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Root and critical point behaviors of certain sums of polynomials

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Abstract. It is known that no two roots of the polynomial equation

$$\prod_{j=1}^{n}(x-r_j)+\prod_{j=1}^{n}(x+r_j)=0, \tag{1}$$

where $0 < r_1 \le r_2 \le \cdots \le r_n$, can be equal and the gaps between the roots of (1) in the upper half-plane strictly increase as one proceeds upward, and for $0 < h < r_k$, the roots of

$$(x-r_k-h)\prod_{\substack{j=1\\ j\neq k}}^{n}(x-r_j)+(x+r_k+h)\prod_{\substack{j=1\\ j\neq k}}^{n}(x+r_j)=0 \tag{2}$$

and the roots of (1) in the upper half-plane lie alternatively on the imaginary axis. In this paper, we study how the roots and the critical points of (1) and (2) are located.

Keywords. Polynomials; sums of polynomials; roots; critical points; root dragging.

Mathematics Subject Classification. 26C10, 30C15.

1. Introduction

There is an extensive literature concerning roots of sums of polynomials. Many authors [5–7] have written about these polynomials. Perhaps the most immediate question of sums of polynomials, $A+B=C$ is: 'given bounds for the roots of A and B, what bounds can be given for the roots of C?' By Fell [3], if all roots of A and B lie in $[-1, 1]$ with A, B monic and $\deg\ A = \deg\ B = n$, then no root of C can have modulus exceeding $\cot(\pi/2n)$, the largest root of $(x+1)^n + (x-1)^n$. This suggests to study polynomials having a form something like $A(x) + B(x)$, where all roots of $A(x)$ are negative and all roots of $B(x)$ are positive.

All (conjugate) roots of the polynomial equation (1) lie on the imaginary axis. Kim [4] showed as follows.

Theorem 1 [4]. *No two roots of* (1) *can be equal and the gaps between the roots of* (1) *in the upper half-plane strictly increase as one proceeds upward.*

Given a polynomial $f(x)$, all of whose roots are real, if we move some of the roots, the critical points also change. A fundamental result in this area is the polynomial root dragging theorem [1] that explains the change qualitatively.

Theorem 2 (Polynomial root dragging theorem). *Let $f(x)$ be a degree n polynomial with n real roots. When we drag some or all of the roots a distance at most ϵ to the right, the critical points will all follow to the right, and each of them will move less than ϵ units.*

Possibly the first question about the polynomial equation (1) in the vein of "root dragging" is how the roots and the critical points of (1) and (2) are arranged, and we will obtain some answers to this question in this paper in section 2. As reference, the polynomial equation (2) is still in the form of (1) so that its roots lie on the imaginary axis and the gaps between the roots in the upper half-plane strictly increase as one proceeds upward. From this motivation, throughout the paper, we let

$$\begin{aligned} p(x) &:= \prod_{j=1}^{n}(x-r_j)+\prod_{j=1}^{n}(x+r_j) = x^c \prod_{j=1}^{\lfloor n/2\rfloor}(x^2+s_j^2),\\ p_h(x) &:= (x-r_k-h)\prod_{\substack{j=1\\ j\neq k}}^{n}(x-r_j)+(x+r_k+h)\prod_{\substack{l=j\\ j\neq k}}^{n}(x+r_j)\\ &= x^c \prod_{j=1}^{\lfloor n/2\rfloor}(x^2+t_j^2),\\ p'(x) &= 2nx^{1-c}\prod_{j=1}^{\lfloor (n-1)/2\rfloor}(x^2+s_j'^2),\\ p_h'(x) &= 2nx^{1-c}\prod_{j=1}^{\lfloor (n-1)/2\rfloor}(x^2+t_j'^2), \end{aligned}$$

where

$$\begin{aligned} &0<r_1\le r_2\le\cdots\le r_n,\\ &0<s_1<s_2<\cdots<s_{\lfloor n/2\rfloor},\\ &0<t_1<t_2<\cdots<t_{\lfloor n/2\rfloor},\\ &0<s_1'<s_2'<\cdots<s_{\lfloor (n-1)/2\rfloor}',\\ &0<t_1'<t_2'<\cdots<t_{\lfloor (n-1)/2\rfloor}' \end{aligned}$$

and

$$c = \begin{cases} 0 & \text{if n is even,} \\ 1 & \text{if n is odd.} \end{cases}$$

About the roots of $p(x)$ and $p_h(x)$, Chong and Kim [2] recently proved that if $0 < h < r_k$, their roots in the upper half-plane lie alternatively on the imaginary axis.

Theorem 3 **[2]**. *If* $0 < h < r_k$, *then*

$$s_1 < t_1 < s_2 < t_2 < \cdots < s_{\lfloor n/2 \rfloor]} < t_{\lfloor n/2 \rfloor}. \tag{3}$$

When we consider $h < 0$ instead of $0 < h < r_k$ in Theorem 3, (3) is replaced by

$$t_1 < s_1 < t_2 < s_2 < \cdots < t_{\lfloor n/2 \rfloor]} < s'_{\lfloor n/2 \rfloor}.$$

In section 2, we will state new results in the form of (3) about the roots and the critical points of $p(x)$ and $p_h(x)$, and section 3 will be devoted to the proofs of all these results.

2. Results and examples

In this section, we state new results, and at the end of the section, we will provide some numerical evidences so that we compare them with our results. First, like (3), the critical points of $p(x)$ and $p_h(x)$ in the upper half-plane also lie alternatively on the imaginary axis.

Theorem 4. *If* $0 < h < r_k$, *then*

$$s'_1 < t'_1 < s'_2 < t'_2 < \cdots .$$

The proof of Theorem 4 will be based on Theorem 2. Another proof for $s'_i < t'_i$ in the elementary way without using Theorem 2 will also be provided. Next, we compare each root of $p(x)$ with its corresponding root of $p_h(x)$. From Theorem 3, when $0 < h < r_k$, $s_i < t_i$ for each i. However we have the following opposite inequality.

Theorem 5. *For each* i,

$$t_i < \sqrt{\frac{r_k + h}{r_k}} s_i,$$

where $h > 0$.

