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Space of invariant bilinear forms

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Abstract. Let $\mathbb{F}$ be a field, V a vector space of dimension n over $\mathbb{F}$. Then the set of bilinear forms on V forms a vector space of dimension n^2 over $\mathbb{F}$. For char $\mathbb{F} \neq 2$, if T is an invertible linear map from V onto V then the set of T-invariant bilinear forms, forms a subspace of this space of forms. In this paper, we compute the dimension of T-invariant bilinear forms over $\mathbb{F}$. Also we investigate similar type of questions for the infinitesimally T-invariant bilinear forms (T-skew symmetric forms). Moreover, we discuss the existence of nondegenerate invariant (resp. infinitesimally invariant) bilinear forms.

Keywords. Vector space; invariant bilinear forms; infinitesimal invariant forms.

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1. Introduction

Let $\mathbb{F}$ be a field of characteristic $\neq 2$, $n \geq 1$ an integer. Let V be a vector space of dimension n over $\mathbb{F}$. Then the set of bilinear forms on V forms a vector space of dimension n^2 over $\mathbb{F}$. If T is an invertible linear map from V onto V then the set $\mathcal{B}$ of T-invariant bilinear forms, forms a subspace of this space of forms. In this paper, we investigate the following question.

Question. What is the dimension of $\mathcal{B}$? Does there exist any non-degenerate form in $\mathcal{B}$?

Let V be a vector space over $\mathbb{F}$, then a bilinear form B on V is said to be invariant (resp. infinitesimally invariant) under a linear map $T : V \to V$, if for all $u, v \in V$, $B(Tu, Tv) = B(u, v)$ (resp. $B(Tu, v) + B(u, Tv) = 0$). A bilinear form B on V is said to be *symmetric*, if for all $u, v \in V$, $B(u, v) = B(v, u)$. If char $\mathbb{F} \neq 2$, then a bilinear form B on V is said to be *skew-symmetric*, if for all $u, v \in V$, $B(u, v) = -B(v, u)$.

The starting point in this study is the existence of such T-invariant forms. This problem has been answered in many ways in the literature. Sergeichuk [6] studied systems of forms and linear mappings by associating with them self-adjoint representations of a category with involution. Gongopadhyay and Kulkarni [2] investigated the question of existence of T-invariant non-degenerate symmetric (resp. skew-symmetric) bilinear forms. They also answered the infinitesimal version of this question. In the literature, the problem to find necessary and sufficient conditions for an endomorphism to be self-adjoint, skew-selfadjoint or orthogonal for at least one non-degenerate quadratic form (or for a symplectic form) has also acquired an important place. Frobenius [1] proved that every endomorphism of a finite dimensional vector space V is self-adjoint for at least one non-degenerate symmetric bilinear form on V. Later, Stenzel [7] determined when an endomorphism could be skew-selfadjoint for a non-degenerate quadratic form, or self-adjoint or skew-selfadjoint for a symplectic form on complex vector spaces. However his results were later generalized to an arbitrary field [3]. Pazzis [5] tackled the case of the automorphisms of a finite dimensional vector space that are orthogonal (resp. symplectic) for at least one non-degenerate quadratic form (resp. symplectic form) over an arbitrary field of characteristics 2.

In this paper, we investigate the question of computing the dimension of the space of bilinear forms invariant (resp. infinitesimal invariant) under a given invertible linear transformation (resp. linear transformation) on a finite dimensional vector space over a field of characteristic $\neq 2$. We have answered this question for several possible types of linear transformations and also discussed about the existence of non-degenerate forms in every case.

2. Preliminaries

Let $f(x) \in \mathbb{F}[x]$, $f(x)$ monic, $\deg f = n$ and $0, \pm 1$ are not the roots. Then $f^*(x) = x^n f(x^{-1})$ is called as the *reciprocal* of f (see also [2]). A monic polynomial $f(x) \in \mathbb{F}[x]$, $f(0) \neq 0 \neq f(\pm 1)$ is said to be *self-reciprocal* if $f(x) = f^*(x)$.

Lemma 2.1. *Let $f(x) \in \mathbb{F}[x]$ be monic, $f(0) \neq 0 \neq f(\pm 1)$. If $\alpha \in \bar{\mathbb{F}}$ is a root of $f(x)$ of multiplicity m, then α^{-1} is a root of $f^*(x)$ of the same multiplicity, i.e., m.*

Proof. Easy to see. □

Lemma 2.2. *Let $f(x) \in \mathbb{F}[x]$ be self-reciprocal*, $\deg f = n$, *then n is even.*

Proof. Clear. □

If $p(x) = \sum_{i=0}^{2n} c_i x^i \in \mathbb{F}[x]$ is a *self-reciprocal* polynomial, then $c_0 = c_{2n} = 1$ and for $1 \leq i \leq n$, $c_i = c_{2n-i}$.

DEFINITION 2.3

A linear operator $T : V \to V$ is said to be *self-reciprocal* if its characteristic polynomial $\chi_T(x)$ is *self-reciprocal*.

It is important to observe that *self-reciprocal* operators exist only on even dimensional vector spaces.

Now for $T \in \mathrm{End}(V)$, let us study bilinear form B on V satisfying

$$B(Tv, w) + B(v, Tw) = 0 \quad \text{for all } v, w \in V \text{ (i.e. } T\text{-skew symmetric forms).}$$

Let $f(x) \in \mathbb{F}[x]$, $f(x)$ monic, deg $f = n$ and 0 is not a root of f, then $f^{-}(x) = (-1)^n f(-x)$ is called as the *additive-dual* of f. Let $f(x) = x^{\epsilon} f_0(x)$, $\epsilon \geq 0$ such that $f_0(0) \neq 0$ and f_0 monic. We call f_0 the reduced part of f. A monic polynomial $f(x) \in \mathbb{F}[x]$, $f(0) \neq 0$ is said to be *additive self-dual* (α-self dual for short) if $f(x) = f^{-}(x)$.

DEFINITION 2.4

A linear operator $T : V \to V$ is said to be α*-self dual* if its characteristic polynomial is so.

Note that all the results in the invariant case have their infinitesimal counterparts. In the last section of this paper we only state the main results in the infinitesimal set up.

2.1 *Dimension invariance over field extensions*

Let V be a vector space over $\mathbb{F}$ of dimension n, $T \in \mathrm{End}(V)$. The characteristic polynomial $\chi_T(x) \in \mathbb{F}[x]$ of T can be expressed uniquely (up to permutations of the factors) as $\chi_T(x) = \Pi_{i=1}^{r} f_i(x)$, $f_i(x) \mid f_{i-1}(x)$ and $T|_{V_{f_i(x)}=\{v\in V \mid f_i(T)v=0\}}$ is cyclic for $2 \leq i \leq r$. The polynomials $f_i(x)$ are the invariant factors of T and this factorization remains invariant under field extensions. There is an ordered basis $(e_1, \ldots, e_n)$ of V over $\mathbb{F}$ with respect to which the matrix representation of T is given as

$$R = \begin{pmatrix} C(f_1(x)) & & & \\ & \ddots & & \\ & & \ddots & \\ & & & C(f_r(x)) \end{pmatrix},$$

where for $1 \leq i \leq r$, $C(f_i(x))$ is a companion matrix of $f_i(x)$ and R is the rational canonical form of T. Under field extensions this form remains the same. In the next proposition, we show that the dimension of T-invariant (resp. infinitesimally invariant) bilinear forms remains invariant under field extensions of $\mathbb{F}$.

PROPOSITION 2.5

Let V be a n dimensional vector space over $\mathbb{F}$, $T \in \mathrm{End}(V)$. Then dimension of the space of T-invariant (resp. infinitesimally invariant) bilinear forms remains invariant under field extensions of $\mathbb{F}$.

Proof. Without loss of generality, take $V = \mathbb{F}^n$. Then for $\mathbb{K} \supseteq \mathbb{F}$ a field extension, $V' = \mathbb{K}^n$ is its extension as a $\mathbb{K}$ vector space. Let $(e_1, \ldots, e_n)$ be the ordered basis of V over $\mathbb{F}$ with respect to which T has the rational canonical form R as expressed above. This will still be an ordered basis of V' over $\mathbb{K}$. Let $T' \in \mathrm{End}(V')$ be determined by R. Let $\mathcal{B} = \{B \mid B(T'u', T'v') = B(u', v')$ (resp. $B(T'u', v') + B(u', T'v') = 0)$ $\forall\, u', v' \in V'\}$ be the space of T'-invariant (resp. infinitesimally T'-invariant) forms over $\mathbb{K}$. Then $\mathcal{B} = \{X = (B(e_i, e_j) = x_{ij}) \in M_n(\mathbb{K}) \mid R^t X R = X$ (resp. $R^t X + X R = 0)\}$, which is the solution space of a system of n^2 linear homogeneous equations having coefficients in $\mathbb{F}$ (as $R \in M_n(\mathbb{F})$), so is a vector space over $\mathbb{F}$ with $\dim \mathcal{B}|_{\mathbb{K}} = \dim \mathcal{B}|_{\mathbb{F}}$. □

3. Invariance under indecomposable operators

Now let us discuss bilinear forms invariant under some indecomposable transformations. We consider two cases: (1) T or $-T$ unipotent and (2) T and $-T$ nonunipotent.

