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On partial sums of arithmetical functions of two variables with absolutely convergent Ramanujan expansions

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Abstract. In this article, we derive an asymptotic formula for the sums of the form $\sum_{n_1,n_2\leq N} f(n_1, n_2)$ with an explicit error term, for any arithmetical function f of two variables with absolutely convergent Ramanujan expansion and Ramanujan coefficients satisfying certain hypothesis.

Keywords. Arithmetic functions; Ramanujan expansions; Ramanujan sums; asymptotic formula.

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1. Introduction

In 1918, Ramanujan [4] introduced certain sums and associated series expansions. He defined the sums as follows:

DEFINITION 1.1

For any positive integers r and n,

$$c_r(n) := \sum_{a\in(\mathbb{Z}/r\mathbb{Z})^*} \zeta_r^{an},$$

where ζ_r denotes a primitive r-th root of unity. These sums are now-a-days known as *Ramanujan sums*.

We can also express Ramanujan sums in terms of the Möbius function μ (for details, see [2]) as follows:

$$c_r(n) := \sum_{d|n,d|r} \mu(r/d)d.$$

Ramanujan sums have various other properties (for details, see [7,9,10]). He used the sums $c_r(n)$ to derive pointwise convergent series expansion of the form $\sum_{r\geq 1} \hat{f}(r)c_r(n)$

for various arithmetic functions. These expansions are known as *Ramanujan expansions*. More precisely, these expansions are defined as follows.

DEFINITION 1.2

We say an arithmetical function f admits a *Ramanujan expansion*, if for each integer $n \geq 1$, the functional value $f(n)$ can be written as a convergent series of the form

$$f(n) = \sum_{r \geq 1} \hat{f}(r) c_r(n)$$

for some appropriate complex numbers $\hat{f}(r)$. The complex number $\hat{f}(r)$ is known as the *r-th Ramanujan coefficient* of f with respect to this expansion.

Using these notions, Ramanujan obtained the following results:

$$\begin{aligned}
\frac{\sigma_s(n)}{n^s} &= \zeta(s+1) \sum_{q=1}^{\infty} \frac{c_q(n)}{q^{s+1}}, \\
\frac{\phi_s(n)}{n^s} &= \frac{1}{\zeta(s+1)} \sum_{q=1}^{\infty} \frac{\mu(q)}{\phi_{s+1}(q)} c_q(n), \\
\tau(n) &= -\sum_{q=1}^{\infty} \frac{\log q}{q} c_q(n), \\
r(n) &= \pi \sum_{q=1}^{\infty} \frac{(-1)^{q-1}}{2q-1} c_{2q-1}(n),
\end{aligned}$$

where $\sigma_s(n) = \sum_{[d|n]} d^s$ with $s > 0$, $\zeta(s)$ is the Riemann zeta function, $\phi_s(n) = n^s \prod_{p|n}(1 - 1/p^s)$, $\tau(n) = \sum_{d|n} 1$, μ is the Möbius function and $r(n)$ is the number of representations of n as the sum of two squares.

Many results concerning Ramanujan expansion of an arithmetic function of one variable have been obtained by many mathematicians until now. However, very few results are known regarding Ramanujan expansions of arithmetic functions of two variables.

Recently, Ushoriya [11] defined Ramanujan expansion for an arithmetical function of two variables in the following way:

$$f(n_1, n_2) = \sum_{q_1, q_2}^{\infty} a_{q_1, q_2} c_{q_1}(n_1) c_{q_2}(n_2),$$

for some complex numbers a_{q_1,q_2}. These complex numbers are called (q_1, q_2) *Ramanujan coefficients* of $f(n_1, n_2)$. He extended Delange's theorem to the function of two variables and provided several examples.

Here, we study arithmetical functions of two variables with absolutely convergent Ramanujan expansions in the context of their partial sums. Following the framework of [5] and [6], we shall study the sum $\sum_{n_1, n_2 \leq N} f(n_1, n_2)$ under certain growth conditions on Ramanujan coefficients and obtain an asymptotic formula with the explicit error term. More precisely, we prove the following theorems.

Theorem 1.3. *Suppose f is an arithmetical function of two variables with absolutely convergent Ramanujan expansions*

$$f(n_1, n_2) = \sum_{q_1, q_2 = 1}^{\infty} a_{q_1, q_2} c_{q_1}(n_1) c_{q_2}(n_2),$$

and Ramanujan coefficients satisfying the following condition:

$$|a_{q_1, q_2}| \ll \frac{1}{[q_1, q_2]^{1+\delta}}$$

for some $\delta > 0$, where $[q_1, q_2]$ denotes the least common multiple of q_1 and q_2. Then, for a positive integer N, we have

$$\sum_{n_1, n_2 \le N} f(n_1, n_2) = \begin{cases} N^2 a_{1,1} + O(N^{2-\delta} (\log N)^{\frac{14-5\delta}{2}}) & \text{if } \delta < 2, \\ N^2 a_{1,1} + O(\log^2 N) & \text{if } \delta = 2, \\ N^2 a_{1,1} + O(1) & \text{if } \delta > 2. \end{cases}$$

In the following theorem, we relax the growth condition and obtain the following.