The next two theorems are about gaps of the roots of $p(x)$ and $p_h(x)$.

Theorem 6. *For any* i *and* j *with* $i > j$,

$$t_i^2 - t_j^2 > s_i^2 - s_j^2.$$

Theorem 7. *For each* i,

$$t_{i+1} - t_i > \sqrt{\frac{r_k}{r_k + h}}(s_{i+1} - s_i).$$

Theorem 7 can be easily obtained from Theorems 5 and 6. But we will give another proof of Theorem 7 based on a result in [4] in section 3. This result in [4] will also play a central role in the proofs of Theorem 10 and Theorem 11 below.

The following theorem explains that the gaps between the critical points of $p(x)$ in the upper half-plane strictly increase as one proceeds upward.

Theorem 8. *We have*

$$s_2' - s_1' < s_3' - s_2' < s_4' - s_3' < \cdots . \tag{4}$$

Let f be a polynomial of degree $n > 2$ with only real, simple roots. Then Riesz's result (see [8]) states that the distance between consecutive roots of f is less than the corresponding quantity associated with f'. In our case, the corresponding inequality is $s_2' - s_1' > s_2 - s_1$ by Theorems 1 and 8. We do not have a proof for this inequality, but we can prove at least $(s_2')^2 - (s_1')^2 > s_2^2 - s_1^2$. In fact, we prove the general case of this as follows.

Theorem 9. *For each* i, $s_{i+1}'^2 - s_i'^2 > s_{i+1}^2 - s_i^2$.

After the proof of the above theorem in Section 3, we will present another elementary proof of $(s_2')^2 - (s_1')^2 > s_2^2 - s_1^2$.

It is not known that $s_{i+1}' - s_i' > s_{i+1} - s_i$ for each i, but we may prove the following.

Theorem 10. *For each* i,

$$s_{i+1}' - s_i' > \left(\frac{\prod_{j=1}^{n-1} r_j}{\prod_{j=1}^{n-1} r_j'} \right) (s_{i+1} - s_i).$$

Let us denote that all roots on the upper-half plane of the j-th derivative of $p(x) = \prod_{i=1}^{n}(x - r_i) + \prod_{i=1}^{n}(x + r_i)$ are

$$is_{(1,j)},\ is_{(2,j)},\ is_{(3,j)},\ \ldots,$$

where $s_{(1,j)} < s_{(2,j)} < s_{(3,j)} < \cdots$. Then we finally present Theorem 11.

Theorem 11. *For each* i, j,

$$s_{(i,j)} < s_{(i,j+2)} < s_{(i+1,j)}.$$

The example below is given to check Theorems 4–10 above with numerical evidences.

Example 12. Consider

$$p(x) = \prod_{j=1}^{10}(x-j) + \prod_{j=1}^{10}(x+j) \quad \text{and}$$

$$p_{0.5}(x) = (x-5.5)\prod_{\substack{j=1\\ j\neq 5}}^{10}(x-j) + (x+5.5)\prod_{\substack{j=1\\ j\neq 5}}^{10}(x+j).$$

Then

$$\begin{aligned}
\{s_i\}_{i=1}^{5} &= \{0.5566\ldots,\ 2.0773\ldots,\ 4.6931\ldots,\ 10.1758\ldots,\ 34.4935\ldots\},\\
\{t_i\}_{i=1}^{5} &= \{0.5605\ldots,\ 2.0976\ldots,\ 4.7465\ldots,\ 10.2833\ldots,\ 34.8139\ldots\},\\
\{s'_i\}_{i=1}^{4} &= \{1.4800\ldots,\ 3.9000\ldots,\ 8.9360\ldots,\ 30.9636\ldots\},\\
\{t'_i\}_{i=1}^{4} &= \{1.4943\ldots,\ 3.9438\ldots,\ 9.0305\ldots,\ 31.2516\ldots\}
\end{aligned}$$

and

$$\begin{aligned}
&\left\{\sqrt{\frac{5.5}{5}}s_i\right\}_{i=1}^{5} = \{0.5838\ldots,\ 2.1787\ldots,\ 4.9222\ldots,\ 10.6725\ldots,\ 36.1771\ldots\},\\
&\{s_{i+1}-s_i\}_{i=1}^{4} = \{1.5206\ldots,\ 2.6158\ldots,\ 5.4826\ldots,\ 24.3177\ldots\},\\
&\{t_{i+1}-t_i\}_{i=1}^{4} = \{1.5409\ldots,\ 2.6692\ldots,\ 5.5900\ldots,\ 24.6381\ldots\},\\
&\left\{\sqrt{\frac{5}{5.5}}(s_{i+1}-s_i)\right\}_{i=1}^{4} = \{1.4498\ldots,\ 2.4941\ldots, 5.2275\ldots, 23.186\ldots\},\\
&\left\{s_{i+1}^2-s_i^2\right\}_{i=1}^{4} = \{4.0053\ldots,\ 17.7105\ldots,\ 81.5216\ldots,\ 1086.25\ldots\},\\
&\left\{t_{i+1}^2-t_i^2\right\}_{i=1}^{4} = \{4.086\ldots,\ 18.1295\ldots,\ 83.2157\ldots,\ 1106.27\ldots\},\\
&\left\{s'_{i+1}-s'_i\right\}_{i=1}^{3} = \{2.4200\ldots,\ 5.0359\ldots,\ 22.0276\ldots\},\\
&\left\{\left(\prod_{j=1}^{n-1} r_j \Big/ \prod_{j=1}^{n-1} r'_j\right)(s_{i+1}-s_i)\right\}_{i=1}^{3} = \{0.5191\ldots,\ 0.8930\ldots,\ 1.8718\ldots\}.
\end{aligned}$$

3. Proofs

We first prove Theorem 4.

