pi((\mathrm{d}K^n+\mathrm{d}J_0^n)(\mathbb{C}[z,z^{-1}])); \ \forall n\geq 1. \tag{3.12}$$

Finally, using equations (3.7), (3.10) and (3.12), we have for all $n\geq 2$,

$$\begin{aligned}\tilde{\Omega}^n_{\Sigma^2 D}(\Sigma^2\mathcal{A}) &\cong \frac{\pi(\Omega^n(\Sigma^2\mathcal{A}))}{\pi(K^n(\Sigma^2\mathcal{A}))+\pi(\mathrm{d}K^{n-1}(\Sigma^2\mathcal{A}))}\\ &\cong \frac{\pi(\Omega^n(\mathcal{A}\otimes\mathcal{S}))\bigoplus\pi(\Omega^n(\mathbb{C}[z,z^{-1}]))}{\pi(\Omega^n(\mathcal{A}\otimes\mathcal{S}))\bigoplus\pi((K^n+\mathrm{d}K^{n-1}+\mathrm{d}J_0^{n-1})(\mathbb{C}[z,z^{-1}]))}\\ &\cong \frac{\pi(\Omega^n(\mathbb{C}[z,z^{-1}]))}{\pi((K^n+\mathrm{d}K^{n-1}+\mathrm{d}J_0^{n-1})(\mathbb{C}[z,z^{-1}]))}\end{aligned}$$

Now, the facts that $\pi_N(\Omega^n(\mathbb{C}[z,z^{-1}])) = \mathbb{C}[z,z^{-1}]$ and $\pi_N(\mathrm{d}J_0^{n-1}(\mathbb{C}[z,z^{-1}])) = \mathbb{C}[z,z^{-1}]$ for all $n\geq 2$ (see Lemmas 3.11 and 3.12 in [2]) completes part (2). □

In view of Theorems 2.14 and 3.16, the conclusion of this article comes as the following final theorem.

**Theorem 3.17.** *There is a category* $\mathcal{S}$pec *of spectral triples such that the Dirac DGA, the FGR DGA and the quantum double suspension, denoted by* $\mathcal{F}, \mathcal{G}, \Sigma^2$ *respectively, become covariant functors. Let* $\mathcal{C}$ *be the subcategory of commutative spectral triples. Restricted to* $\mathcal{C}$*, both the functor* $\mathcal{F}$ *and* $\mathcal{G}$ *are equal to the de-Rham DGA. Unlike* $\mathcal{F}\circ\Sigma^2$*, the functor* $\mathcal{G}\circ\Sigma^2$ *becomes a constant functor on* $\mathcal{S}$pec.

## Acknowledgements

The first author (PSC) acknowledges support of DST, India through Swarnajayanti Fellowship Award Project No. DST/SJF/MSA-01/2012-13 and the second author (SG) acknowledges support of DST, India through INSPIRE Faculty Award (Award No. DST/INSPIRE/04/2015/000901).

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COMMUNICATING EDITOR: B V Rajarama Bhat