Theorem 1.4. *Suppose f is an arithmetical function of two variables with absolutely convergent Ramanujan expansions*

$$f(n_1, n_2) = \sum_{q_1, q_2, = 1}^{\infty} a_{q_1, q_2} c_{q_1}(n_1) c_{q_2}(n_2),$$

and Ramanujan coefficients satisfying the following condition:

$$|a_{q_1, q_2}| \ll \frac{1}{[q_1, q_2] \log^{\alpha} [q_1, q_2]}$$

for some $\alpha > 7$. Then, for a positive integer N, we have

$$\sum_{n_1, n_2 \le N} f(n_1, n_2) = N^2 a_{1,1} + O\left(\frac{N^2}{(\log N)^{\alpha - 7}} \right).$$

For any real number $\delta > 0$, Ushiroya [11] proved that

$$\frac{\sigma_{-1+\delta}((n_1, n_2))}{((n_1, n_2))^{-1+\delta}} = \zeta(2+\delta) \sum_{q_1, q_2 = 1}^{\infty} \frac{1}{[q_1, q_2]^{1+\delta}} c_{q_1}(n_1) c_{q_2}(n_2)$$

and

$$\frac{\phi_{-1+\delta}((n_1, n_2))}{((n_1, n_2))^{1+\delta}} = \frac{1}{\zeta(2+\delta)} \sum_{q_1, q_2 = 1}^{\infty} \frac{\mu([q_1, q_2])}{\phi_{1+\delta}([q_1, q_2])} c_{q_1}(n_1) c_{q_2}(n_2),$$

where (n_1, n_2) denotes the greatest common divisor of n_1 and n_2. By taking

$$f(n_1, n_2) = \frac{\sigma_{-1+\delta}((n_1, n_2))}{((n_1, n_2))^{-1+\delta}} \quad \text{and} \quad f(n_1, n_2) = \frac{\phi_{-1+\delta}((n_1, n_2))}{((n_1, n_2))^{1+\delta}}$$

in Theorem 1.3, we get the following corollaries.

COROLLARY 1.5

Let $\delta > 0$ be any number. Then, for a positive integer N, we have

$$\sum_{\substack{n_1 \le N \\ n_2 \le N}} \frac{\sigma_{-1+\delta}((n_1, n_2))}{((n_1, n_2))^{-1+\delta}} = \begin{cases} N^2\zeta(\delta+1) + O(N^{2-\delta}(\log N)^{\frac{14-5\delta}{2}}) & \text{if } \delta < 2, \\ N^2\zeta(3) + O(\log^2 N) & \text{if } \delta = 2, \\ N^2\zeta(\delta+1) + O(1) & \text{if } \delta > 2. \end{cases}$$

COROLLARY 1.6

Let $\delta > 0$ be a given real number. Then, for a positive integer N, we have

$$\sum_{\substack{n_1 \le N \\ n_2 \le N}} \frac{\phi_{-1+\delta}((n_1, n_2))}{((n_1, n_2))^{-1+\delta}} = \begin{cases} \frac{N^2}{\zeta(\delta+1)} + O(N^{2-\delta}(\log N)^{\frac{14-5\delta}{2}}) & \text{if } \delta < 2, \\ \frac{N^2}{\zeta(3)} + O(\log^2 N) & \text{if } \delta = 2, \\ \frac{N^2}{\zeta(\delta+1)} + O(1) & \text{if } \delta > 2. \end{cases}$$

2. Preliminaries

In this section, we record some results which are useful to prove the main results. We shall start with the well-known partial summation formula, which will be used frequently, as follows.

PROPOSITION 2.1

Let $a : \mathbb{N} \to \mathbb{C}$ be an arithmetic function. Let $x \ge 1$ be a real number and let $f : [1, x] \to \mathbb{C}$ be a function with continuous derivative on $[1, x]$. Then, we have

$$\sum_{n \le x} a(n) f(n) = A(x) f(x) - \int_1^x A(t) f'(t) \mathrm{d}t,$$

where

$$A(t) = \sum_{n \le t} a(n).$$

Let $d_k(n)$ be the number of ways of writing n as a product of k numbers. Note that when $k = 2$, we get $d_2(n) = d(n)$ divisor function, which counts the number of divisors of n. When $k = 4$, we get

$$d_4(n) = \sum_{\substack{a,b \\ ab=n}} d(a)d(b).$$

We need the following asymptotic formula for the average order of the arithmetical function $d_k(\cdot)$, which can be deduced using partial summation formula.

