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Integral pentavalent Cayley graphs on abelian or dihedral groups

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Abstract. A graph is called integral, if all of its eigenvalues are integers. In this paper, we give some results about integral pentavalent Cayley graphs on abelian or dihedral groups.

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1. Introduction

We say that a graph is integral if all the eigenvalues of its adjacency matrix are integers. The notion of integral graphs was first introduced by Harary and Schwenk [8]. Bussemaker and Cvetković [5] proved that there are exactly 13 connected cubic integral graphs. The same result was indepedently proved by Schwenk [12] who unlike the effort in Bussemaker and Cvetković [5] avoids the use of computer search to examine all the possibilities. In [3] it is shown that the total number of matrices of integral graphs with n vertices is less than or equal to $2^{\frac{n(n-1)}{2}-\frac{n}{400}}$ for a sufficiently large n.

Stevanović [14] determined all connected 4-regular integral graphs avoiding ± 3 in the spectrum. Sander [11] proved that Sudoku graphs are integral. It is known that the size of a connected k-regular graph with diameter d is bounded above by $\frac{k(k-1)^d-2}{k-2}$ (see, for example, [7]). In [6], it is noted that if the graph is integral then $d \leq 2k$ because there are at most $2k+1$ distinct eigenvalues. Consequently, the upper bound of the size of a connected k-regular integral graph is

$$n \leq \frac{k(k-1)^{2k}-2}{k-2}.$$

Let G be a non-trivial group, $S \subseteq G - \{1\}$ and $S = S^{-1} = \{s^{-1} \mid s \in S\}$. The Cayley graph of G denoted by $\mathrm{Cay}(G, S)$ is a graph with vertex set G and two vertices a and b are adjacent if $ab^{-1} \in S$. If S generates G then $\mathrm{Cay}(G, S)$ is connected. A Cayley graph is simple and vertex transitive. Let G be a group. An element $g \in G$ is said to be an involution, if its order is 2. The main question that we are concerned here is the following: Which Cayley graphs are integral?

It is clear that if $S = G - \{1\}$, then $\mathrm{Cay}(G, S)$ is the complete graph with $|G|$ vertices and so it is integral. Klotz and Sander [10] showed that all nonzero eigenvalues of $\mathrm{Cay}(\mathbb{Z}_n, U_n)$ are integers dividing the value $\varphi(n)$ of the Euler totient function, where $\mathbb{Z}_n$ is the cyclic group of order n and U_n is the subset of all elements of $\mathbb{Z}_n$ of order n. So [13] characterized integral graphs among circulant graphs. Abdollahi and Vatandoost [1, 2], determined integral cubic and tetravalent Cayley graphs on abelian groups. By using a result of Babai [4] which presented the spectrum of a Cayley graph in terms of irreducible characters of the underlying group, we give some results on integral pentavalent Cayley graphs on abelian or dihedral groups.

2. Preliminaries

In this section we give some results which will be used in the next section.

PROPOSITION 2.1

Let $\mathrm{Aut}(G)$ *denote the automorphism group of* G. *Also let* $\alpha \in \mathrm{Aut}(G)$. *Then* $\mathrm{Cay}(G, S)$ *is isomorphic to* $\mathrm{Cay}(G, S^\alpha)$.

Proof. It is easy to see that the map $\psi : g \mapsto g^\alpha$ is an isomorphism between two Cayley graphs. □

PROPOSITION 2.2 [4]

Let G be a finite group of order n whose irreducible characters (over $\mathbb{C}$) are $\chi_1, \ldots, \chi_h$ with respective degrees $n_1, \ldots n_h$. Then the spectrum of the Cayley graph $\mathrm{Cay}(G, S)$ *can be arranged as $\Lambda = \{\lambda_{ijk} \mid i = 1, \ldots, h;\ j; k = 1, \ldots, n_i\}$ such that $\lambda_{ij1} = \cdots = \lambda_{ijn_i}$ (this common value will be denoted by λ_{ij}), and*

$$\lambda_{i1}^t + \cdots + \lambda_{in_i}^t = \sum_{s_1, \ldots, s_t \in S} \chi_i \left(\prod_{l=1}^{t} s_l \right) \tag{1}$$

for any natural number t.

PROPOSITION 2.3 [9]

Let C_n be the cyclic group generated by a of order n. Then the irreducible characters of C_n are $\rho_j(a^k) = \omega^{jk}$, where $j, k = 0, 1, \ldots, n-1$.