Case 1. *T or $-T$ unipotent.* Since invariant forms of T or $-T$ are the same, we may assume T as unipotent. Let $V = \mathbb{F}^n$, $e_1, \ldots, e_n \in V$ be the standard bases elements. Let $T : V \to V$ be a linear transformation with the Jordan block as

$$C = \begin{pmatrix} 1 & 0 & 0 & & \cdots & 0 \\ 1 & 1 & 0 & & \cdots & 0 \\ 0 & 1 & 1 & & \cdots & 0 \\ \vdots & & & \ddots & \ddots & \\ 0 & 0 & 0 & & \cdots & 1 \end{pmatrix}.$$

Then we have $Te_n = e_n$ and $\forall\, i,\ 1 \leq i \leq n-1,\ \ Te_i = e_i + e_{i+1}$. Now let $B : V \times V \to k$ be any bilinear form invariant under T. Again, we have the equation

$$\big(B(e_i, e_j)\big) = X = C^t X C = \big(B(Ce_i, Ce_j)\big) = \big(B(Te_i, Te_j)\big).$$

Here $C = I + N$, thus the above equation becomes $(I + N^t)X(I + N) = X$ which implies and is implied by $N^t X + XN + N^t XN = 0$. Similarly for $C = -I + N$, we have $-N^t X - XN^t + N^t XN = 0$.

Theorem 3.1. *Let V be a vector space of dimension n over $\mathbb{F}$. If T is an indecomposable invertible transformation from V onto V such that T or $-T$ is unipotent, then the space of T-invariant bilinear forms is of dimension n. If n is even (resp. odd) then there exists a nondegenerate T-invariant skew symmetric (resp. symmetric) but not a symmetric (resp. skew-symmetric) bilinear form.*

Proof. Let us consider C as above with the standard basis $\{e_1, \ldots, e_n\}$ of V over $\mathbb{F}$ and B be a T -invariant bilinear form with $(B(e_i, e_j)) = X$.

Case i. $i = n$: $x_{n,j} = B(e_n, e_j) = B(Ce_n, Ce_j)$ for $1 \leq j \leq n$. Thus we have $x_{n,j} = x_{n,j} + x_{n,j+1}$ for $1 \leq j \leq n-1$, i.e., $x_{n,j+1} = 0$ for $1 \leq j \leq n-1$.

Case ii. $j = n$: $x_{i,n} = B(e_i, e_n) = B(Ce_i, Ce_n)$ for $1 \leq i \leq n$. Thus we have $x_{i,n} = x_{i,n} + x_{i+1,n}$ for $1 \leq i \leq n-1$, i.e., $x_{i+1,n} = 0$ for $1 \leq i \leq n-1$.

Case iii. $1 \leq i, j \leq n-1$: $x_{i,j} = B(e_i, e_j) = B(Ce_i, Ce_j) = B(e_i + e_{i+1}, e_j + e_{j+1})$ i.e., $x_{i,j} = x_{i,j} + x_{i,j+1} + x_{i+1,j} + x_{i+1,j+1}$, which says that $x_{i,j+1} = -x_{i+1,j} - x_{i+1,j+1}$ for $1 \leq i, j \leq n-1$.

Thus recursively we have $x_{n-i,j+1} = 0$ for $i + 1 \leq j \leq n-1$ and $x_{n-(s-1),s} = (-1)^{s-1} x_{n,1}$ for $1 \leq s \leq n-1$. So,

$$X = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & \cdots & x_{1,n-1} & (-1)^{n-1} x_{n,1} \\ x_{2,1} & x_{2,2} & x_{2,3} & \cdots & (-1)^{n-2} x_{n,1} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n-1,1} & (-1)x_{n,1} & 0 & \cdots & 0 & 0 \\ x_{n,1} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Note that $\det X = x_{n,1}^n$, B is non degenerate when $x_{n,1}$ is nonzero.

Further, we have $x_{n-2,2} = (-1)x_{n-1,1} + (-1)^2 x_{n,1}$, $x_{n-3,3} = (-1)^2 x_{n-1,1} + 2(-1)^3 x_{n,1}$ and thus recursively we get for $1 \le t \le n-1$, $x_{n-t,t} = (-1)^{t-1} x_{n-1,1} + (t-1)(-1)^t x_{n,1}$, so

$$x_{1,n-1} = (-1)^{n-2} x_{n-1,1} + (n-2)(-1)^{n-1} x_{n,1}. \tag{3.1}$$

Similarly $x_{n-3,2} = -x_{n-2,1} - x_{n-2,2} = (-1)x_{n-2,1} + (-1)^2 x_{n-1,1} + (-1)^3 x_{n,1}$, $x_{n-4,3} = -x_{n-3,2} - x_{n-3,3} = (-1)^2 x_{n-2,1} + 2(-1)^3 x_{n-1,1} + 3(-1)^4 x_{n,1}$ and thus recursively for $2 \le t \le n-1$, we get $x_{n-t,t-1} = (-1)^{t-2} x_{n-2,1} + (t-2)(-1)^{t-1} x_{n-1,1} + (t-1)(-1)^t x_{n,1}$, so

$$x_{1,n-2} = (-1)^{n-3} x_{n-2,1} + (n-3)(-1)^{n-2} x_{n-1,1} + (n-2)(-1)^{n-1} x_{n,1}. \tag{3.2}$$

Repeating this process recursively, we get for every $1 \le s \le n-2$ and $s \le t \le n-1$,

$$x_{n-t,t-s+1} = (-1)^{t-s} x_{n-s,1} + \sum_{i=0}^{s-1} (t-s+i)(-1)^{t-s+i+1} x_{n-s+i+1,1}, \tag{3.3}$$

i.e., for $1 \le s \le n-2$, we have

$$x_{1,n-s} = (-1)^{n-s-1} x_{n-s,1} + \sum_{i=0}^{s-1} (n-s-1+i)(-1)^{n-s+i} x_{n-s+1+i,1}. \tag{3.4}$$

Thus basis of the space of invariant forms over $\mathbb{F}$ is

$$\begin{aligned} \Bigg\{ & e_{n,1} + \sum_{s=1}^{n-2} \sum_{t=s}^{n-1} (t-1)(-1)^t e_{n-t,t-s+1}, e_{n-1,1} + \sum_{t=2}^{n-1} (-1)^{t-1} e_{n-t,t} \\ & + \sum_{s=2}^{n-2} \sum_{t=s}^{n-1} (t-2)(-1)^{t-1} e_{n-t,t-s+1}, \ldots, e_{n-l,1} + \sum_{t=l+1}^{n-1} (-1)^{t-l} e_{n-t,t-l+1} \\ & + \sum_{s=l+1}^{n-2} \sum_{t=s}^{n-1} (t-l-1)(-1)^{t-l} e_{n-t,t-s+1}, \ldots, e_{2,1} - e_{1,2}, e_{1,1} \Bigg\} \end{aligned}$$

and X has a unique expression as a linear combination of these vectors with $x_{n,1}, x_{n,2}, \ldots, x_{2,1}, x_{1,1}$ respectively as free coefficients.

Note that if n is even then B symmetric (i.e., $X^t = X$) implies $x_{n,1} = 0$, i.e., B cannot be nondegenerate symmetric form. However we have plenty of nondegenerate T-invariant skew symmetric forms obtainable by taking $x_{n,1} \neq 0$, $x_{i,j} = -x_{j,i}$ and substituting in X.

Similarly if n is odd, then B skew symmetric (i.e. $X^t = -X$) implies $x_{n,1} = 0$, so B cannot be a nondegenerate skew symmetric form. However we have plenty of nondegenerate T-invariant symmetric forms obtainable by taking $x_{n,1} \neq 0$, $x_{i,j} = x_{j,i}$ and substituting in X. $\square$

COROLLARY 3.2

With the hypothesis as in the theorem, if n is odd, then the subspace of T-invariant symmetric forms is of dimension $\frac{n+1}{2}$ and the subspace of T-invariant skew symmetric forms is of dimension $\frac{n-1}{2}$.