Proof of Theorem 4. Let

$$p(x) = \prod_{j=1}^{n}(x-r_j) + \prod_{j=1}^{n}(x+r_j) =: p_1(x) + p_2(x)$$

and

$$p_h(x) = (x - r_k - h)\prod_{\substack{j=1\\ j\neq k}}^{n}(x - r_j) + (x + r_k + h)$$
$$\prod_{\substack{j=1\\ j\neq k}}^{n}(x + r_j) =: p_{1,h}(x) + p_{2,h}(x), \quad \text{say.}$$

Then

$$p'(x) = p_1'(x) + p_2'(x) \quad \text{and} \quad p_h'(x) = p_{1,h}'(x) + p_{2,h}'(x).$$

By Theorem 2, the roots

$$r'_{1,h},\ r'_{2,h},\ \ldots\ r'_{n-1,h},$$

of $p'_{1,h}(x)$ all follow to the right, and each of them moves less than h units and so for each i,

$$r'_i < r'_{i,h},$$

where the r'_is are the roots of $p'_1(x)$. On the one hand, from symmetry, each root of $p'_{2,h}(x)$ follows to the left with the same distance as that of the corresponding root movement of $p'_{1,h}(x)$. Then Theorem 3 completes the proof. □

Remark 13. We may prove an inequality $s'_i < t'_i$ for each i in an elementary way without using Theorem 2 as follows. Assume that n is even. Then

$$p'(x) = \frac{d}{dx^2}\prod_{j=1}^{n/2}(x^2 + s_j^2)\frac{dx^2}{dx} = 2x\prod_{j=1}^{n/2}(x^2 + s_j^2)\sum_{j=1}^{n/2}\frac{1}{x^2 + s_j^2}.$$

Since $p'(is'_i) = 0$ for $1 \leq i \leq n/2 - 1$, we choose any i with

$$2is'_i\prod_{j=1}^{n/2}(-(s'_i)^2 + s_j^2)\sum_{j=1}^{n/2}\frac{1}{-(s'_i)^2 + s_j^2} = 0$$

and then fix it. By Theorem 1, all roots of $p(x)$ are simple, and

$$\prod_{j=1}^{n/2}(-(s'_i)^2 + s_j^2) \neq 0,$$

which implies that

$$\sum_{j=1}^{n/2}\frac{1}{-(s'_i)^2 + s_j^2} = 0.$$

In the same way, we have

$$\sum_{j=1}^{n/2} \frac{1}{-(t_i')^2 + t_j^2} = 0.$$

But by Theorem 3,

$$s_j^2 < t_j^2$$

for all j. If $s_i' \geq t_i'$, then

$$-(s_i')^2 + s_j^2 < -(t_i')^2 + t_j^2$$

which is a contradiction to

$$\sum_{j=1}^{n/2} \frac{1}{-(s_i')^2 + s_j^2} = \sum_{j=1}^{n/2} \frac{1}{-(t_i')^2 + t_j^2},$$

and so $s_i' < t_i'$. The case n odd can be proved by the same method.

Next we prove Theorems 5, 6, 7. To prove these, we need a lemma.

Lemma 14. *For each* i,

$$\frac{\partial s_i}{\partial r_k} = \frac{s_i}{s_i^2 + r_k^2} \frac{1}{\sum_{j=1}^{n} \frac{2r_i}{r_j^2 + s_i^2}}.$$

Proof. Taking a partial derivative with respect to the k-th derivative r_k of each side of

$$(p(s_i i) =) \prod_{j=1}^{n} (s_i i - r_j) + \prod_{j=1}^{n} (s_i i + r_j) = 0$$

yields

$$i \frac{\partial s_i}{\partial r_k} \left[\left(\sum_{j=1}^{n} \frac{1}{s_i i - r_j} \right) \prod_{j=1}^{n} (s_i i - r_j) + \left(\sum_{j=1}^{n} \frac{1}{s_i i + r_j} \right) \prod_{j=1}^{n} (s_i i + r_j) \right]$$
$$= \prod_{\substack{j=1 \\ j \neq k}}^{n} (s_i i - r_j) - \prod_{\substack{j=1 \\ j \neq k}}^{n} (s_i i + r_j).$$

Since $\prod_{j=1}^{n}(s_i i - r_j) + \prod_{j=1}^{n}(s_i i + r_j) = 0$,

$$\prod_{\substack{j=1\\ j\neq k}}^{n}(s_i i - r_j) - \prod_{\substack{j=1\\ j\neq k}}^{n}(s_i i + r_j) = \left(-\frac{s_i i + r_k}{s_i i - r_k} - 1\right)\prod_{\substack{j=1\\ j\neq k}}^{n}(s_i i + r_j)$$

$$= \frac{-2s_i i}{s_i i - r_k}\prod_{\substack{j=1\\ j\neq k}}^{n}(s_i i + r_j).$$

So

$$i\frac{\partial s_i}{\partial r_k}p'(s_i i) = \frac{2s_i i\prod_{j=1}^{n}(s_i i + r_j)}{s_i^2 + r_k^2}$$

and

$$\left|p'(s_i i)\right| = \frac{2s_i\prod_{j=1}^{n}|s_i i + r_j|}{s_i^2 + r_k^2}\frac{1}{\left|\frac{\partial s_i}{\partial r_k}\right|}. \qquad (5)$$

But

$$p'(s_i i) = \left(\sum_{j=1}^{n}\frac{1}{s_i i + r_j}\right)\prod_{j=1}^{n}(s_i i + r_j) + \left(\sum_{j=1}^{n}\frac{1}{s_i i - r_j}\right)\prod_{j=1}^{n}(s_i i - r_j)$$

$$= \left(\sum_{j=1}^{n}\frac{1}{s_i i + r_j}\right)\prod_{j=1}^{n}(s_i i + r_j) - \left(\sum_{j=1}^{n}\frac{1}{s_i i - r_j}\right)\prod_{j=1}^{n}(s_i i + r_j)$$

$$= \sum_{j=1}^{n}\frac{2r_j}{s_i^2 + r_j^2}\prod_{j=1}^{n}(s_i i + r_j)$$

and

$$\left|p'(s_i i)\right| = 2\sum_{j=1}^{n}\frac{2r_j}{s_i^2 + r_j^2}\prod_{j=1}^{n}|s_i i + r_j|. \qquad (6)$$

By Theorem 3, $\partial s_i/\partial r_k > 0$, and comparing (5) and (6) gives

$$\frac{\partial s_i}{\partial r_k}\sum_{j=1}^{n}\frac{2r_j}{s_i^2 + r_j^2} = \frac{s_i}{s_i^2 + r_k^2}$$

and finally

$$\frac{\partial s_i}{\partial r_k} = \frac{s_i}{s_i^2 + r_k^2}\frac{1}{\sum_{j=1}^{n}\frac{2r_j}{s_i^2 + r_j^2}}.$$

□

Using the above, we prove Theorem 5 and Theorem 6.