PROPOSITION 2.4 [9]

Let $G = C_{n_1} \times \cdots \times C_{n_r}$ and $C_{n_i} = \langle a_i \rangle$, so that for any $i, j \in \{1, \ldots, r\}$, $(n_i, n_j) \neq 1$. If $\omega_t = e^{\frac{2\pi i}{n_t}}$, then $n_1 \ldots n_r$ irreducible characters of G are

$$\rho_{l_1 \ldots l_r}(a_1^{k_1}, \ldots, a_r^{k_r}) = \omega_1^{l_1 k_1} \omega_2^{l_2 k_2} \ldots \omega_r^{l_r k_r}$$

where $l_i = 0, 1, \ldots, n_i - 1$ and $i = 1, 2, \ldots, r$.

PROPOSITION 2.5 [1]

Let $G = \langle S \rangle$ *be a group*, $|G| = n$, $|S| = 2$, $1 \notin S = S^{-1}$. *Then* $\mathrm{Cay}(G, S)$ *is an integral graph if and only if* $n \in \{3, 4, 6\}$.

PROPOSITION 2.6 [1]

Let G *be the cyclic group* $\langle a \rangle$, $|G| = n > 3$ *and let* S *be a generating set of* G *such that* $|S| = 3$, $S = S^{-1}$ *and* $1 \notin S$. *Then* $\mathrm{Cay}(G, S)$ *is an integral graph if and only if* $n \in \{4, 6\}$.

3. Results

Lemma 3.1. *Let* G_1 *and* G_2 *be two non-trivial abelian groups and* $G = G_1 \times G_2$ *such that* $X = \mathrm{Cay}(G, S)$ *is integral and* $G = \langle S \rangle$, *where* $|S| = 5$, $S = S^{-1}$ *and* $1 \notin S$. *Let* $S_1 = \{s_1 \mid (s_1, g_2) \in S, g_2 \in G_2\} - \{1\}$. *Then* $\mathrm{Cay}(G_1, S_1)$ *is a connected integral graph.*

Proof. Let χ_0 and ρ_0 be the trivial irreducible characters of G_1 and G_2, respectively. Let λ_{i0} and λ_i be the eigenvalues of $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(G_1, S_1)$ corresponding to irreducible characters of $\chi_i \times \rho_0$ and χ_i, respectively. We have $|S_1| \in \{1, 2, 3, 4, 5\}$. If $|S_1| = 1$, then $|G_1| = 2$ and so $\mathrm{Cay}(G_1, S_1)$ is the complete graph K_2 with two vertices which is an integral graph. By Proposition 2.2,

$$\lambda_{io} = \sum_{(g_1, g_2) \in S} (\chi_i \times \rho_0)(g_1, g_2).$$

We have the following cases:

Case 1. If $|S_1| = 5$, then $\lambda_{i0} = \lambda_i$. It follows that $\mathrm{Cay}(G_1, S_1)$ is integral.

Case 2. Let $|S_1| = 4$ and suppose that either $S = \{(a, x), (a^{-1}, x^{-1}), (b, y), (b^{-1}, y^{-1}), (1, z)\}$, where $o(z) = 2$ or $S = \{(a, x), (b, y), (c, z), (d, w), (1, f)\}$, where $o(a) = o(b) = o(c) = o(d) = o(x) = o(y) = o(z) = o(w) = o(f) = 2$ or $S = \{(a, x), (b, y), (c, z), (c^{-1}, z^{-1}), (1, f)\}$, where $o(a) = o(b) = o(x) = o(y) = o(f) = 2$ or $S = \{(a, x), (b, y), (c, z), (c, z^{-1}), (d, w)\}$, where $o(a) = o(b) = o(c) = o(d) = o(x) = o(y) = o(w) = 2$ or $S = \{(a, x), (a^{-1}, x^{-1}), (b, y), (b, y^{-1}), (c, z)\}$, where $o(b) = o(c) = o(z) = 2$. Thus either $\lambda_{i0} = \lambda_i + \chi_i(1)$ or $\lambda_{i0} = \lambda_i + \chi_i(b)$, or $\lambda_{i0} = \lambda_i + \chi_i(c)$, respectively. Since $2|\chi_i(b) - \chi_i(1)$ and $2|\chi_i(c) - \chi_i(1)$, $\chi_i(b)$ and $\chi_i(c)$ are integers and so $\mathrm{Cay}(G_1, S_1)$.