Proof. Let $X = (x_{i,j})$ be a T-invariant bilinear form. If X is symmetric, then as n is odd from equation (3.4), we get

$$\begin{aligned} x_{n-s,1} = x_{1,n-s} &= (-1)^s x_{n-s,1} + (n-s-1)(-1)^{s-1} x_{n-s+1,1} \\ &\quad + \cdots + (n-2)x_{n,1}. \end{aligned}$$

If s is odd, i.e., $n-s$ is even, then

$$x_{n-s,1} = \frac{1}{2}[(n-s-1)(-1)^{s-1} x_{n-s+1,1} + \cdots + (n-2)x_{n,1}],$$

however there is no condition on $x_{n-s,1}$ for $n-s$ odd. Thus with similar steps as in the theorem one may calculate the basis with $\{x_{1,1}, x_{3,1}, \ldots, x_{n,1}\}$ respectively as the free coefficients for the T-invariant symmetric forms. So subspace of T-invariant symmetric bilinear forms is of dimension $\frac{n+1}{2}$.

Similarly $\{x_{2,1}, x_{4,1}, \ldots, x_{n-1,1}\}$ is the largest set of free coefficients for the T-invariant skew symmetric forms. So the subspace of T-invariant skew symmetric forms is of dimension $\frac{n-1}{2}$. □

COROLLARY 3.3

With the hypothesis as in the theorem, if n is even, then the subspace of T-invariant symmetric forms is of dimension $\frac{n}{2}$ and the subspace of T-invariant skew symmetric forms is of dimension $\frac{n}{2}$.

Proof. Similar. □

Case 2. *T and $-T$ nonunipotent*. In light of Proposition 2.5, for $\mathbb{F}$ algebraically closed (and hence a perfect field) we know (Theorem 15.3, J E Humphreys, Linear Alg. Groups, p. 99) that for $T \in O(V, B)$ (the orthogonal group $O(V, B)$, [2] being an algebraic group), if $T = T_s U$ is the Jordan decomposition of T with T_s, U semisimple and unipotent respectively, then $T_s, U \in O(V, B)$. In the next lemma, we use this fact to find the bilinear forms invariant under nonunipotent linear operators.

Lemma 3.4. *Let V be a vector space of dimension n over $\mathbb{F}$. If T is an indecomposable, non-unipotent invertible transformation from V onto V such that the characteristic polynomial $\chi_T(x) = (x-\alpha)^n$, $\pm 1 \neq \alpha \in \mathbb{F}$, then the space of T-invariant bilinear forms $\mathcal{B} = 0$.*

Proof. Clear (see [2]). □

4. Invariance under decomposable operators

Let us denote minimal polynomial of a linear operator T by $m_T(x)$.

Lemma 4.1. *Let V be a finite dimensional vector space over $\mathbb{F}$, $T : V \to V$ an invertible linear operator, $\alpha, \beta \in \mathbb{F}$, $\alpha\beta \neq 1$. If $(x-\alpha)^r$ and $(x-\beta)^s$ occur as elementary divisors of $\chi_T(x)$, then for B a T-invariant bilinear form on V for all $u \in V_{(x-\alpha)^r}$, $v \in V_{(x-\beta)^s}$, we have $B(u, v) = 0$.*

Proof. Since $T \in O(V, B)$ implies $T_s \in O(V, B)$, so for all $u \in V_{(x-\alpha)^r}$, $v \in V_{(x-\beta)^s}$, we have $B(T_s u, T_s v) = B(u, v)$ or $(\alpha\beta - 1)B(u, v) = 0$, i.e., $B(u, v) = 0$. □

Lemma 4.2. *Let V be a finite dimensional vector space over $\mathbb{F}$, $T : V \to V$ an invertible linear operator and $\pm 1 \neq \alpha$, $\beta \in \mathbb{F}$, $\alpha\beta = 1$. If $(x-\alpha)^r$ and $(x-\beta)^s$ occur as elementary divisors of $\chi_T(x) = (x-\alpha)^r(x-\beta)^s = m_T(x)$, then dimension of the space of T-invariant bilinear forms is $2 \cdot \min\{r, s\}$.*

Proof. Without loss of generality, we can assume that $V = V_{(x-\alpha)^r(x-\beta)^s}$. Then $\dim V = r + s = n$, $V = V_{(x-\alpha)^r} \oplus V_{(x-\beta)^s}$ and the Jordan form of T is

$$C = \left(\begin{array}{cccc|cccc} \alpha & & & & & & & \\ 1 & \ddots & & & & & & \\ & \ddots & \ddots & & & & & \\ & & 1 & \alpha & & & & \\ \hline & & & & \beta & & & \\ & & & & 1 & \ddots & & \\ & & & & & \ddots & \ddots & \\ & & & & & & 1 & \beta \end{array}\right).$$

Let $\{e_1, \ldots, e_r, e_{r+1}, \ldots, e_{n=r+s}\}$ be the standard basis with $\{e_1, \ldots, e_r\}$ as a basis for $V_{(x-\alpha)^r}$ and $\{e_{r+1}, \ldots, e_{r+s}\}$ as a basis for $V_{(x-\beta)^s}$. Then we have for $1 \leq i \leq r-1$, $Te_i = \alpha e_i + e_{i+1}$, $Te_r = \alpha e_r$ and for $1 \leq j \leq s-1$, $Te_{r+j} = \beta e_{r+j} + e_{r+j+1}$, $Te_n = \beta e_n$.

In light of Proposition 2.5, without loss of generality, we may assume that $\mathbb{F}$ is algebraically closed and hence a perfect field. Let $T = T_s U$ be the Jordan decomposition of T and B for any T-invariant bilinear form given by $X = (x_{ij}) = (B(e_i, e_j))$. Then $T_s \in O(V, B)$, so for $v, w \in V_{(x-\alpha)^r}$, $B(v, w) = B(T_s v, T_s w) = \alpha^2 B(v, w)$, but as $\alpha \neq \pm 1$, $B(v, w) = 0$. Thus $B|_{V_{(x-\alpha)^r}} = 0$ and similarly one says that $B|_{V_{(x-\beta)^s}} = 0$.

Now again, without loss of generality, one assumes that $r \leq s$. Let us find the $r \times s$ block of X.

Case i. For $j = n$ and $1 \leq i \leq r-1$, $B(Te_i, Te_n) = B(e_i, e_n) = x_{in}$ implies $x_{i+1n} = 0$.

Case ii. For $i = r$ and $1 \leq j \leq s-1$, $B(Te_r, Te_{r+j}) = B(e_r, e_{r+j}) = x_{rr+j}$ implies $x_{rr+j+1} = 0$.

Case iii. For $1 \leq i \leq r-1$ and $1 \leq j \leq s-1$, $B(Te_i, Te_{r+j}) = B(e_i, e_{r+j}) = x_{ir+j}$ implies $\alpha x_{ir+j+1} + \beta x_{i+1r+j} + x_{i+1r+j+1} = 0$, i.e.,

$$x_{ir+j+1} = -\frac{\beta}{\alpha} x_{i+1r+j} - \frac{1}{\alpha} x_{i+1r+j+1}.$$

So $\{e_{ir+1} + \sum_{j=1,k>r+1}^{i-1} u_{jk}(\alpha,\beta)e_{jk} \ : \ 1 \le i \le r\}$ forms a basis for this $r \times s$ block, where e_{ij} is an elementary matrix of order n with 1 at ij-th entry and 0 elsewhere and $u_{ij}(\alpha,\beta)$ are some particular values in $\mathbb{Q}(\alpha,\beta)$.

Now let us find the $s \times r$ block of $X = (B(e_i, e_j))$.

Case i. For $i = n,\ 1 \le j \le r-1$, $B(Te_n, Te_j) = B(e_n, e_j) = x_{nj}$ implies $x_{nj+1} = 0$.

Case ii. For $j = r,\ 1 \le i \le s-1$, $B(Te_{r+i}, Te_r) = B(e_{r+i}, e_r) = x_{r+i\,r}$ implies $x_{r+i\,r} + \alpha x_{r+i+1\,r} = x_{r+i\,r}$, i.e., $x_{r+i+1\,r} = 0$.