Lemma 2.2. *For any real number $x \geq 1$ and for any integer $k \geq 2$, we have*

$$\sum_{n \leq x} d_k(n) = \frac{x \log^{k-1} x}{(k-1)!} + O(x \log^{k-2} x).$$

Lemma 2.3 [3]. *For all $x \geq 1$, we have*

$$\sum_{n \leq x} d^2(n) \sim \frac{1}{\pi^2} x (\log^3 x).$$

For any positive integer $n \geq 1$, we define

$$N(n) := \#\{(a,b) \in \mathbb{N} \times \mathbb{N} : [a,b] = n\},$$

the number of ordered pairs of positive integers a and b whose least common multiple $[a,b] = n$. Then, the asymptotic formula for the partial sum of this function is given as follows.

Lemma 2.4 [8]. *For all real numbers $x \geq 2$, there exist absolute constants c_1 and c_2 such that*

$$\sum_{n \leq x} N(n) = \frac{1}{2\zeta(2)} x \log^2 x + c_1 x \log x + c_2 x + O(x^{\frac{1}{2}+\epsilon})$$

for any $\epsilon > 0$.

Regarding the Ramanujan sums, we need the following estimates.

Lemma 2.5. *For all positive integers $N \geq 1$ and $r \geq 1$, we have*

$$\sum_{n \leq N} c_r(n) \leq N d(r).$$

Proof. By substituting $s = 1$ in Lemma 2 of [1] and using $c_1(n) = 1$, the proof follows. □

Lemma 2.6. *For all positive integers $r \geq 2$ and $N \geq 1$, we have*

$$\left| \sum_{n \leq N} c_r(n) \right| \leq r \log r.$$

Proof. By substituting $s = 1$ in Lemma 2 of [1], the proof follows. □

Lemma 2.7. *For any integer $r \geq 1$ and for a positive integers n, we have*

$$|c_r(n)| \ll n \log n.$$

Proof. We can write $c_r(n) = \sum_{d|n, d|r} \mu(r/d)d$. By taking modulus on both sides and estimating, we get

$$|c_r(n)| \ll \sigma_1((n,r)) \leq \sigma_1(n) \ll n \log n.$$

□

3. Proof of Theorem 1.3

Let U be a parameter which tends to infinity be chosen later. For any natural numbers n_1 and n_2, we consider

$$\begin{aligned} f(n_1, n_2) &= \sum_{q_1,q_2=1}^{\infty} a_{q_1,q_2} c_{q_1}(n_1) c_{q_2}(n_2) \\ &= \sum_{[q_1,q_2] \leq U} a_{q_1,q_2} c_{q_1}(n_1) c_{q_2}(n_2) \\ &\quad + \sum_{[q_1,q_2] > U} a_{q_1,q_2} c_{q_1}(n_1) c_{q_2}(n_2) \\ &= \sum_{[q_1,q_2] \leq U} a_{q_1,q_2} c_{q_1}(n_1) c_{q_2}(n_2) \\ &\quad + O\left(\sum_{[q_1,q_2] > U} \frac{n_1 n_2 \log n_1 \log n_2}{[q_1, q_2]^{1+\delta}} \right) \quad \text{(using Lemma 2.7)} \\ &= \sum_{[q_1,q_2] \leq U} a_{q_1,q_2} c_{q_1}(n_1) c_{q_2}(n_2) \\ &\quad + O\left(n_1 \log n_1 n_2 \log n_2 \sum_{[q_1,q_2] > U} \frac{1}{[q_1, q_2]^{1+\delta}} \right). \end{aligned}$$

We first break the following sum into two sums as

$$\begin{aligned} A := \sum_{[q_1,q_2] > U} \frac{1}{[q_1, q_2]^{1+\delta}} &= \sum_{[q_1,q_2] > U, (q_1,q_2)=1} \frac{1}{[q_1, q_2]^{1+\delta}} \\ &\quad + \sum_{[q_1,q_2] > U, (q_1,q_2)>1} \frac{1}{[q_1, q_2]^{1+\delta}}. \end{aligned}$$

Since $nm = [n, m](n, m)$ for any natural numbers n and m, we see that the first sum in A becomes

$$\sum_{[q_1,q_2] > U, (q_1,q_2)=1} \frac{1}{[q_1, q_2]^{1+\delta}} = \sum_{t > U} \frac{d(t)}{t^{1+\delta}},$$

where $d(t)$ is the divisor function. Since $\sum_{t\le x} d(t) = x\log x + (2\gamma - 1)x + O(\sqrt{x})$, where γ is the Euler's constant, then for any real number $x \ge 2$, using Proposition 2.1, we can estimate as

$$\sum_{[q_1,q_2]>U,(q_1,q_2)=1} \frac{1}{[q_1,q_2]^{1+\delta}} = \sum_{t>U} \frac{d(t)}{t^{1+\delta}} = O\left(\frac{\log U}{U^\delta}\right).$$