Case 3. Now assume that $|S_1| = 3$. Then either $S = \{(a, x), (a, x^{-1}), (b, y), (b, y^{-1}), (c, z)\}$, where $o(a) = o(b) = o(c) = o(z) = 2$ or $S = \{(a, x), (a^{-1}, x^{-1}), (b, y), (b, y^{-1}), (1, z)\}$, where $o(b) = o(z) = 2$ or $S = \{(a, x), (b, y), (c, z), (c, z^{-1}), (1, w)\}$, where $o(a) = o(b) = o(c) = o(x) = o(y) = o(w) = 2$ or $S = \{(a, x), (b, y), (c, z), (1, e), (1, f)\}$, where $o(a) = o(b) = o(c) = o(x) = o(y) = o(z) = o(e) = o(f) = 2$ or $S = \{(a, x), (a^{-1}, x^{-1}), (b, y), (1, e), (1, f)\}$, where $o(b) = o(y) = o(e) = o(f) = 2$. Therefore either $\lambda_{i0} = \lambda_i + \chi_i(b) + \chi_i(a)$ or $\lambda_{i0} = \lambda_i + \chi_i(b) + \chi_i(1)$ or $\lambda_{i0} = \lambda_i + \chi_i(c) + \chi_i(1)$ or $\lambda_{i0} = \lambda_i + \chi_i(1) + \chi_i(1)$. So $\mathrm{Cay}(G_1, S_1)$ is integral again. We must note that $S = S^{-1}$ and if the elements e and f are not involutions then again we have the same results.

Case 4. Finally assume that $|S_1| = 2$. Then either $S = \{(a, x), (a^{-1}, x^{-1}), (1, y), (1, z), (1, w)\}$, where $o(y) = o(z) = o(w) = 2$ or $S = \{(a, x), (b, y), (1, z), (1, w), (1, r)\}$, where $o(a) = o(b) = o(x) = o(y) = o(z) = o(w) = o(r) = 2$ or $S = \{(a, x), (b, y), (b, y^{-1}), (1, w), (1, z)\}$, where $o(a) = o(b) = o(x) = o(z) = o(w) = 2$ or $S = \{(a, x), (a, x^{-1}), (b, y), (b, y^{-1}), (1, z)\}$, where $o(a) = o(b) = o(z) = 2$. Therefore either $\lambda_{i0} = \lambda_i + \chi_i(1) + \chi_i(1) + \chi_i(1)$ or $\lambda_{i0} = \lambda_i + \chi_i(b) + \chi_i(1) + \chi_i(1)$ or $\lambda_{i0} = \lambda_i + \chi_i(a) + \chi_i(b) + \chi_i(1)$. So $\mathrm{Cay}(G_1, S_1)$ is integral again. We must note that $S = S^{-1}$ and if the elements of the form $(1, t)$, where $t \in \{y, z, w, r\}$ are not involutions then again we have the same results.

Lemma 3.2. *Let G be the cyclic group $\langle a\rangle$, $|G| = n > 4$ and let S be a generating set of G such that $|S| = 5$, $S = S^{-1}$ and $1 \notin S$. Then $a^{n/2} \in S$. Also let $a^r \in S$ and $a^t \in S$, where $o(a^r) = m > 2$ and $o(a^t) = n > 2$. Then we have one of the following cases*:

(*i*) $(r, n) = 1$ *or* $(r, n/2) = 1$;
(*ii*) $(t, n) = 1$ *or* $(t, n/2) = 1$;
(*iii*) $(t, n/2, r) = 1$.

Proof. Since $S = S^{-1}$, then S has at least one involution. Thus n is even and $a^{n/2} \in S$. Therefore we may assume that $S = \{a^r, a^{-r}, a^t, a^{-t}, a^{n/2}\}$. Suppose on the contrary that none of the above cases happen. So we may suppose that $(n/2, r) = d$ and $(n/2, t) = d'$. Thus $\langle a^r, a^t, a^{n/2}\rangle \subseteq \langle a^d, a^{d'}\rangle$. Since $(t, n/2, r) \neq 1$, it follows that $(d, d') = d'' \neq 1$. Thus $\langle a^d, a^{d'}\rangle \subseteq \langle a^{d''}\rangle \neq G$, a contradiction. □

Theorem 3.3. *Let G be a finite abelian group such that it is not cyclic and let $G = \langle S\rangle$, where $|S| = 5$, $S = S^{-1}$ and $1 \notin S$. Also let $\mathrm{Cay}(G, S)$ is integral. Then* $|G| \in \{8, 16, 18, 24, 32, 36, 40, 48, 50, 64, 72, 80, 96, 100, 120, 128, 144, 160, 192, 200, 240, 288\}$.