Proof of Theorem 5. Consider a function $f(r_1, r_2, \ldots, r_k, \cdots, r_n) = \dfrac{s_i}{\sqrt{r_k}}$. Then it is enough to show that f is a decreasing function with respect of r_k. To prove this, we partially differentiate f by r_k and ensure that for all possible r_k, the value of f is less than 0. In this case,

$$\frac{\partial f(r_k)}{\partial r_k} = \frac{\dfrac{\partial s_i}{\partial r_k}\sqrt{r_k} - s_i \dfrac{1}{2\sqrt{r_k}}}{r_k} = \frac{1}{\sqrt{r_k}}\left(\frac{\partial s_i}{\partial r_k} - \frac{s_i}{2r_k}\right)$$

holds, and by Lemma 14,

$$\frac{\partial s_i}{\partial r_k} = \frac{s_i}{s_i^2 + r_k^2} \frac{1}{\displaystyle\sum_{j=1}^{n} \frac{2r_j}{s_i^2 + r_j^2}} < \frac{s_i}{s_i^2 + r_k^2} \frac{1}{\dfrac{2r_k}{s_i^2 + r_k^2}} = \frac{s_i}{2r_k}.$$

So for $r_k > 0$,

$$\frac{\partial f(r_k)}{\partial r_k} < 0,$$

which concludes the proof. □

Proof of Theorem 6. By Lemma 14,

$$\frac{\partial s_i^2}{\partial r_k} = 2s_i \frac{\partial s_i}{\partial r_k} = \frac{2s_i^2}{s_i^2 + r_k^2} \frac{1}{\sum_{j=1}^{n} \dfrac{2r_j}{s_i^2 + r_j^2}},$$

and this increases as i increases, which follows $t_i^2 - s_i^2 > t_j^2 - s_j^2$ for $i > j$. □

Proof of Theorem 7. By Theorems 5 and 6,

$$t_{i+1}^2 - t_i^2 > s_{i+1}^2 - s_i^2 \quad \text{and} \quad \sqrt{\frac{r_k + h}{r_k}}(s_{i+1} + si) > t_{i+1} + t_i.$$

Combining these two inequalities completes the proof. □

We give another proof of Theorem 7. This proof is basically based on the following proposition from [4]. This proposition will also play a central role in the proofs of Theorem 10 and Theorem 11.

PROPOSITION 15

If $\alpha_j(s_k)$ denotes the angle formed at the real number r_j by the triangle joining r_j, is_k and the origin, then the sums

$$\theta_k = \sum_{j=1}^{n} \alpha_j(s_k)$$

for $k = \lfloor n/2 \rfloor, \ldots, 2, 1$ are, respectively, the numbers

$$\frac{\pi(n-1)}{2}, \quad \frac{\pi(n-3)}{2}, \quad \frac{\pi(n-5)}{2}, \quad \cdots, \quad \frac{\pi c}{2},$$

where $c = 0$ if n is odd and $c = 1$ if n is even. In particular, these are independent of the r_j's. Moreover, for each k,

$$\theta_{k+1} - \theta_k = \pi \tag{7}$$

We give Lemmas 16–20 for another proof of Theorem 7.

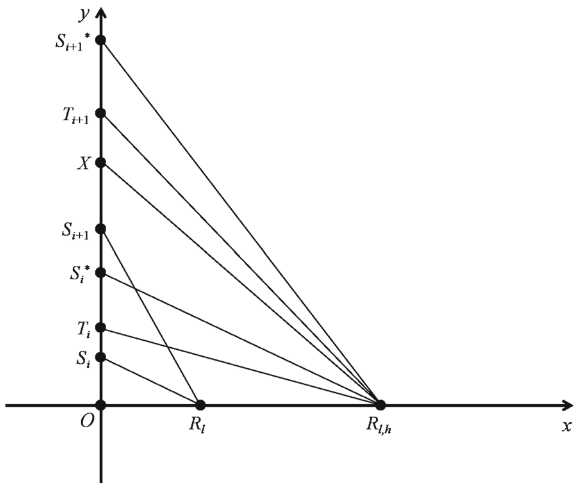
Lemma 16. *Let r be a fixed positive number, and ia, ib the purely imaginary numbers above the real axis, where $b - a = m$ is a fixed positive number. Let O, R, A and B be the points in the complex plane that represent the origin, r, ia and ib, respectively. Then $\angle ARB$ is increasing as a is decreasing.*

Proof. Since

$$\tan(\angle ARB) = \tan(\angle ORB - \angle ORA) = \frac{\tan(\angle ORB) - \tan(\angle ORA)}{1 + \tan(\angle ORB)\tan(\angle ORA)}$$
$$= \frac{\dfrac{m+a}{r} - \dfrac{a}{r}}{1 + \left(\dfrac{m+a}{r}\right)\dfrac{a}{r}} = \frac{mr}{r^2 + (m+a)a},$$

$\tan(\angle ARB)$ is a decreasing function of a, and $\angle ARB$ is increasing as a is decreasing. □

Recall, from $p(x)$, if we shift r_l to the right by a distance h and $-r_l$ to the left by h where $0 < h < r_l$, then a zero is_i of $p(x)$ is shifted to it_i and so is is_{i+1} to it_{i+1}. Let R_l, $R_{l,h}$, S_i, S_{i+1}, T_i and T_{i+1} be the points that represent r_l, $r_l + h$, is_i, is_{i+1}, it_i and it_{i+1}, respectively. We also denote by S_i^* and S_{i+1}^* the points so that $S_i R_l$ and $S_{i+1} R_l$ are parallel to $S_i^* R_{l,h}$ and $S_{i+1}^* R_{l,h}$, respectively. The points S_i^* and S_{i+1}^* will correspond to the purely imaginary numbers is_i^* and is_{i+1}^*, respectively.