Now, we consider the second sum in A. Since $(q_1, q_2) = \ell > 1$, we see that $q_1 = \ell r_0$ and $q_2 = \ell s_0$. Therefore, we get

$$[q_1,q_2] = \frac{q_1 q_2}{(q_1,q_2)} = \frac{\ell r_0 \ell s_0}{\ell} = \ell r_0 s_0.$$

Thus,

$$\sum_{[q_1,q_2]>U,(q_1,q_2)>1} \frac{1}{[q_1,q_2]^{1+\delta}} = \sum_{\ell\le U} \frac{1}{\ell^{1+\delta}} \sum_{r_0 s_0>U/\ell} \frac{1}{(r_0 s_0)^{1+\delta}} + \sum_{\ell>U} \frac{1}{\ell^{1+\delta}} \sum_{r_0 s_0>U/\ell} \frac{1}{(r_0 s_0)^{1+\delta}}.$$

Put $r_0 s_0 = t$ to get

$$\sum_{[q_1,q_2]>U,(q_1,q_2)>1} \frac{1}{[q_1,q_2]^{1+\delta}} = \sum_{\ell\le U} \frac{1}{\ell^{1+\delta}} \sum_{t>U/\ell} \frac{d(t)}{t^{1+\delta}} + \sum_{\ell>U} \frac{1}{\ell^{1+\delta}} \sum_{t>U/\ell} \frac{d(t)}{t^{1+\delta}}.$$

Note that in the above expression, the second sum is $O\left(\frac{1}{U^\delta}\right)$. Therefore, we can write

$$\sum_{[q_1,q_2]>U,(q_1,q_2)>1} \frac{1}{[q_1,q_2]^{1+\delta}} = \sum_{\ell\le U} \frac{1}{\ell^{1+\delta}} \frac{(U/\ell)\log(U/\ell)}{(U/\ell)^{1+\delta}} + O\left(\frac{1}{U^\delta}\right).$$

By Proposition 2.1, we estimate as

$$\sum_{[q_1,q_2]>U,(q_1,q_2)>1} \frac{1}{[q_1,q_2]^{1+\delta}} = O\left(\frac{\log^2 U}{U^\delta}\right).$$

Thus, the sum A can be estimated as

$$A = O\left(\frac{\log^2 U}{U^\delta}\right).$$

Hence,

$$f(n_1,n_2) = \sum_{[q_1,q_2]\le U} a_{q_1,q_2} c_{q_1}(n_1) c_{q_2}(n_2) + O\left((n_1\log n_1)(n_2\log n_2)\frac{\log^2 U}{U^\delta}\right).$$

Now, let N be a large enough positive integer and by summing $f(n_1, n_2)$ over all the natural numbers n_1 and $n_2 \le N$, we get

$$\sum_{n_1,n_2\le N} f(n_1,n_2) = \sum_{n_1,n_2\le N}\sum_{[q_1,q_2]\le U} a_{q_1,q_2} c_{q_1}(n_1) c_{q_2}(n_2) + O\left(\frac{N^4(\log N)^2(\log U)^2}{U^\delta}\right)$$

$$= N^2 a_{1,1} + B + O\left(\frac{N^4(\log N)^2(\log U)^2}{U^\delta}\right),$$

where

$$B = \sum_{2\le[q_1,q_2]\le U} a_{q_1,q_2} \sum_{n_1\le N} c_{q_1}(n_1) \sum_{n_2\le N} c_{q_2}(n_2).$$

Now, we shall estimate the sum B as follows:

$$\begin{aligned}
B &= \sum_{2\le[q_1,q_2]\le U} a_{q_1,q_2} \sum_{n_1\le N} c_{q_1}(n_1) \sum_{n_2\le N} c_{q_2}(n_2) \\
&\ll \sum_{2\le[q_1,q_2]\le U} \frac{q_1q_2 \log q_1 \log q_2}{[q_1,q_2]^{1+\delta}} \qquad \text{(using Lemma 2.6)} \\
&\le \left((\log U)^2 \sum_{2\le[q_1,q_2]\le U} \frac{q_1q_2}{[q_1,q_2]^{1+\delta}}\right) \\
&= \left((\log U)^2 \sum_{\substack{2\le[q_1,q_2]\le U \\ (q_1,q_2)=1}} \frac{q_1q_2}{[q_1,q_2]^{1+\delta}}\right) + \left((\log U)^2 \sum_{\substack{2\le[q_1,q_2]\le U \\ (q_1,q_2)=\ell>1}} \frac{q_1q_2}{[q_1,q_2]^{1+\delta}}\right).
\end{aligned}$$

By Proposition 2.1, we see that

$$\sum_{[q_1,q_2]\le U,(q_1,q_2)=1} \frac{q_1q_2}{[q_1,q_2]^{1+\delta}} = \sum_{t\le U} \frac{td(t)}{t^{1+\delta}} = \sum_{t\le U} \frac{d(t)}{t^\delta} = O(U^{1-\delta}\log U).$$