Case iii. For $1 \le i \le s-1$ and $1 \le j \le r-1$, $B(Te_{r+i}, Te_j) = B(e_{r+i}, e_j) = x_{r+i\,j}$ implies $\beta x_{r+i\,j+1} + \alpha x_{r+i+1\,j} + x_{r+i+1\,j+1} = 0$, i.e.,

$$x_{r+i+1\,j} = -\frac{\beta}{\alpha} x_{r+i\,j+1} - \frac{1}{\alpha} x_{r+i+1\,j+1}.$$

So $\{e_{r+1i} + \sum_{k>r+1,j=1}^{i-1} u_{kj}(\alpha,\beta)e_{kj} \ : \ 1 \le i \le r\}$ forms a basis for this $s \times r$ block, where $u_{ij}(\alpha,\beta) = u_{ji}(\alpha,\beta)$.

Thus dimension of space of T-invariant bilinear forms is $2r = 2 \cdot \min\{r,s\}$. □

Note that the T-invariant bilinear forms obtained in the above lemma is always degenerate for $s \ne r$ as against, we have lots of nondegenerate forms for $s = r$ obtained in the paper.

COROLLARY 4.3

Under the hypothesis of the lemma, if $(x-\alpha)^r$ and $(x-\beta)^s$ occur as elementary divisors of $\chi_T(x) = (x-\alpha)^r(x-\beta)^s$, then the dimension of the subspace of T-invariant symmetric (resp. skew symmetric) bilinear forms is $\min\{r,s\}$.

Proof. Again, without loss of generality, we can assume that $V = V_{(x-\alpha)^r(x-\beta)^s}$ and denote the space of T-invariant bilinear forms by $\mathcal{B} = \{X = (x_{ij})\}$. Then by the lemma, $\{e_{ir+1} + e_{r+1i} + \sum_{j=1,k>r+1}^{j=i-1} u_{jk}(\alpha,\beta)(e_{jk} + e_{kj}) \ : \ 1 \le i \le r\}$ forms a basis for $\mathcal{B}$ if $\mathcal{B}$ is symmetric and $\{e_{ir+1} - e_{r+1i} + \sum_{j=1,k>r+1}^{j=i-1} u_{jk}(\alpha,\beta)(e_{jk} - e_{kj}) \ : \ 1 \le i \le r\}$ forms a basis if $\mathcal{B}$ is skew symmetric. So subspace of symmetric (resp. skew symmetric) forms is of dimension $r = \min\{r,s\}$. □

Lemma 4.4. *Let V be a finite dimensional vector space over $\mathbb{F}$, $T : V \to V$ an invertible linear operator. For $r + s = n$, $\alpha = \pm 1$, if $(x-\alpha)^r$ and $(x-\alpha)^s$, occur as elementary divisors of $\chi_T(x) = (x-\alpha)^r(x-\alpha)^s = m_T(x)$, then dimension of the space of T-invariant bilinear forms is $n + 2 \cdot \min\{r,s\}$.*

Proof. Without loss of generality, we can assume that $V = V_{(x-\alpha)^r} \oplus V_{(x-\alpha)^s}$. Then $\dim V = r + s = n$.

□

COROLLARY 4.5

With the hypothesis as in the lemma, if n is odd, then the subspace of T-invariant symmetric forms is of dimension $\frac{n+1}{2} + \min\{r,s\}$ and the subspace of T-invariant skew symmetric forms is of dimension $\frac{n-1}{2} + \min\{r,s\}$.

Proof. As n is odd we have two cases:

Case 1. *r odd and s even*. By Corollaries 3.2 and 3.3, we have $\dim \mathcal{B}_{\text{sym}}|_{V_{(x-1)^r}} = \frac{r+1}{2}$, $\dim \mathcal{B}_{\text{skew}}|_{V_{(x-1)^r}} = \frac{r-1}{2}$, $\dim \mathcal{B}_{\text{sym}}|_{V_{(x-1)^s}} = \frac{s}{2}$ and $\dim \mathcal{B}_{\text{skew}}|_{V_{(x-1)^s}} = \frac{s}{2}$. So

$$\begin{aligned}\dim \mathcal{B}_{\text{sym}} &= \dim B_{\text{sym}}|_{V_{(x-1)^r}} + \dim B_{\text{sym}}|_{V_{(x-1)^s}} + r\\ &= \frac{r+1}{2} + \frac{s}{2} + r = \frac{n+1}{2} + \min\{r, s\}\end{aligned}$$

and

$$\begin{aligned}\dim \mathcal{B}_{\text{skew}} &= \dim B_{\text{skew}}|_{V_{(x-1)^r}} + \dim B_{\text{skew}}|_{V_{(x-1)^s}} + r\\ &= \frac{r-1}{2} + \frac{s}{2} + r = \frac{n-1}{2} + \min\{r, s\}.\end{aligned}$$

Case 2. *r even and s odd*. Again by Corollaries 3.2 and 3.3, we have $\dim \mathcal{B}_{\text{sym}}|_{V_{(x-1)^r}} = \frac{r}{2}$, $\dim \mathcal{B}_{\text{skew}}|_{V_{(x-1)^r}} = \frac{r}{2}$, $\dim \mathcal{B}_{\text{sym}}|_{V_{(x-1)^s}} = \frac{s+1}{2}$ and $\dim \mathcal{B}_{\text{skew}}|_{V_{(x-1)^s}} = \frac{s-1}{2}$. So

$$\begin{aligned}\dim \mathcal{B}_{\text{sym}} &= \dim B_{\text{sym}}|_{V_{(x-1)^r}} + \dim B_{\text{sym}}|_{V_{(x-1)^s}} + r\\ &= \frac{r}{2} + \frac{s+1}{2} + r = \frac{n+1}{2} + \min\{r, s\}\end{aligned}$$

and

$$\begin{aligned}\dim \mathcal{B}_{\text{skew}} &= \dim B_{\text{skew}}|_{V_{(x-1)^r}} + \dim B_{\text{skew}}|_{V_{(x-1)^s}} + r\\ &= \frac{r}{2} + \frac{s-1}{2} + r = \frac{n-1}{2} + \min\{r, s\}.\end{aligned}$$

Thus in both the cases, $\dim \mathcal{B}_{\text{sym}} = \frac{n+1}{2} + \min\{r, s\}$ and $\dim \mathcal{B}_{\text{skew}} = \frac{n-1}{2} + \min\{r, s\}$. □

COROLLARY 4.6

With the hypothesis as in the lemma, let n be even. Then if r is odd, the subspaces of T-invariant symmetric forms and T-invariant skew symmetric forms are of dimensions $\frac{n}{2} + 1 + \min\{r, s\}$ and $\frac{n}{2} - 1 + \min\{r, s\}$ respectively. If r is even, they are of dimensions $\frac{n}{2} + \min\{r, s\}$ and $\frac{n}{2} + \min\{r, s\}$ respectively.

Proof. Similar. □

Lemma 4.7. *Let V be a finite dimensional vector space over $\mathbb{F}$, $T : V \to V$ a linear operator. If for $\pm 1 \neq \alpha, \beta \in \mathbb{F}, \alpha\beta = 1$ the minimal and characteristic polynomials of T are $m_T(x) = (x-\alpha)(x-\beta)$ and $\chi_T(x) = (x-\alpha)^r(x-\beta)^s$ respectively, then dimension of the space $\mathcal{B}$ of T-invariant bilinear forms is $2rs$.*

Proof. Without loss of generality, assume that $r \leq s, r+s = n$,

$$V = \oplus_{i=1}^{r}(V_{(x-\alpha)} \oplus V_{(x-\beta)}) \oplus (\oplus_{j=1}^{s-r} V_{(x-\beta)})$$

and

$$T = \left(\begin{array}{ccccccc|ccc}
\alpha & & & & & & & & & \\
& \beta & & & & & & & & \\
& & \alpha & & & & & & & \\
& & & \beta & & & & & & \\
& & & & \ddots & & & & & \\
& & & & & \alpha & & & & \\
& & & & & & \beta & & & \\
\hline
& & & & & & & \beta & & \\
& & & & & & & & \ddots & \\
& & & & & & & & & \beta
\end{array}\right),$$

where the first diagonal block is of order $2r$ and last is of order $s-r$.

Let $\{e_1, \ldots, e_n\}$ be the standard basis for V. Let $B \in \mathcal{B}$ be a T-invariant bilinear form on V given by $X = (x_{i,j}) = (B(e_i, e_j))$.