Then it is obvious that

$$s_i < s_i^* \quad \text{and} \quad s_{i+1} < s_{i+1}^*$$

and by Theorem 3,

$$s_i < t_i < s_{i+1} < t_{i+1}.$$

Lemma 17. *For each* i,

$$\frac{t_i}{r_l + h} < \frac{s_i}{r_l}.$$

Proof. We consider two sums of angles

$$\sum_{j=1}^{n} \angle S_i R_j O$$

and

$$\sum_{\substack{j=1 \\ j \neq l}}^{n} \angle T_i R_j O + \angle T_i R_{l,h} O,$$

which is obtained after dragging r_l by a distance h $(0 < h < r_l)$ to the right. Then by Proposition 15, we see that

$$\sum_{j=1}^{n} \angle S_i R_j O = \sum_{\substack{j=1 \\ j \neq l}}^{n} \angle T_i R_j O + \angle T_i R_{l,h} O.$$

Since $OT_i > OS_i$, we have $\angle T_i R_j O > \angle S_i R_j O$ for every $j \neq l$, and so

$$\angle S_i R_l O - \angle T_i R_{l,h} O = \sum_{\substack{j=1 \\ j \neq l}}^{n} (\angle T_i R_j O - \angle S_i R_j O) > 0,$$

i.e., $\angle T_i R_{l,h} O < \angle S_i R_l O$. This implies that

$$\frac{t_i}{r_l + h} = \tan(\angle T_i R_{l,h} O) < \tan(\angle S_i R_l O) = \frac{s_i}{r_l}.$$

□

Remark 18. Since the triangles $\triangle S_i R_l O$ and $\triangle S_i^* R_{l,h} O$ are similar,

$$\frac{r_l + h}{r_l} = \frac{s_i^*}{s_i}.$$

By the above lemma, $t_i < s_i \frac{r_l + h}{r_l}$, which implies that

$$t_i < s_i^*.$$

Lemma 19. *If* $s_{i+1} - s_i > t_{i+1} - t_i$, *then*

$$\angle T_i R_{l,h} S_i^* > \angle T_{i+1} R_{l,h} S_{i+1}^*.$$

Proof. Let R_j denote the point for representing r_j. By Proposition 15,

$$\sum_{j=1}^{n} \angle S_i R_j S_{i+1} = \left(\sum_{\substack{j=1 \\ j \neq l}}^{n} \angle T_i R_j T_{i+1} \right) + \angle T_i R_{l,h} T_{i+1} = \pi$$

and so

$$\sum_{\substack{j=1 \\ j \neq l}}^{n} \left(\angle S_i R_j S_{i+1} - \angle T_i R_j T_{i+1} \right) = \angle T_i R_{l,h} T_{i+1} - \angle S_i R_l S_{i+1}.$$

But it follows from by Lemma 16 and $s_{i+1} - s_i > t_{i+1} - t_i$ that for any j with $j \neq l$,

$$\angle S_i R_j S_{i+1} - \angle T_i R_j T_{i+1} > 0.$$

So

$$\angle T_i R_{l,h} T_{i+1} > \angle S_i R_l S_{i+1} = \angle S_i^* R_{l,h} S_{i+1}^* \qquad (8)$$

because S_iR_l and $S_{i+1}R_l$ are parallel to $S_i^*R_{l,h}$ and $S_{i+1}^*R_{l,h}$, respectively. Due to the inequalities $t_i < s_i^*$ and $t_{i+1} < s_{i+1}^*$, there are two possible ways of ordering four points T_i, S_i^*, T_{i+1}, S_{i+1}^*. These possible orders are, starting from the origin,

$$T_i,\ S_i^*,\ T_{i+1},\ S_{i+1}^*$$

and

$$T_i,\ T_{i+1},\ S_i^*,\ S_{i+1}^*.$$

When $s_i^* < t_{i+1}$, subtracting the angle $\angle T_{i+1}R_{l,h}S_i^*$ on each side of (8) induces

$$\angle T_iR_{l,h}S_i^* > \angle T_{i+1}R_{l,h}S_{i+1}^*.$$

When $t_{i+1} < s_i^*$, by adding the angle $\angle T_{i+1}R_{l,h}S_i^*$ on each side of (8), we get the same inequality. □

Lemma 20. *If* $s_{i+1} - s_i > t_{i+1} - t_i$, *then*

$$t_i + t_{i+1} < \sqrt{\frac{r_l + h}{r_l}}(s_i + s_{i+1}).$$

Proof. By Lemma 17,

$$t_i < s_i^* \quad \text{and} \quad t_{i+1} < s_{i+1}^*.$$

So

$$\angle T_iR_{l,h}S_i^* > \angle T_{i+1}R_{l,h}S_{i+1}^*.$$

Let X be the point below S_{i+1}^* that represents a pure imaginary number ix above the real axis so that

$$\angle T_iR_{l,h}S_i^* = \angle S_{i+1}^*R_{l,h}X.$$

Then

$$\tan(\angle T_iR_{l,h}S_i^*) = \tan(\angle S_{i+1}^*R_{l,h}X)$$

and

$$\tan(\angle S_i^*R_{l,h}O - \angle T_iR_{l,h}O) = \tan(\angle S_{i+1}^*R_{l,h}O - \angle XR_{l,h}O).$$

Using tangent formula for $\tan(a - b)$, we may get

$$\frac{\dfrac{s_i^*}{r_l + h} - \dfrac{t_i}{r_l + h}}{1 + \dfrac{s_i^*}{r_l + h}\dfrac{t_i}{r_l + h}} = \frac{\dfrac{s_{i+1}^*}{r_l + h} - \dfrac{x}{r_l + h}}{1 + \dfrac{s_{i+1}^*}{r_l + h}\dfrac{x}{r_l + h}},$$

and so

$$\frac{s_i^* - t_i}{(r_l + h)^2 + s_i^* t_i} = \frac{s_{i+1}^* - x}{(r_l + h)^2 + s_{i+1}^* x}$$

and

$$((r_l + h)^2 + s_i^* t_i)(s_{i+1}^* - x) = ((r_l + h)^2 + s_{i+1}^* x)(s_i^* - t_i).$$