Now, consider

$$\begin{aligned}
\sum_{[q_1,q_2]\le U,(q_1,q_2)>1} \frac{q_1q_2}{[q_1,q_2]^{1+\delta}} &= \sum_{\ell\le U} \sum_{r_0s_0\le U/\ell} \frac{\ell r_0 s_0}{\ell^\delta (r_0s_0)^{1+\delta}} \\
&= \sum_{\ell\le U} \ell^{1-\delta} \sum_{t\le U/\ell} \frac{td(t)}{t^{1+\delta}} = \sum_{\ell\le U} \ell^{1-\delta} \sum_{t\le U/\ell} \frac{d(t)}{t^\delta} \\
&= \sum_{\ell\le U} \ell^{1-\delta} \frac{U}{\ell} \frac{\log(U/\ell)}{(U/\ell)^\delta} = U^{1-\delta} \sum_{\ell\le U} \log(U/\ell).
\end{aligned}$$

By evaluating the above two sums, we get

$$B = \begin{cases} O\left(U^{2-\delta}\log^2 U\right) & \text{if } \delta < 2, \\ O(\log^2 U) & \text{if } \delta = 2, \\ O(1) & \text{if } \delta > 2, \end{cases} \tag{3.1}$$

and hence,

$$\sum_{n_1,n_2\le N} f(n_1,n_2) = \begin{cases} N^2 a_{1,1} + O(U^{2-\delta}\log^2 U) + O\left(\frac{N^4\log^2 N\log^2 U}{U^\delta}\right) & \text{if } \delta < 2, \\ N^2 a_{1,1} + O(\log^2 U) + O\left(\frac{N^4\log^2 N\log^2 U}{U^\delta}\right) & \text{if } \delta = 2, \\ N^2 a_{1,1} + O(1) + O\left(\frac{N^4\log^2 N\log^2 U}{U^\delta}\right) & \text{if } \delta > 2. \end{cases} \tag{3.2}$$

In order to optimize the error term, we choose the parameter U as

$$U = \begin{cases} N^2\log N & \text{if } \delta < 2, \\ N^2\log N & \text{if } \delta = 2, \\ N^{\frac{4}{\delta}}(\log N)^{\frac{4}{\delta}} & \text{if } \delta > 2. \end{cases} \tag{3.3}$$

This gives us

$$\sum_{n_1,n_2\le N} f(n_1,n_2) = \begin{cases} N^2 a_{1,1} + O(N^{4-2\delta}(\log N)^{4-\delta}) & \text{if } \delta < 2, \\ N^2 a_{1,1} + O(\log^2 N) & \text{if } \delta = 2, \\ N^2 a_{1,1} + O(1) & \text{if } \delta > 2. \end{cases}$$

This proves Theorem 1.3 for all $\delta \ge 2$. Note that when $\delta < 2$, we see for $0 < \delta \le 1$ the error term is of bigger order than that of the main term and hence we cannot obtain the required asymptotic formula in this case. In order to resolve this problem, we shall introduce another parameter $V < U$, tending to infinity which is to be chosen later and rewrite the sum B as follows:

$$\begin{aligned} &\sum_{n_1,n_2\le N}\ \sum_{2\le[q_1,q_2]\le U} a_{q_1,q_2} c_{q_1}(n_1)c_{q_2}(n_2) \\ &\quad = \sum_{2\le[q_1,q_2]\le V} a_{q_1,q_2}\sum_{n_1\le N} c_{q_1}(n_1)\sum_{n_2\le N} c_{q_2}(n_2) \\ &\qquad + \sum_{V<[q_1,q_2]\le U} a_{q_1,q_2}\sum_{n_1\le N} c_{q_1}(n_1)\sum_{n_2\le N} c_{q_2}(n_2). \end{aligned} \tag{3.4}$$

Now, we shall evaluate the first term of the right-hand side of equation(3.4). Note that the first term is nothing but B with U replaced by V. So, we get the similar expression as before. Now, we estimate the second term of the right-hand side of equation (3.4).

By Lemma 2.5 and the hypothesis on Ramanujan coefficients, we get

$$\begin{aligned} &\sum_{V<[q_1,q_2]\le U} a_{q_1,q_2}\sum_{n_1\le N} c_{q_1}(n_1)\sum_{n_2\le N} c_{q_2}(n_2) \\ &\quad \le N^2 \sum_{V<[q_1,q_2]\le U} \frac{d(q_1)d(q_2)}{[q_1,q_2]^{1+\delta}} \end{aligned}$$

$$= N^2 \sum_{\substack{V < [q_1, q_2] \leq U \\ (q_1, q_2) = 1}} \frac{d(q_1)d(q_2)}{[q_1, q_2]^{1+\delta}} + N^2 \sum_{\substack{V < [q_1, q_2] \leq U \\ (q_1, q_2) > 1}} \frac{d(q_1)d(q_2)}{[q_1, q_2]^{1+\delta}}.$$