Let us first determine $B|_{\oplus_{i=1}^r (V_{(x-\alpha)} \oplus V_{(x-\beta)})}$. For $1 \le i, j \le 2r$,

$$\begin{aligned} x_{ij} &= B(e_i, e_j) = B(Te_i, Te_j) \\ &= \begin{cases} \alpha^2 B(e_i, e_j) & \text{if } i \text{ and } j \text{ are odd,} \\ \alpha\beta B(e_i, e_j) & \text{if } i \text{ and } j \text{ are of distinct parity,} \\ \beta^2 B(e_i, e_j) & \text{if } i \text{ and } j \text{ are even,} \end{cases} \end{aligned}$$

but as $\alpha^2 \ne 1$ and $\beta^2 \ne 1$, we have for $1 \le i, j \le 2r$, $x_{ij} = 0$ if i and j are of the same parity, whereas no condition if i and j are of different parities. So $S_1 = \{e_{ij},\ 1 \le i, j \le 2r\ :\ 2 \nmid i + j\}$ forms a basis for $\mathcal{B}|_{\oplus_{i=1}^r (V_{(x-\alpha)} \oplus V_{(x-\beta)})}$, with $\dim \mathcal{B}|_{\oplus_{i=1}^r (V_{(x-\alpha)} \oplus V_{(x-\beta)})} = \binom{r}{1}\binom{r}{1} + \binom{r}{1}\binom{r}{1} = 2r^2$.

Now let us determine $B|_{\oplus_{i=1}^{s-r} V_{(x-\beta)}}$. For $u, v \in \oplus_{i=1}^{s-r} V_{(x-\beta)}$, $B(u, v) = B(Tu, Tv) = \beta^2 B(u, v)$, again as $\beta^2 \ne 1$, $B(u, v) = 0$. Thus $\mathcal{B}|_{\oplus_{i=1}^{s-r} V_{(x-\beta)}} = 0$.

The $2r \times (s-r)$ block is determined as follows: For $1 \le i \le 2r$ and $1 \le j \le s-r$,

$$x_{i,2r+j} = B(e_i, e_{2r+j}) = B(Te_i, Te_{2r+j}) = \begin{cases} \alpha\beta B(e_i, e_{2r+j}) & \text{if } i \text{ is odd,} \\ \beta^2 B(e_i, e_{2r+j}) & \text{if } i \text{ is even,} \end{cases}$$

i.e., $1 \le j \le s-r$, $x_{i,2r+j} = 0$ if i is even. So $S_2 = \{e_{i,2r+j},\ 1 \le i \le 2r,\ 1 \le j \le s-r\ :\ i \text{ is odd}\}$ forms a basis for this $2r \times (s-r)$ block and its dimension is $\binom{r}{1}\binom{s-r}{1} = r(s-r)$.

Finally the $(s-r) \times 2r$ block is determined as follows: For $1 \le j \le 2r$ and $1 \le i \le s-r$,

$$x_{2r+i,j} = B(e_{2r+i}, e_j) = B(Te_{2r+i}, Te_j) = \begin{cases} \alpha\beta B(e_{2r+i}, e_j) & \text{if } j \text{ is odd,} \\ \beta^2 B(e_{2r+i}, e_j) & \text{if } j \text{ is even,} \end{cases}$$

i.e., $1 \le i \le s-r$, $x_{2r+i,j} = 0$ if j is even. So $S_3 = \{e_{2r+i,j},\ 1 \le i \le s-r,\ 1 \le j \le 2r\ :\ j \text{ is odd}\}$ forms a basis for this $(s-r) \times 2r$ block and its dimension is $\binom{r}{1}\binom{s-r}{1} = r(s-r)$.

Thus $\dim \mathcal{B} = 2r^2 + 0 + r(s-r) + r(s-r) = 2rs$, with the disjoint union $S_1 \cup S_2 \cup S_3$ as a basis. □

COROLLARY 4.8

Under the hypothesis of the lemma, dimension of the subspace $\mathcal{B}_{\rm sym}$ of T-invariant symmetric bilinear forms is $rs = \frac{1}{2}\dim \mathcal{B}$.

Proof. Follows from the lemma. □

COROLLARY 4.9

Under the hypothesis of the lemma, dimension of the subspace $\mathcal{B}_{\rm skew}$ of T-invariant skew symmetric bilinear forms is $rs = \frac{1}{2}\dim \mathcal{B}$.

Proof. Similar. □

Remark 4.10. If $r = s$, then there are nondegenerate T-invariant bilinear forms (symmetric as well as skew symmetric) for T as described in the lemma.

If $r = s$ and

$$T = \begin{pmatrix} \alpha & & & & \\ & \beta & & & \\ & & \ddots & & \\ & & & \alpha & \\ & & & & \beta \end{pmatrix},$$

let $\sigma \in S_{2r}$ be a permutation given by

$$\sigma = \begin{cases} \Pi_{i=1,\ i\ \text{even}}^{r}(i,\ r+i-1) & \text{if } r \text{ is even},\\ \Pi_{i=1,\ i\ \text{even}}^{r}(i,\ r+i) & \text{if } r \text{ is odd}. \end{cases}$$

Let P be the matrix representation of the permutation σ with respect to the standard basis $(P(e_i) = e_{\sigma(i)})$. Then P gives an isomorphism of V onto V and for $B \in \mathcal{B}$ as obtained in the lemma, we have $P^t BP = \left(\begin{array}{c|c} O & C \\ \hline D & O \end{array}\right)$, where $O,\ C,\ D \in M_r(\mathbb{F})$, O is a zero matrix. Thus for $C,\ D \in Gl_r(\mathbb{F})$, B is nondegenerate. As an example $C = D = I_r$ gives rise to a nondegenerate T-invariant bilinear form. Also we can take $D = C^t \in Gl_r(\mathbb{F})$. Then B thus obtained will be a nondegenerate T-invariant symmetric form, similarly we can take $C \in Gl_r(\mathbb{F})$ and $D = -C^t$, then B thus obtained will be a nondegenerate T-invariant skew symmetric bilinear form.

Further, note that if $r \neq s$, we can assume $r < s$ and P as above. Then for B a T-invariant bilinear form, we have

$$H = P^t BP = \left(\begin{array}{cc} \boxed{\begin{array}{cc} O & C \\ D & O \end{array}} & \begin{array}{c} E \\ O_1 \end{array} \\ \begin{array}{cc} F & O_2 \end{array} & O_3 \end{array}\right) \in M_{r+s}(\mathbb{F}),$$

where $O, C, D \in M_r(\mathbb{F})$, $E, O_1 \in M_{r\times s-r}(\mathbb{F})$, $F, O_2 \in M_{s-r\times r}(\mathbb{F})$, $O_3 \in M_{s-r}(\mathbb{F})$, O, O_1, O_2, O_3 are zero matrices. The first row consisting of the blocks O, C, E is of rank atmost r, the last two rows are of rank same as the column rank of the matrix $\begin{pmatrix} D \\ F \end{pmatrix}$ which can be atmost r. Thus in all the matrix H is of rank atmost $2r < r + s$, so B cannot be nondegenerate.

5. Main results

5.1 *Invariance under self-reciprocal operators*

Recall that for V a n dimensional vector space over $\mathbb{F}$, $T \in \mathrm{End}(V)$ is cyclic if there exists $v \in V$ such that the set $\{v, Tv, \ldots, T^{n-1}v\}$ forms a basis for V.

Theorem 5.1. *Let V be a vector space of dimension $2n$ over $\mathbb{F}$, $T : V \to V$ an invertible, cyclic, self-reciprocal transformation. Then the space of T-invariant bilinear forms is of dimension $2n$ over $\mathbb{F}$.*

Proof. Let $m_T(x)$ be the minimal polynomial of T. Since T is cyclic $\chi_T(x) = m_T(x)$. Let $\chi_T(x) = p(x) = \sum_{i=0}^{2n} c_i x^i \in \mathbb{F}[x]$. As T is *self-reciprocal*, we have $c_0 = 1$, $c_i = c_{2n-i}$ for all i, $0 \le i \le n$.