Proof. Let $\mathrm{Cay}(G, S)$ be integral. If all elements of S are involutions, then $G \cong \mathbb{Z}_2^3$ or $\mathbb{Z}_2^4$ or $\mathbb{Z}_2^5$. So $|G| = 8$, 16 or 32. Otherwise $G = \mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_2$ or $G = \mathbb{Z}_m \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. First suppose that $G = \mathbb{Z}_m \times \mathbb{Z}_n \times \mathbb{Z}_2$. By Lemma 3.1, $\mathrm{Cay}(\mathbb{Z}_m, S_1)$ and $\mathrm{Cay}(\mathbb{Z}_n \times \mathbb{Z}_2, S_2)$ are integral graphs where $S_1 = \{s_1 \in \mathbb{Z}_m \mid \exists\, x \in \mathbb{Z}_n \times \mathbb{Z}_2, (s_1, x) \in S\} - \{1\}$ and $S_2 = \{s_2 \in \mathbb{Z}_n \times \mathbb{Z}_2 \mid \exists\, x \in \mathbb{Z}_m, (x, s_2) \in S\} - \{1\}$. Also since $\mathrm{Cay}(\mathbb{Z}_n \times \mathbb{Z}_2, S_2)$ is integral it follows that $\mathrm{Cay}(\mathbb{Z}_n, S_2')$ is integer where $S_2' = \{s_2' \in \mathbb{Z}_n \mid \exists\, x \in \mathbb{Z}_2, (s_2', x) \in S_2\} - \{1\}$.

By Lemmas 2.7 and 2.9 of [1] and Lemma 2.14, Corollary 2.16 of [2], $m, n \in \{3, 4, 5, 6, 8, 10, 12\}$. Since $(m, n) \neq 1$, we have $|G| \in \{2 \times 9, 2 \times 16, 2 \times 18, 2 \times 24, 2 \times 25, 2 \times 32, 2 \times 36, 2 \times 40, 2 \times 48, 2 \times 50, 2 \times 60, 2 \times 64, 2 \times 72, 2 \times 80, 2 \times 96, 2 \times 100, 2 \times 120, 2 \times 144\}$. Now suppose that $G = \mathbb{Z}_m \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. By Lemma 3.1, $\mathrm{Cay}(\mathbb{Z}_m, S_1)$ is integral where $m \in \{3, 4, 5, 6, 8, 10, 12\}$. So $|G| \in \{2 \times 2 \times 2 \times 3, 2 \times 2 \times 2 \times 4, 2 \times 2 \times 2 \times 5, 2 \times 2 \times 2 \times 6, 2 \times 2 \times 2 \times 8, 2 \times 2 \times 2 \times 10, 2 \times 2 \times 2 \times 12\}$. Now the proof is complete. □

The following results are the generalization of results obtained recently by Abdollahi and Vatandoost [1, 2].

Lemma 3.4. *Let $D_{2n} = \langle a, b \mid a^n = b^2 = 1, (ab)^2 = 1\rangle$, $n = 2m+1$ and $\mathrm{Cay}(D_{2n}, S)$ be connected integral graph, where $S = S^{-1}$ and $|S| = k$. Then $-k$ is the simple eigenvalue of $\mathrm{Cay}(D_{2n}, S)$ if and only if all of elements of S are of order two.*

Proof. Let $-k$ be the simple eigenvalue of $\mathrm{Cay}(D_{2n}, S)$. By Proposition 2.2 and using character table of D_{2n}, $-k$ is the eigenvalue of $\mathrm{Cay}(D_{2n}, S)$ corresponding to the irreducible character χ_{m+1}. So all elements of S are in conjugacy class of b. Conversely, if all the elements of S are of order two, then $S \subseteq \overline{b}$ (the bar indicates conjugacy class). By Proposition 2.2 and using the character table of D_{2n}, the eigenvalue of $\mathrm{Cay}(D_{2n}, S)$ corresponding to irreducible character χ_{m+1} is $-k$. □

Lemma 3.5. *Let* $D_{2n} = \langle a, b \mid a^n = b^2 = 1, (ab)^2 = 1\rangle$, $n = 2m + 1$ *and* $\mathrm{Cay}(D_{2n}, S)$ *be integral, where* $G = \langle S\rangle$, $|S| = 4$, $S = S^{-1}$ *and* $1 \notin S$. *If* $3 \nmid n$, *then* $\mathrm{Cay}(D_{2n}, S)$ *is bipartite.*