Now, consider the sum

$$\sum_{\substack{V < [q_1, q_2] \leq U \\ (q_1, q_2) = 1}} \frac{N^2 d(q_1)d(q_2)}{[q_1, q_2]^{1+\delta}} = N^2 \sum_{t \leq U} \frac{d_4(t)}{t^{1+\delta}} - N^2 \sum_{t \leq V} \frac{d_4(t)}{t^{1+\delta}}$$
$$= O\left(\frac{N^2 \log^3 U}{U^\delta}\right) + O\left(\frac{N^2 \log^3 V}{V^\delta}\right).$$

The above estimation is done using Proposition 2.1 and Lemma 2.2. Suppose $(q_1, q_2) = \ell \geq 2$ and hence, $q_1 = \ell r_0$ and $q_2 = \ell s_0$. This gives

$$N^2 \sum_{\substack{V < [q_1, q_2] \leq U \\ (q_1, q_2) > 1}} \frac{d(q_1)d(q_2)}{[q_1, q_2]^{1+\delta}} \leq N^2 \sum_{\ell \leq U} \sum_{r_0 s_0 \leq U/\ell} \frac{d^2(\ell)d(r_0)d(s_0)}{\ell^{1+\delta}(r_0 s_0)^{1+\delta}}$$
$$- N^2 \sum_{\ell \leq V} \sum_{r_0 s_0 \leq V/\ell} \frac{d^2(\ell)d(r_0)d(s_0)}{\ell^{1+\delta}(r_0 s_0)^{1+\delta}}$$
$$= N^2 \sum_{\ell \leq U} \frac{d^2(\ell)}{\ell^{1+\delta}} \sum_{t \leq U/\ell} \frac{d_4(t)}{t^{1+\delta}}$$
$$- N^2 \sum_{\ell \leq V} \frac{d^2(\ell)}{\ell^{1+\delta}} \sum_{t \leq V/\ell} \frac{d_4(t)}{t^{1+\delta}}$$
$$= N^2 \sum_{\ell \leq U} \frac{d^2(\ell)}{\ell^{1+\delta}} \frac{\ell^\delta}{U^\delta} \log^3(U/\ell)$$
$$+ N^2 \sum_{\ell \leq V} \frac{d^2(\ell)}{\ell^{1+\delta}} \frac{\ell^\delta}{V^\delta} \log^3(V/\ell)$$
$$\leq N^2 \frac{\log^3 U}{U^\delta} \sum_{\ell \leq U} \frac{d^2(\ell)}{\ell} + N^2 \frac{\log^3 V}{V^\delta} \sum_{\ell \leq V} \frac{d^2(\ell)}{\ell}$$
$$= O\left(\frac{N^2 \log^7 U}{U^\delta}\right) + O\left(\frac{N^2 \log^7 V}{V^\delta}\right).$$

The above estimation is done using Proposition 2.1 and Lemmas 2.2–2.3 repeatedly. Thus, we get

$$\sum_{V < [q_1, q_2] \leq U} a_{q_1, q_2} \sum_{n_1 \leq N} c_{q_1}(n_1) \sum_{n_2 \leq N} c_{q_2}(n_2) = O\left(\frac{N^2 \log^7 U}{U^\delta}\right)$$
$$+ O\left(\frac{N^2 \log^7 V}{V^\delta}\right).$$

Therefore, for the case $0 < \delta < 2$, we obtain

$$\sum_{n_1,n_2\leq N} f(n_1,n_2) = N^2 a_{1,1} + O(V^{2-\delta}\log^2 V) + O\left(\frac{N^2\log^7 V}{V^\delta}\right) + O\left(\frac{N^2\log^7 U}{U^\delta}\right) + O\left(\frac{N^4\log^2 N\log^2 U}{U^\delta}\right). \tag{3.5}$$

Now, we choose the parameters U and V as

$$U = \exp((\log N^N)^{\frac{2}{5}}) \quad \text{and} \quad V = N(\log N)^{\frac{5}{2}}.$$

Putting the values of U and V in (3.5), we get the required asymptotic formula and hence the theorem.