As T is cyclic, there exists a vector $v \in V$ such that orbit of v with respect to T spans V, i.e., $(e_1 = v, e_2 = Tv, \ldots, e_{2n} = T^{2n-1}v)$ forms a frame for V and $T^{2n}v = -\sum_{i=0}^{2n-1} c_i T^i v$. Fix the frame $\underline{e} = (e_1, \ldots, e_n)$. Let the matrix of T with respect to $\underline{e}$ be $[T]_e$ and the matrix of B be $X = [B]_e = (B(e_i, e_j))$. Then T-invariance amounts to $[T]_e^t [B]_e [T]_e = [B]_e$. So we have

$$C = [T]_e = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & 0 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -c_{2n-1} \end{pmatrix}.$$

Let $C^t = (c^t_{ij})$ be the transpose of the matrix C, $c^t_{ij} = c_{ji}$.

$$\begin{aligned}(C^t X C)_{ij} &= \sum_{k,l=1}^{2n} c^t_{ik} x_{kl} c_{lj} = \sum_{k,l=1}^{2n} c_{ki} B(e_k, e_l) c_{lj} = B\left(\sum_{k=1}^{2n} c_{ki} e_k, \sum_{l=1}^{2n} c_{lj} e_l\right) \\ &= B(Te_i, Te_j).\end{aligned}$$

So we have

$$(B(e_i, e_j)) = X = C^t X C = (B(Te_i, Te_j)).$$

Thus we get $x_{ij} = x_{i+1\, j+1}$ for all i, j, $1 \le i, j \le 2n-1$. Let $B(e_1, e_i) = \alpha_i$, $1 \le i \le 2n$ and $B(e_j, e_1) = \beta_j$, $1 \le j \le 2n$. Clearly $\alpha_1 = \beta_1$. So

$$X = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots \alpha_{2n} \\ \beta_2 & \alpha_1 & \alpha_2 & \cdots \alpha_{2n-1} \\ \vdots & \ddots & \ddots & \ddots \vdots \\ \vdots & & \ddots & \ddots \vdots \\ \beta_{2n} & \beta_{2n-1} & \beta_{2n-2} & \cdots \alpha_1 \end{pmatrix}.$$

Now for $1 \le i \le n$, we have

$$\begin{aligned} \alpha_{2n-i+1} &= B(e_1, e_{2n-i+1}) = B(e_2, e_{2n-i+2}) = \cdots = B(e_i, e_{2n}) \\ &= B(Te_i, Te_{2n}) = B(e_{i+1}, T^{2n}v) \\ &= B(e_{i+1}, -\sum_{j=0}^{2n-1} c_j e_{j+1}) = -\sum_{j=0}^{2n-1} c_j B(e_{i+1}, e_{j+1}) \\ &= -\sum_{j=i}^{2n-1} c_j B(e_{i+1}, e_{j+1}) - \sum_{j=0}^{i-1} c_j B(e_{i+1}, e_{j+1}) \\ &= -\sum_{j=i}^{2n-1} c_j B(e_1, e_{j-i+1}) - \sum_{j=0}^{i-1} c_j B(e_{i+1}, e_{j+1}) \\ &= -\sum_{j=i}^{2n-1} c_j \alpha_{j-i+1} - \sum_{j=0}^{i-1} c_j B(e_{i+1}, e_{j+1}). \end{aligned} \tag{5.1}$$

Similarly for $1 \le k \le n$, we have

$$\beta_{2n-k+1} = -\sum_{j=k}^{2n-1} c_j \beta_{j-k+1} - \sum_{j=0}^{k-1} c_j B(e_{j+1}, e_{k+1}). \tag{5.2}$$

Now for a general bilinear form B, equations (5.1) and (5.2) can be expressed as

$$\alpha_{2n-i+1} = -\sum_{j=i}^{2n-1} c_j \alpha_{j-i+1} - \sum_{j=0}^{i-1} c_j \beta_{i-j+1}, \tag{5.3}$$

$$\beta_{2n-i+1} = -\sum_{j=i}^{2n-1} c_j \beta_{j-i+1} - \sum_{j=0}^{i-1} c_j \alpha_{i-j+1} \tag{5.4}$$

respectively for $1 \le i \le n$. Thus for $i = t \le n$ and $i = s \le n$, from equations (5.3) and (5.4), we get

$$\begin{aligned} \alpha_{2n-t+1} = -\beta_{t+1} &- \sum_{j=1}^{t-1} c_j (\beta_{t-j+1} + \alpha_{2n-j-t+1}) \\ &- \sum_{j=t}^{n-1} c_j (\alpha_{j-t+1} + \alpha_{2n-j-t+1}) - c_n \alpha_{n-t+1} \end{aligned} \tag{5.5}$$

and

$$\beta_{2n-s+1} = -\alpha_{s+1} - \sum_{j=1}^{s-1} c_j(\alpha_{s-j+1} + \beta_{2n-j-s+1}) \\ - \sum_{j=s}^{n-1} c_j(\beta_{j-s+1} + \beta_{2n-j-s+1}) - c_n\beta_{n-s+1} \tag{5.6}$$

respectively. Thus $\alpha_{n+1}, \ldots, \alpha_{2n}, \beta_{n+2}, \ldots, \beta_{2n}$ are expressible in terms of $\alpha_1, \ldots, \alpha_n$, $\beta_2, \ldots, \beta_{n+1}$.

It follows that the dimension is $2n$. □

Remark 5.2. Remark that 'space of T-invariant forms = subspace of symmetric T-invariant forms $\oplus$ subspace of skew symmetric T-invariant forms'.

COROLLARY 5.3

Under the hypothesis of the theorem, the space of T-invariant symmetric bilinear forms is of dimension n.

Proof. If B is a symmetric bilinear form, then for $1 \leq j \leq 2n, \alpha_j = \beta_j$ and for $1 \leq i \leq n$ from equation (5.3), we have

$$\begin{aligned} \alpha_{2n-i+1} &= -\sum_{j=i}^{2n-1} c_j\alpha_{j-i+1} - \sum_{j=0}^{i-1} c_j B(e_{j+1}, e_{i+1}) \\ &= -\sum_{j=i}^{2n-1} c_j\alpha_{j-i+1} - \sum_{j=0}^{i-1} c_j B(e_1, e_{i-j+1}) \\ &= -\sum_{j=i}^{2n-1} c_j\alpha_{j-i+1} - \sum_{j=0}^{i-1} c_j\alpha_{i-j+1}. \end{aligned}$$

Thus for $i = t \leq n$, we have

$$\alpha_{2n-t+1} = -\alpha_{t+1} - \sum_{j=1}^{t-1} c_j(\alpha_{t-j+1} + \alpha_{2n-j-t+1}) \\ - \sum_{j=t}^{n-1} c_j(\alpha_{j-t+1} + \alpha_{2n-j-t+1}) - c_n\alpha_{n-t+1}.$$

These are the only non identically zero equations in this case. So out of $\{\alpha_1 \cdots \alpha_{n+1}\}$ (as appearing in the theorem), $\alpha_1 \cdots \alpha_n$ are the free scalars. Thus the set of T-invariant symmetric bilinear forms turns out to be a vector space over $\mathbb{F}$ of dimension n. □

COROLLARY 5.4

Under the hypothesis of the theorem, the space of T-invariant skew symmetric bilinear forms is of dimension n.

Proof. Follows from the theorem, remark and the previous corollary. □

Note that there are always nondegenerate T-invariant bilinear forms for T as in the theorem. As an example, if we take for $1 \leq i \leq 2n$, $\alpha_i = -\beta_i$, $\alpha_j = 0$ for all $1 \leq j \leq n$ and $\alpha_{n+1} \neq 0$ (the choice of this non-zero value is assured from equations (5.5) and (5.6)), then X is

$$X = \begin{pmatrix} O & A \\ -A^t & O \end{pmatrix},$$

where O is the zero square matrix of order n, A is the upper triangular square matrix of order n with α_{n+1} as entries on the main diagonal. Clearly X is skew symmetric and as $\det X = \det^2 A = \alpha_{n+1}^{2n} \neq 0$, it is nondegenerate. Now let $Y = (C - C^{-1})^t X$, then we have

$$\begin{aligned} C^t Y C &= C^t (C - C^{-1})^t X C = C^t (C - C^{-1})^t (C^{-1})^t C^t X C \\ &= (C^{-1}(C - C^{-1})C)^t X = (C - C^{-1})^t X = Y, \end{aligned}$$

i.e., Y gives a T-invariant bilinear form. Now we will see that Y is symmetric. As $Y = (C - C^{-1})^t X = C^t X - (C^{-1})^t X = C^t X - (C^{-1})^t C^t X C = C^t X - X C$, we have $Y^t = X^t C - C^t X^t = -XC + C^t X = C^t X - XC = Y$. Now since by hypothesis none of ± 1 is an eigenvalue of C, $C - C^{-1}$ is invertible and as X is invertible, we get Y as a nondegenerate T-invariant symmetric bilinear form.

Remark 5.5. There exist symmetric nondegenerate and skew symmetric nondegenerate forms invariant under the linear operator as in the theorem.

5.2 *Invariance under indecomposable operators*

Theorem 5.6. *Let V be a vector space of dimension n over $\mathbb{F}$. If T is an indecomposable, self-reciprocal transformation on V, then the space of T-invariant bilinear forms is of dimension n over $\mathbb{F}$.*

Proof. As T is self-reciprocal, n is even. T is invertible and as T is indecomposable it is cyclic also. So by Theorem 5.1, this theorem follows. □

COROLLARY 5.7

Under the hypothesis of the theorem the subspace of T-invariant symmetric (resp. skew-symmetric) forms is of dimension $\frac{n}{2}$ and this subspace contains a nondegenerate form.