Solving this in x easily gives

$$x = \frac{(r_l + h)^2 s_{i+1}^* + s_i^* s_{i+1}^* t_i + t_i (r_l + h)^2 - (r_l + h)^2 s_i^*}{s_i^* s_{i+1}^* + (r_l + h)^2 + s_i^* t_i - s_{i+1}^* t_i}.$$

By assumption,

$$x - t_i < t_{i+1} - t_i < s_{i+1} - s_i,$$

which implies that

$$\begin{aligned} x - t_i &= \frac{(r_l + h)^2 s_{i+1}^* + s_i^* s_{i+1}^* t_i + t_i (r_l + h)^2 - (r_l + h)^2 s_i^* - s_{i+1}^* s_i^* t_i - (r_l + h)^2 t_i - s_i^* t_i^2 + s_{i+1}^* t_i^2}{s_i^* s_{i+1}^* + (r_l + h)^2 + s_i^* t_i - s_{i+1}^* t_i} \\ &< s_{i+1} - s_i. \end{aligned}$$

So

$$((r_l + h)^2 + t_i^2)(s_{i+1}^* - s_i^*) < (s_{i+1} - s_i)(s_i^* s_{i+1}^* + (r_l + h)^2 + s_i^* t_i - s_{i+1}^* t_i).$$

But due to the similarity of two triangles $\triangle R_l O S_{i+1}$, $\triangle R_{l,h} O S_{i+1}^*$ and $\triangle R_l O S_i$, $\triangle R_{l,h} O S_i^*$, respectively, we have

$$s_{i+1}^* = \frac{r_l + h}{r_l} s_{i+1} \quad \text{and} \quad s_i^* = \frac{r_l + h}{r_l} s_i,$$

and so

$$\begin{aligned} \frac{r_l + h}{r_l}((r_l + h)^2 + t_i^2) &< \left(\frac{r_l + h}{r_l}\right)^2 s_i s_{i+1} + (r_l + h)^2 \\ &\quad + \frac{r_l + h}{r_l} s_i t_i - \frac{r_l + h}{r_l} s_{i+1} t_i \end{aligned}$$

and

$$(r_l + h)^2 + t_i^2 < \frac{r_l + h}{r_l} s_i s_{i+1} + r_l (r_l + h) + s_i t_i - s_{i+1} t_i.$$

Thus

$$t_i^2 + (s_{i+1} - s_i) t_i - \frac{r_l + h}{r_l} s_i s_{i+1} < r_l (r_l + h) - (r_l + h)^2 = -h(r_l + h) < 0.$$

Write $\dfrac{r_l + h}{r_l} = p$ so that

$$t_i^2 + (s_{i+1} - s_i)t_i - ps_i s_{i+1} < 0,$$

and solving this gives

$$t_i < \frac{(s_i - s_{i+1}) + \sqrt{(s_{i+1} + s_i)^2 + 4(p-1)s_i s_{i+1}}}{2}. \tag{9}$$

We now repeat the above process from setting the point Y below S_i^* for a purely imaginary number iy above the real axis such that

$$\angle S_i^* R_{l,h} Y = \angle S_{i+1}^* R_{l,h} T_{i+1}.$$

Then using $y > t_i$, it can be shown that

$$t_{i+1} < \frac{(s_{i+1} - s_i) + \sqrt{(s_{i+1} + s_i)^2 + 4(p-1)s_i s_{i+1}}}{2}. \tag{10}$$

Adding each side of (9) and (10) gives

$$\begin{aligned} t_i + t_{i+1} &< \sqrt{(s_{i+1} + s_i)^2 + 4(p-1)s_i s_{i+1}} \\ &< \sqrt{(s_{i+1} + s_i)^2 + (p-1)(s_i + s_{i+1})^2} \\ &= \sqrt{p}(s_i + s_{i+1}), \end{aligned}$$

which completes the proof. □

We are now ready to give another proof of Theorem 7.

Another proof of Theorem 7. If $s_{i+1} - s_i \leq t_{i+1} - t_i$, then (7) holds because $\frac{r_l+h}{r_l} > 1$. Suppose that $s_{i+1} - s_i > t_{i+1} - t_i$. Then by Theorem 6,

$$t_{i+1}^2 - t_i^2 > (s_{i+1} - s_i)(s_{i+1} + s_i).$$

But Lemma 20 gives

$$\sqrt{\frac{r_l + h}{r_l}}(s_{i+1} + s_i) > t_{i+1} + t_i,$$

and so we obtain the inequality

$$t_{i+1} - t_i > \sqrt{\frac{r_l}{r_l + h}}(s_{i+1} - s_i),$$

which concludes the proof. □

Proof of Theorem 8. Let

$$p(x) = \prod_{j=1}^{n}(x - r_j) + \prod_{j=1}^{n}(x + r_j) =: p_1(x) + p_2(x)$$

and

$$p_h(x) = (x - r_k - h)\prod_{\substack{j=1\\ l\neq k}}^{n}(x - r_j) + (x + r_k + h)\prod_{\substack{j=1\\ l\neq k}}^{n}(x + r_j) =: p_{1,h}(x) + p_{2,h}(x).$$

Then

$$p'(x) = p_1'(x) + p_2'(x) \quad \text{and} \quad p_h'(x) = p_{1,h}'(x) + p_{2,h}'(x).$$

The roots of $p_1'(x)$ lies between the roots of $p_1(x)$ and

$$p_1'(x) = n\prod_{j=1}^{n-1}(x - r_j').$$

By symmetry,

$$p_2'(x) = n\prod_{j=1}^{n-1}(x + r_j').$$

Then Theorem 1 asserts the result. □

Proof of Theorem 9. Consider two polynomials

$$p(x) = \prod_{j=1}^{n}(x - r_j) + \prod_{j=1}^{n}(x + r_j) \quad \text{and}$$

$$xp'(x) = 2nx\left[\prod_{j=1}^{n-1}(x - r_j') + \prod_{j=1}^{n-1}(x + r_j')\right].$$