4. Proof of Theorem 1.4

Let U be the parameter to be chosen later. For the given integers n_1 and n_2, we consider

$$\begin{aligned} f(n_1,n_2) &= \sum_{q_1,q_2=1}^{\infty} a_{q_1,q_2} c_{q_1}(n_1) c_{q_2}(n_2) \\ &= \sum_{[q_1,q_2]\leq U} a_{q_1,q_2} c_{q_1}(n_1) c_{q_2}(n_2) + \sum_{[q_1,q_2]>U} a_{q_1,q_2} c_{q_1}(n_1) c_{q_2}(n_2) \\ &= \sum_{[q_1,q_2]\leq U} a_{q_1,q_2} c_{q_1}(n_1) c_{q_2}(n_2) \\ &\quad + O\left(\sum_{[q_1,q_2]>U} \frac{n_1 n_2 \log n_1 \log n_2}{[q_1,q_2]\,(\log[q_1,q_2])^\alpha}\right) \text{(using Lemma 2.7)}. \end{aligned}$$

By the definition of $N(t)$, we get

$$\sum_{[q_1,q_2]>U} \frac{1}{[q_1,q_2](\log[q_1,q_2])^\alpha} = \sum_{t>U} \frac{N(t)}{t\log^\alpha t} = O\left(\frac{1}{\log^{\alpha-2} U}\right).$$

The above estimation is done using Lemma 2.1 and Lemma 2.4. Therefore, we get

$$\begin{aligned} \sum_{n_1,n_2\leq N} f(n_1,n_2) &= \sum_{n_1,n_2\leq N} \sum_{[q_1,q_2]\leq U} a_{q_1,q_2} c_{q_1}(n_1) c_{q_2}(n_2) + O\left(\frac{N^4(\log N)^2}{(\log U)^{\alpha-2}}\right) \\ &= N^2 a_{1,1} + C + O\left(\frac{N^4(\log N)^2}{(\log U)^{\alpha-2}}\right), \end{aligned}$$

where

$$C = \sum_{2\leq[q_1,q_2]\leq U} a_{q_1,q_2} \sum_{n_1\leq N} c_{q_1}(n_1) \sum_{n_2\leq N} c_{q_2}(n_2).$$

By Lemma 2.6 and the hypothesis on Ramanujan coefficients, we get

$$
\begin{aligned}
C &\ll \sum_{2\leq[q_1,q_2]\leq U} \frac{q_1q_2 \log q_1 \log q_2}{[q_1,q_2]\log^{\alpha}[q_1,q_2]}\\
&\leq \left((\log U)^2 \sum_{2\leq[q_1,q_2]\leq U} \frac{q_1q_2}{[q_1,q_2]\log^{\alpha}[q_1,q_2]}\right)\\
&= \left((\log U)^2 \sum_{\substack{2\leq[q_1,q_2]\leq U\\ (q_1,q_2)=1}} \frac{q_1q_2}{[q_1,q_2]\log^{\alpha}[q_1,q_2]}\right)\\
&\quad + \left((\log U)^2 \sum_{\substack{2\leq[q_1,q_2]\leq U\\ (q_1,q_2)=\ell}} \frac{q_1q_2}{[q_1,q_2]\log^{\alpha}[q_1,q_2]}\right)\\
&= \sum_{2\leq r\leq U} \frac{d(r)}{r\log^{\alpha} r} + \sum_{\ell\leq U}\ell \sum_{2\leq r\leq \frac{U}{l}} \frac{d(r)}{r\log^{\alpha} r}.
\end{aligned}
$$

On evaluating the above two sums using Proposition 2.1 and Lemma 2.2, we get

$$
C = O\left(\frac{U^2}{\log^{\alpha-2} U}\right).
$$

This gives us

$$
\sum_{n_1,n_2\leq N} f(n_1,n_2) = N^2 a_{1,1} + O\left(\frac{U^2}{(\log U)^{\alpha-2}}\right) + O\left(\frac{N^4(\log N)^2}{(\log U)^{\alpha-2}}\right).
$$

In order to optimize the error term, we choose the parameter U as

$$
U = N^2 \log N.
$$

This choice of U gives us

$$
\sum_{n_1,n_2\leq N} f(n_1,n_2) = N^2 a_{1,1} + O\left(\frac{N^4}{(\log N)^{\alpha-4}}\right).
$$

In the above expression for partial sums of $f(n_1,n_2)$, the error term is of a bigger order than that of the main term and hence we cannot get the required asymptotic formula. In order to resolve this problem, we shall introduce another parameter $V < U$ which tends to infinity and is to be chosen later. We rewrite the sum C as follows:

$$
\begin{aligned}
&\sum_{2\le[q_1,q_2]\le U} a_{q_1,q_2}\sum_{n_1\le N} c_{q_1}(n_1)\sum_{n_2\le N} c_{q_2}(n_2)\\
&\quad=\sum_{2\le[q_1,q_2]\le V} a_{q_1,q_2}\sum_{n_1\le N} c_{q_1}(n_1)\sum_{n_2\le N} c_{q_2}(n_2)\\
&\qquad+\sum_{V<[q_1,q_2]\le U} a_{q_1,q_2}\sum_{n_1\le N} c_{q_1}(n_1)\sum_{n_2\le N} c_{q_2}(n_2). \qquad (4.1)
\end{aligned}
$$