Proof. Clear from the corollaries of Theorem 5.1. □

Theorem 5.8. *Let V be a vector space of dimension n over $\mathbb{F}$. If T is an indecomposable, invertible, non-unipotent, non self-reciprocal transformation on V, then the space of T-invariant bilinear forms is* 0.

Proof. Since T is indecomposable, invertible, the characteristic polynomial of T is $\chi_T(x) = p(x)^d$, $p(x) \in \mathbb{F}[x]$ irred., $d \in \mathbb{Z}^+$. So $V = \ker(p(x)^d)$. Now due to Proposition 2.5, we can pass our investigation to $\bar{\mathbb{F}}$. Then we have

$$V = \oplus_{\alpha\in\bar{\mathbb{F}}\text{ a root of } p(x)} V_\alpha, \text{ where } V_\alpha = \ker\,(x-\alpha)^{dk},$$

with

$$k = \begin{cases} 1, & \text{if char } \mathbb{F} = 0, \\ (\text{ char } \mathbb{F})^s,\ s \in \mathbb{Z}^+, & \text{otherwise.} \end{cases}$$

Let B be a T-invariant bilinear form on V. If $u \in V_\alpha$ and $v \in V_\beta$, then $B(Tu, Tv) = B(u, v)$ amounts to $B(T_s u, T_s v) = B(u, v)$, i.e., $(\alpha\beta - 1)B(u, v) = 0$. As $p(x)$ is nonunipotent $\alpha \neq \pm 1 \neq \beta$ also as $p(x)$ is non self-reciprocal $\beta \neq \alpha^{-1}$. Thus we have $B(u, v) = 0$. Also by the previous lemma, $\mathcal{B}_{V_\alpha} = 0$. Therefore the space of T-invariant forms is 0. □

5.3 *Invariant under decomposable operators*

Theorem 5.9. *Let V be a finite dimensional vector space over $\mathbb{F}$, $T : V \to V$ an invertible linear operator. If for $p(x) \in \mathbb{F}[x]$ irreducible with $p(\pm 1) \neq 0$, $p(x)^r$ and $p^*(x)^s$ occur as elementary divisors of $\chi_T(x) = p(x)^r p^*(x)^s$. Then the dimension of the space of T-invariant bilinear forms is $2 \cdot \deg p(x) \cdot \min\{r, s\}$.*

Proof. Since $p(x) \in \mathbb{F}[x]$ is irreducible, all its roots are of same multiplicity, say k. So by Lemma 2.1, all roots of $p^*(x)$ are also of the same multiplicity, i.e., of k. So $\dim V = kr + ks$. Let $R = kr$ and $S = ks$. As invariances of χ_T does not change over the extensions of the field, over $\bar{\mathbb{F}}$, it will remain the same and over $\bar{\mathbb{F}}$, we have

$$V = V_{p(x)^r p^*(x)^s} = \oplus_{\alpha\beta=1,\, p(\alpha)=0} (V_{(x-\alpha)^R} \oplus V_{(x-\beta)^S}).$$

Since $\alpha,\ \beta \neq \pm 1$, by Lemma 4.2, we have

$$\dim \mathcal{B}|_{V_{(x-\alpha)^R} \oplus V_{(x-\beta)^S}} = 2 \cdot \min\{R, S\}.$$

So

$$\begin{aligned}\dim \mathcal{B} &= \sum_{\alpha\beta=1,\, p(\alpha)=0} \dim \mathcal{B}|_{V_{(x-\alpha)^R} \oplus V_{(x-\beta)^S}} \\ &= \sum_{\text{distinct roots of } p(x)} 2 \cdot \min\{R, S\} \\ &= \#\,\text{distinct roots of}\, p(x) \times 2 \cdot \min\{R, S\} \\ &= \frac{\deg p(x)}{k} \cdot 2 \cdot \min\{R, S\} = 2 \deg p(x) \cdot \min\{r, s\}.\end{aligned}$$

□

COROLLARY 5.10

Under the hypothesis of the theorem, the dimension of the subspace of T-invariant symmetric (resp. skew symmetric) bilinear forms is $\deg p(x) \min\{r, s\}$.

Proof. Again, without loss of generality, assume that $V = V_{p(x)^r p^*(x)^s}$ and $T = T|_{V_{p(x)^r p^*(x)^s}}$. Then as in the proof of the theorem, we obtain

$$V = V_{p(x)^r p^*(x)^s} = \oplus_{\alpha\beta=1,\, p(\alpha)=0}(V_{(x-\alpha)^R} \oplus V_{(x-\beta)^S}).$$

Since $\alpha,\ \beta \neq \pm 1$, by Corollary 4.3, we have

$$\dim \mathcal{B}_{\text{sym}}|_{V_{(x-\alpha)^R} \oplus V_{(x-\beta)^S}} = \min\{R, S\} = \frac{1}{2} \dim \mathcal{B}|_{V_{(x-\alpha)^R} \oplus V_{(x-\beta)^S}},$$

and

$$\dim \mathcal{B}_{\text{skew}}|_{V_{(x-\alpha)^R} \oplus V_{(x-\beta)^S}} = \min\{R, S\} = \frac{1}{2} \dim \mathcal{B}|_{V_{(x-\alpha)^R} \oplus V_{(x-\beta)^S}}.$$

So

$$\dim \mathcal{B}_{\text{sym}} = \frac{1}{2} \dim \mathcal{B} = \dim \mathcal{B}_{skew},$$

hence by the theorem

$$\dim \mathcal{B}_{\text{sym}} = \dim \mathcal{B}_{\text{skew}} = \deg p(x) \cdot \min\{r, s\}.$$

□

Theorem 5.11. *Let V be a finite dimensional vector space over $\mathbb{F}$, $T : V \to V$ a linear operator. Let the minimal and characteristic polynomials of T are $m_T(x) = f(x) f^*(x)$ and $\chi_T(x) = f(x)^r f^*(x)^s$, $f(x) \in \mathbb{F}[x]$ is separable with $f(\pm 1) \neq 0$ and for $u \in \bar{\mathbb{F}}$ a root of $f(x)$, $f(u^{-1}) \neq 0$. Then dimension of the space of T-invariant bilinear forms is* $2 \cdot \deg f(x) rs$.

Proof. As $f(x)$ is separable, by Lemma 2.1, $f^*(x)$ is also separable and so according to the hypothesis, $m_T(x)$ has distinct roots. Hence over $\bar{\mathbb{F}}$, we have

$$V = \oplus_{\alpha\beta=1, \alpha \text{ a root of } f(x)} (V_{(x-\alpha)^r} \oplus V_{(x-\beta)^s})$$

and for every root α of $f(x)$, we have

$$T|_{V_{(x-\alpha)^r}\oplus V_{(x-\beta)^s}} = \left(\begin{array}{ccccccc|ccc} \alpha &&&&&&&&& \\ & \beta &&&&&&&& \\ && \alpha &&&&&&& \\ &&& \beta &&&&&& \\ &&&& \ddots &&&&& \\ &&&&& \alpha &&&& \\ &&&&&& \beta &&& \\ \hline &&&&&&& \beta && \\ &&&&&&&& \ddots & \\ &&&&&&&&& \beta \end{array}\right).$$

Let $B \in \mathcal{B}$ be a T-invariant bilinear form. For $\alpha,\ \alpha'$ two distinct roots of $f(x)$, if $u \in V_{(x-\alpha)^r}\oplus V_{(x-\beta)^s}$ and $v \in V_{(x-\alpha')^r}\oplus V_{(x-\beta')^s}$, then $u = u_1+u_2$ and $v = v_1+v_2$ with $u_1 \in V_{(x-\alpha)^r}$, $u_2 \in V_{(x-\beta)^s}$, $v_1 \in V_{(x-\alpha')^r}$, $v_2 \in V_{(x-\beta')^s}$ uniquely. $B(u_1, v_1) = B(Tu_1, Tv_1)$ implies $(\alpha\alpha' - 1)B(u_1, v_1) = 0$, as $\alpha\alpha' \neq 1$, $B(u_1, v_1) = 0$. By similar arguments, we see that $B(u_i, v_j) = 0 = B(v_j, u_i)$ for $1 \leq i, j \leq 2$, so $B(u, v) = 0 = B(v, u)$.