Then we may regard $p(x)$ as a dragged polynomial from $xp'(x)/(2n)$, dragging from 0 to r_1 and from r_j' to r_{j+1} for $1 \leq j \leq n-1$, since each r_j' lies between r_j and r_{j+1}. This directly leads to the theorem because the square gap increases when dragged by Theorem 6. □

Another proof of $(s_2')^2 - (s_1')^2 > s_2^2 - s_1^2$. Assume n is even. By (4), it suffices to show that for any i with $1 \leq i \leq n/2 - 1$,

$$(s_{i+1}')^2 - (s_i')^2 > s_2^2 - s_1^2.$$

Observe that

$$\frac{p'}{p}(x) = 2x\sum_{j=1}^{n/2}\frac{1}{x^2 + s_j^2}.$$

Let is_i' and is_{i+1}' be successive two zeros of $p'(x)$ in the upper half plane. Suppose

$$(s_{i+1}')^2 - (s_i')^2 \leq s_{l+1}^2 - s_l^2$$

for all $1 \leq l \leq n/2 - 1$. Then

$$\frac{1}{-(s'_{i+1})^2 + s^2_{l+1}} \leq \frac{1}{-(s'_i)^2 + s^2_l} \tag{11}$$

since both $-(s'_{i+1})^2 + s^2_{l+1}$ and $-(s'_i)^2 + s^2_l$ are either positive or negative. But

$$\sum_{j=1}^{n/2} \frac{1}{-(s'_{i+1})^2 + s^2_l} = \sum_{j=1}^{n/2} \frac{1}{-(s'_i)^2 + s^2_l} = 0$$

and so

$$\begin{aligned} 0 = & \frac{1}{-(s'_{i+1})^2 + s^2_1} - \frac{1}{-(s'_i)^2 + s^2_{n/2}} \\ & + \sum_{j=1}^{n/2-1} \left(\frac{1}{-(s'_{i+1})^2 + s^2_{j+1}} - \frac{1}{-(s'_i)^2 + s^2_j} \right). \end{aligned} \tag{12}$$

Since

$$\frac{1}{-(s'_{i+1})^2 + s^2_l} < 0 \quad \text{and} \quad -\frac{1}{-(s'_i)^2 + s^2_{n/2}} < 0,$$

it follows from (11) that the right hand side of the equality (12) is negative, which is a contradiction. This implies that

$$(s'_{i+1})^2 - (s'_i)^2 > s^2_{l+1} - s^2_l$$

for some l, $1 \leq l \leq n/2 - 1$. By Theorem 1,

$$(s'_{i+1})^2 - (s'_i)^2 > s^2_2 - s^2_1$$

The case n odd can be proved in the same way. □

Using Proposition 15, we will prove Theorem 10 and Theorem 11. The below lemma will be useful for the proof of Theorem 10.

Lemma 21. *On xy-plane, let $X_1, X_2, \ldots, X_n$ be distinct points on the x-axis, and A, B, C, D distinct points on the positive y-axis such that for some positive numbers ϕ, ϕ' and c with $\phi \leq \phi'$,*

$$\sum_{j=1}^{n} \angle AX_jO = \phi, \quad \sum_{j=1}^{n} \angle CX_jO = \phi + c, \quad \sum_{j=1}^{n-1} \angle BX_jO = \phi', \sum_{j=1}^{n-1} \angle DX_jO = \phi' + c.$$

Then $AC < BD$.

Proof. From the conditions, it is obvious that

$$\sum_{j=1}^{n} \angle AX_jC = c \quad \text{and} \quad \sum_{j=1}^{n-1} \angle BX_jD = c.$$

Since $\sum_{j=1}^{n} \angle AX_jO < \sum_{j=1}^{n} \angle BX_jO$, we have $OA < OB$ and similarly, $OC < OD$. Now, suppose $AC > BD$. Then, for all j, $\angle AX_jC > \angle BX_jD$ by Lemma 16. Then, $c = \sum_{j=1}^{n} \angle AX_jC > \sum_{j=1}^{n-1} \angle BX_jD = c$, which leads to a contradiction. □

Proof of Theorem 10. Assume that n is even. We recall that $is_1,\ is_2,\ \ldots,\ is_{n/2}$ are roots on the upper half-plane of

$$p(x) = \prod_{j=1}^{n}(x - r_j) + \prod_{j=1}^{n}(x + r_j),$$

and $S_1,\ S_2,\ \ldots,\ S_{n/2}$ represent the points $is_1,\ is_2,\ \ldots,\ is_{n/2}$, and $R_1,\ R_2,\ \ldots,\ R_n$ represent the points $r_1, r_2, \ldots, r_n$ on the complex plane, respectively. We now consider the polynomial

$$q(x) = \prod_{j=1}^{n-1}(x - r_j) + \prod_{j=1}^{n-1}(x + r_j),$$

and say their roots that are not on the lower half-plane are $0, iu_1, iu_2, \ldots, iu_{n/2-1}$ with the corresponding points $O,\ U_1,\ U_2,\ \ldots,\ U_{n/2-1}$ on the complex plane. Then using Proposition 15, we may compute that for each i,

$$\sum_{j=1}^{n} \angle S_i R_j O = \frac{\pi}{2} + (i-1)\pi, \quad \sum_{j=1}^{n} \angle S_{i+1} R_j O = \frac{\pi}{2} + i\pi,$$
$$\sum_{j=1}^{n-1} \angle U_i R_j O = i\pi, \quad \sum_{j=1}^{n-1} \angle U_{i+1} R_j O = (i+1)\pi.$$