Note that the first term of the right-hand side of equation (4.1) is nothing but C with U being replaced by V. Hence, as before, we get

$$
\sum_{2\le[q_1,q_2]\le V} a_{q_1,q_2}\sum_{n_1\le N} c_{q_1}(n_1)\sum_{n_2\le N} c_{q_2}(n_2)=O\left(\frac{V^2}{\log^{\alpha-2}V}\right).
$$

Now, we shall estimate the second term of the right-hand side of equation (4.1). Using Lemma 2.5 and the hypothesis on the Ramanujan coefficients, we get

$$
\begin{aligned}
&\sum_{V<[q_1,q_2]\le U} a_{q_1,q_2}\sum_{n_1\le N} c_{q_1}(n_1)\sum_{n_2\le N} c_{q_2}(n_2)\\
&\quad\ll N^2\sum_{V<[q_1,q_2]\le U}\frac{d(q_1)d(q_2)}{[q_1,q_2]\log^{\alpha}[q_1,q_2]}\\
&\quad= N^2\sum_{\substack{V<[q_1,q_2]\le U\\ (q_1,q_2)=1}}\frac{d(q_1)d(q_2)}{[q_1,q_2]\log^{\alpha}[q_1,q_2]}\\
&\qquad+N^2\sum_{\substack{V<[q_1,q_2]\le U\\ (q_1,q_2)>1}}\frac{d(q_1)d(q_2)}{[q_1,q_2]\log^{\alpha}[q_1,q_2]}.
\end{aligned}
$$

The first term of the above expression becomes

$$
\begin{aligned}
N^2\sum_{\substack{V<[q_1,q_2]\le U\\ (q_1,q_2)=1}}\frac{d(q_1)d(q_2)}{[q_1,q_2]\log^{\alpha}[q_1,q_2]}&=N^2\sum_{r\le U}\frac{d_4(r)}{r\log^{\alpha}r}-N^2\sum_{r\le V}\frac{d_4(r)}{r\log^{\alpha}r}\\
&=O\left(\frac{N^2}{\log^{\alpha-3}U}\right)+O\left(\frac{N^2}{\log^{\alpha-3}V}\right),
\end{aligned}
$$

The above estimation is done using Proposition 2.1 and Lemma 2.2. Consider the other sum

$$
\begin{aligned}
N^2\sum_{\substack{V<[q_1,q_2]\le U\\ (q_1,q_2)=\ell>1}}\frac{d(q_1)d(q_2)}{[q_1,q_2]\log^{\alpha}[q_1,q_2]}&=N^2\sum_{\ell\le U}\sum_{r_0s_0\le U/\ell}\frac{d(\ell r_0)d(\ell s_0)}{\ell r_0s_0\log^{\alpha}(\ell r_0s_0)}\\
&\quad-N^2\sum_{\ell\le V}\sum_{r_0s_0\le V/\ell}\frac{d(\ell r_0)d(\ell s_0)}{\ell r_0s_0\log^{\alpha}(\ell r_0s_0)}
\end{aligned}
$$

$$
\begin{aligned}
&\le N^2 \sum_{\ell \le U} \frac{d^2(\ell)}{\ell} \sum_{t \le U/\ell} \frac{d_4(t)}{t \log^\alpha(\ell t)} \\
&\quad + N^2 \sum_{\ell \le V} \frac{d^2(\ell)}{\ell} \sum_{t \le V/\ell} \frac{d_4(t)}{t \log^\alpha(\ell t)} \\
&= O\left(\frac{N^2}{\log^{\alpha-7} U}\right) + O\left(\frac{N^2}{\log^{\alpha-7} V}\right).
\end{aligned}
$$

The above estimation is done using Proposition 2.1 and Lemmas 2.2–2.3 repeatedly. Thus, we conclude that

$$
\sum_{V<[q_1,q_2]\le U} a_{q_1,q_2} \sum_{n_1 \le N} c_{q_1}(n_1) \sum_{n_2 \le N} c_{q_2}(n_2) = O\left(\frac{N^2}{\log^{\alpha-7} U}\right) + O\left(\frac{N^2}{\log^{\alpha-7} V}\right).
$$

Hence by the above calculation, we get

$$
\begin{aligned}
\sum_{n_1,n_2 \le N} f(n_1,n_2) &= N^2 a_{1,1} + O\left(\frac{V^2}{(\log V)^{\alpha-2}}\right) + O\left(\frac{N^2}{(\log V)^{\alpha-7}}\right) \\
&\quad + O\left(\frac{N^2}{(\log U)^{\alpha-7}}\right) + O\left(\frac{N^4 (\log N)^2}{(\log U)^{\alpha-2}}\right).
\end{aligned}
$$

In order to optimize the error term, we choose our parameters $U = \exp((\log N^N)^{\frac{2}{5}})$ and $V = N \log^{\frac{5}{2}} N$, and hence we get

$$
\sum_{n_1,n_2 \le N} f(n_1,n_2) = N^2 a_{1,1} + O\left(\frac{N^2}{(\log N)^{\alpha-7}}\right).
$$

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