Thus $\dim \mathcal{B} = \sum_{\alpha\beta=1,\alpha \text{ a root of } f(x)} \dim \mathcal{B}|_{V_{(x-\alpha)^r}\oplus V_{(x-\beta)^s}}$. But by Lemma 4.8, $\dim \mathcal{B}|_{V_{(x-\alpha)^r}\oplus V_{(x-\beta)^s}} = 2rs$ and we get

$$\dim \mathcal{B} = \sum_{\alpha \text{ a root of } f(x)} 2rs = 2rs \deg f(x).$$

□

COROLLARY 5.12

Under the hypothesis of the theorem, dimension of the subspace of T-invariant symmetric bilinear forms is $\deg f(x)rs$.

Proof. Clear. □

COROLLARY 5.13

Under the hypothesis of the theorem, dimension of the subspace of T-invariant skew symmetric bilinear forms is $\deg f(x)rs$.

Proof. Similar. □

Remark 5.14. If $r = s$ and T as described in the theorem, then there are nondegenerate T-invariant bilinear forms (symmetric as well as skew symmetric).

Let $B \in \mathcal{B}$ be a T-invariant bilinear form. Let $\deg f(x) = k$ and $\alpha_1, \ldots, \alpha_k$ be the distinct roots of $f(x)$. Then for every α_i assuming $V_i = V_{(x-\alpha_i)^r} \oplus V_{(x-\beta_i)^r}$, we have $B_i = B|_{V_i}$ is a $T_i = T|_{V_i}$-invariant form. By Remark 4.10, for every i, $1 \leq i \leq k$, we have an isomorphism $P_i : V_i \to V_i$ such that $P_i{}^t B_i P_i = \left(\begin{array}{c|c} O & C_i \\ \hline D_i & O \end{array}\right)$, with $O,\ C_i,\ D_i \in M_r(\mathbb{F})$, O is a zero matrix. Now let

$$P = \begin{pmatrix} P_1 & & \\ & \ddots & \\ & & P_k \end{pmatrix}, \text{ then } P^t B P = \begin{pmatrix} (P_1{}^t B_1 P_1) & & \\ & \ddots & \\ & & (P_k{}^t B_k P_k) \end{pmatrix},$$

i.e., $$P^t B P = \begin{pmatrix} \boxed{\begin{matrix} O & C_1 \\ D_1 & O \end{matrix}} & & \\ & \ddots & \\ & & \boxed{\begin{matrix} O & C_k \\ D_k & O \end{matrix}} \end{pmatrix}.$$

Thus for all $i,\ 1 \leq i \leq k$, we can choose $C_i,\ D_i \in Gl_r(\mathbb{F})$ and get B a nondegenerate T-invariant bilinear form. Further, we can choose $C_i \in Gl_r(\mathbb{F})$ and $D_i = C_i{}^t$ (resp. $-C_i{}^t$) and get B a nondegenerate T-invariant symmetric (resp. skew symmetric) bilinear form. As an explicit example, we can choose $C_i = I_r$ and $D_i = I_r$ (resp. $-I_r$) and obtain a T-invariant nondegenerate symmetric (resp. skew symmetric) bilinear form.

Note that if $r \neq s$, by Remark 4.10, $P_i^t B_i P_i$ are degenerate for all $i,\ 1 \leq i \leq k$; hence B is always degenerate.

5.4 *Infinitesimal version*

DEFINITION 5.15

A linear transformation is said to be α-self dual if its characteristic polynomial is so.

In the next theorem, we characterize the bilinear forms infinitesimally invariant under a cyclic, α-self dual transformation $T \in \mathrm{End}(V)$. Since T is cyclic $\chi_T(x) = m_T(x)$.

Theorem 5.16. *Let V be a vector space of dimension $2n$ over $\mathbb{F}$, $T \in \mathrm{End}(V)$ cyclic, α-self dual, then the space of T-invariant infinitesimal bilinear forms is of dimension $2n$.*

COROLLARY 5.17

With the hypothesis of the theorem, the subspace of T-invariant infinitesimally symmetric bilinear forms is of dimension n.

COROLLARY 5.18

With the hypothesis of the theorem, the subspace of T-invariant infinitesimally skew symmetric bilinear forms is of dimension n.

COROLLARY 5.19

Under the hypothesis of the theorem, there exists a nondegenerate T-invariant infinitesimally symmetric (resp. skew symmetric) bilinear form.

Now let us discuss bilinear forms infinitesimally invariant under indecomposables. We consider two cases: (1) T nilpotent and (2) T nonnilpotent.

Case 1. *T nilpotent.* Let $S : V \to V$ be a linear nilpotent cyclic transformation. The Jordan form may be chosen as

$$D = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Then we have $Se_n = 0,\ Se_i = e_{i+1} \, \forall\, i,\ 1 \le i \le n-1$.

Theorem 5.20. *Let V be a vector space of dimension n over $\mathbb{F}$. If $S \in \mathrm{End}\,(V)$ is an indecomposable nilpotent transformation, then the space of infinitesimally T-invariant bilinear forms is of dimension n. If n is even (resp. odd), this space has a nondegenerate skew symmetric (resp. symmetric) form but any symmetric (resp. skew symmetric) form.*

COROLLARY 5.21

With the hypothesis as in the theorem, if n is odd, then the subspace of infinitesimally T-invariant symmetric forms is of dimension $\frac{n+1}{2}$ and the subspace of infinitesimal T-invariant skew symmetric forms is of dimension $\frac{n-1}{2}$.

COROLLARY 5.22

With the hypothesis as in the theorem, if n is even, then the subspace of infinitesimally T-invariant symmetric forms is of dimension $\frac{n}{2}$ and the subspace of infinitesimally T-invariant skew symmetric forms is of dimension $\frac{n}{2}$.

Case 2. *T nonnilpotent.* Again, in the light of Proposition 2.5, without loss of generality, we may assume that the base field is algebraically closed. By Theorem 15.3 (J E Humphreys, Linear Algebraic Groups, p. 98) (see also, [4]) for $T \in \mathcal{O}(V, B)$, with Jordan decomposition $T = T_s + N$, we have $T_s \in \mathcal{O}(V, B)$.

Theorem 5.23. *Let V be a vector space of dimension n over $\mathbb{F}$. If T is an indecomposable, α-self dual transformation on V, then the space of T-infinitesimally invariant bilinear forms is of dimension n over $\mathbb{F}$.*

COROLLARY 5.24

Under the hypothesis of the theorem the subspace of T-infinitesimally invariant symmetric (resp. skew symmetric) forms is of dimension $\frac{n}{2}$. Also it contains a nondegenerate form.

Theorem 5.25. *Let V be a vector space of dimension n over $\mathbb{F}$. If T is an indecomposable, non-nilpotent, non α-self dual transformation on V, then the space of T-infinitesimally invariant bilinear forms is* 0.

Theorem 5.26. *Let V be a finite dimensional vector space over $\mathbb{F}$, $T \in \mathrm{End}\,(V)$. If for $p(x) \in \mathbb{F}[x]$ irreducible with $p(0) \neq 0$, $p(x)^r$ and $p^-(x)^s$ occur as elementary divisors of $\chi_T(x)$, then the dimension of the space of $T|_{V_{p(x)^r p^-(x)^s}}$-invariant infinitesimally bilinear forms is* $2 \cdot \deg p(x) \cdot \min\{r, s\}$.

COROLLARY 5.27

Under the hypothesis of the theorem, the dimension of the subspace of $T|_{V_{p(x)^r p^-(x)^s}}$-invariant symmetric (resp. skew symmetric) infinitesimally bilinear forms is $\deg p(x) \cdot \min\{r, s\}$.

Theorem 5.28. *Let V be a finite dimensional vector space over $\mathbb{F}$, $T \in \mathrm{End}\,(V)$. Let the minimal and characteristic polynomials of T be $m_T(x) = f(x)f^-(x)$ and $\chi_T(x) = f(x)^r f^-(x)^s$, $f(x) \in \mathbb{F}[x]$ is separable with $f(0) \neq 0$ and for $u \in \bar{\mathbb{F}}$ a root of $f(x)$, $f(-u) \neq 0$. Then dimension of the space of T-invariant infinitesimally bilinear forms is* $2 \cdot \deg f(x)rs$.

COROLLARY 5.29

Under the hypothesis of the theorem dimension of the subspace of T-invariant symmetric (resp. skew symmetric) infinitesimally bilinear forms is $\deg f(x)rs$.

Remark 5.30. If $r = s$ and T as described in the theorem, then there are nondegenerate T-invariant infinitesimally bilinear forms (symmetric as well as skew symmetric) and if $r \neq s$, there are no nondegenerate T-invariant infinitesimally forms. Explanations are similar to the invariance version.

Future scope. As a future work, the investigation of invariant forms under a group of linear transformations over a field may be taken.

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