As $S_i,\ S_{i+1},\ U_i,\ U_{i+1}$ satisfy the conditions of Lemma 21, we see that $U_iU_{i+1} > S_iS_{i+1}$, which means

$$u_{i+1} - u_i > s_{i+1} - s_i. \tag{13}$$

We now consider relations between $u_{i+1} - u_i$ and $s'_{i+1} - s'_i$. The polynomial

$$\frac{p'(x)}{2n} = \prod_{j=1}^{n-1}(x - r'_j) + \prod_{j=1}^{n-1}(x + r'_j)$$

with the roots $0,\ is'_1,\ is'_2,\ \ldots,\ is'_{n/2-1}$ that are not on the lower-half plane is a dragged polynomial from

$$q(x) = \prod_{j=1}^{n-1}(x - r_j) + \prod_{j=1}^{n-1}(x + r_j).$$

So by Theorem 7, we get

$$s'_{i+1} - s'_i > \left(\frac{\prod_{j=1}^{n-1} r_j}{\prod_{j=1}^{n-1} r'_j} \right) (u_{i+1} - u_i).$$

Combining this with (13) gives

$$s'_{i+1} - s'_i > \left(\frac{\prod_{j=1}^{n-1} r_j}{\prod_{j=1}^{n-1} r'_j} \right) (s_{i+1} - s_i)$$

which is desired. The case n odd can be proved in the same way. □

The two lemmas below will be used to prove Theorem 11.

Lemma 22. *For two points X_1, X_2 on the x-axis and a point Y on the y-axis that are not the origin, if $OX_1 < OX_2$, $\angle YX_1O > YX_2O$.*

Proof. Since $\tan(\angle YX_1O) = \frac{OY}{OX_1}$ and $\tan(\angle YX_2O) = \frac{OY}{OX_2}$, we have

$$\tan(\angle YX_1O) > \tan(\angle YX_2O).$$

This completes the proof because both angles are less than $\pi/2$. □

Lemma 23. *Let $f(x) = \prod_{j=1}^{n}(x - a_j)$, where $a_1 < a_2 < \cdots < a_n$, and its inflection points are $a''_1, a''_2, \ldots, a''_{n-2}$ in ascending order. Then for all i, we have*

$$a_i < a''_i < a_{i+2}.$$

Proof. By Rolle's theorem, for each i,

$$a_i < a'_i < a_{i+1} < a'_{i+1} < a_{i+2} \quad \text{and} \quad a'_i < a''_i < a'_{i+1},$$

which follows the result. □

Let us denote that all roots on the upper-half plane of the j-th derivative of $p(x) = \prod_{i=1}^{n}(x - r_i) + \prod_{i=1}^{n}(x + r_i)$ are

$$is_{(1,j)},\ is_{(2,j)},\ is_{(3,j)},\ \ldots,$$

where $s_{(1,j)} < s_{(2,j)} < s_{(3,j)} < \ldots$.

Proof of Theorem 11. We note that

$$p'(x) = k_1 \left[\prod_{i=1}^{n-1} (x + r_i') + \prod_{i=1}^{n-1} (x - r_i') \right],$$

$$p''(x) = k_2 \left[\prod_{i=1}^{n-2} (x + r_i'') + \prod_{i=1}^{n-2} (x - r_i'') \right]$$

for some integers k_1 and k_2, where r_i's are the critical points of the polynomial $\prod_{i=1}^{n}(x - r_i)$ and r_i''s are the inflection points of $\prod_{i=1}^{n}(x - r_i)$. Now, we denote the corresponding points on the complex plane to $s_{(i,1)}$ and $s_{(i,2)}$ by S_i' and S_i'', respectively. Let

$$\phi = \sum_{j=1}^{n} \angle S_i R_j O \qquad (1 \leq i \leq \lfloor n/2 \rfloor - 2).$$

Then by Proposition 15,

$$\sum_{j=1}^{n} \angle S_{i+1} R_j O = \phi + \pi \tag{14}$$

and

$$\sum_{j=1}^{n-2} \angle S_i'' R_j'' O = \phi$$

since n and $n-2$ have the same parity. Now, by Lemma 23, $OR_{j+2} > OR_j''$, so Lemma 22 leads to

$$\angle S_{i+1} R_{j+2} O < \angle S_{i+1} R_j'' O$$

for all j with $1 \leq j \leq n-2$, and summing up similar angles we get

$$\sum_{j=3}^{n} S_{i+1} R_j O < \sum_{j=1}^{n-2} S_{i+1} R_j'' O.$$

Since $\angle S_{i+1} R_j O < \dfrac{\pi}{2}$, we have

$$\sum_{j=3}^{n} \angle S_{i+1} R_j O > \sum_{j=1}^{n} \angle S_{i+1} R_j O - \pi = \phi$$

by (14), and

$$\phi < \sum_{j=1}^{n-2} \angle S_{i+1} R_j'' O.$$

But $\sum_{j=1}^{n-2} \angle S_i'' R_j'' O = \phi$ and so

$$\sum_{j=1}^{n-2} \angle S_i'' R_j'' O < \sum_{j=1}^{n-2} \angle S_{i+1} R_j'' O.$$

This inequality directly leads to $OS_i'' < OS_{i+1}$, and

$$s_{(i,2)} < s_{(i+1,0)}. \tag{15}$$

Similarly, by Lemma 23, $OR_j'' > OR_j$ and Lemma 22 leads to

$$\angle S_i R_j O > \angle S_i R_j'' O$$

for all j. Summing up all the similar angles we get

$$\sum_{j=1}^{n-2} S_i R_j O > \sum_{j=1}^{n-2} S_i R_j'' O.$$

Since $\angle S_i R_j O > 0$, we have

$$\phi > \sum_{j=1}^{n-2} S_i R_j O > \sum_{j=1}^{n-2} S_i R_j'' O.$$

But $\sum_{j=1}^{n-2} \angle S_i'' R_j'' O = \phi$, so

$$\sum_{j=1}^{n-2} \angle S_i'' R_j'' O > \sum_{j=1}^{n-2} S_i R_j'' O.$$

This inequality directly leads to $OS_i'' > OS_i$, and

$$s_{(i,0)} < s_{(i,2)}. \tag{16}$$

Thus by (15) and (16), we have

$$s_{(i,0)} < s_{(i,2)} < s_{(i+1,0)}$$

and in fact we may generalize this to

$$s_{(i,j)} < s_{(i,j+2)} < s_{(i+1,j)}.$$

□

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