Proof. Let $a^r \in S$, where $1 \le r \le m$. Then either $S = \{a^r, a^{-r}, a^s, a^{-s}\}$, where $1 \le r, s < n$ or $S = \{a^r, a^{-r}, a^i b, a^j b\}$, where $1 \le i, j \le n$, $1 \le r < n$ and $i \neq j$. Since X is connected the former case cannot happen. So we may suppose that $S = \{a^r, a^{-r}, a^i b, a^j b\}$. If $(r, n) = 1$, then since $n \neq 3$ it implies that $2\cos\frac{2\pi r}{n}$ is not integer. Let λ_{11} and λ_{12} be eigenvalues of $\mathrm{Cay}(D_{2n}, S)$ corresponding to irreducible character χ_1. By Proposition 2.2 and using character table of D_{2n}, $\lambda_{11} + \lambda_{12} = 4\cos\frac{2\pi r}{n}$. This contradicts the fact that $\mathrm{Cay}(D_{2n}, S)$ is integral. Now let $(r, n) \neq 1$. Also let λ_{11} and λ_{12} be eigenvalues of $\mathrm{Cay}(D_{2n}, S)$ corresponding to the irreducible character χ_1. By Proposition 2.2, $\lambda_{11} + \lambda_{12} = 4\cos\frac{2\pi r}{n}$ and $\lambda_{11}^2 + \lambda_{12}^2 = 8 + 4\cos\frac{4\pi r}{n} + 4\cos\frac{2\pi(i-j)}{n}$. Note that λ_{11} and λ_{12} are integers. If $4\cos\frac{2\pi r}{n}$ is not an integer we have a contradiction. So suppose that $4\cos\frac{2\pi r}{n}$ is an integer. Thus we must have $i = j$, a contradiction. So $S \subseteq \overline{b}$ and hence -4 is an eigenvalue of $\mathrm{Cay}(D_{2n}, S)$. Therefore $\mathrm{Cay}(D_{2n}, S)$ is bipartite.

Lemma 3.6. *Let* $D_{2n} = \langle a, b \mid a^n = b^2 = 1, (ab)^2 = 1\rangle$, $n = 2m + 1$ *and* $\mathrm{Cay}(D_{2n}, S)$ *be integral, where* $G = \langle S\rangle$, $|S| = 5$, $S = S^{-1}$ *and* $1 \notin S$. *If* $3 \nmid n$, *then* $\mathrm{Cay}(D_{2n}, S)$ *is bipartite.*

Proof. Let $a^r \in S$, where $1 \le r \le n$. Then either $S = \{a^r, a^{-r}, a^s, a^{-s}, a^i b\}$, where $1 \le r, s < n$ and $1 \le i \le n$ or $S = \{a^r, a^{-r}, a^i b, a^j b, a^k b\}$, where $1 \le r < n$ and $1 \le i, j, k \le n$. First suppose that $S = \{a^r, a^{-r}, a^s, a^{-s}, a^i b\}$. Since $\mathrm{Aut}(G)$ acts transitively on involution, by Proposition 2.1, we may suppose that $S = \{a^r, a^{-r}, a^s, a^{-s}, b\}$. Since X is connected, without loss of generality, we may suppose that $(r, n) = 1$. Thus $\cos\frac{2\pi}{n} r$ is not integral. Let λ_{11} and λ_{12} be eigenvalues of $\mathrm{Cay}(D_{2n}, S)$ corresponding to irreducible character χ_1. By Proposition 2.2 and using character table of D_{2n}, $\lambda_{11}+\lambda_{12} = 4\cos\frac{2\pi r}{n} + 4\cos\frac{2\pi s}{n}$. First suppose that $-\cos\frac{2\pi r}{n} = \cos\frac{2\pi s}{n}$. Therefore $\cos(\pi + \frac{2\pi r}{n}) = \cos\frac{2\pi s}{n}$ and

Character table of D_{2n}, $n = 2m + 1$ odd.

	1	a^r	b
χ_j	2	$\omega^{jr} + \omega^{-jr}$	0
χ_{m+1}	1	1	−1
χ_{m+2}	1	1	1

$w = e^{\frac{2\pi i}{n}}$, $1 \le j \le m$ and $1 \le r \le m$.

so $2\pi s = 2k\pi n \pm (n\pi + 2\pi r)$, a contradiction. Now suppose that $-\cos\frac{2\pi r}{n} \neq \cos\frac{2\pi s}{n}$. Since $\cos\frac{2\pi}{n}r$ is not integral and $\lambda_{11} + \lambda_{12}$ is integral, we have a contradiction. $\square$

Now suppose that $S = \{a^r, a^{-r}, a^i b, a^j b, a^k b\}$. If $(r, n) = 1$, then since $n \neq 3$ it implies that $2\cos\frac{2\pi}{n}r$ is not an integer. Let λ_{11} and λ_{12} be eigenvalues of $\mathrm{Cay}(D_{2n}, S)$ corresponding to irreducible character χ_1. By Proposition 2.2 and using character table of D_{2n}, $\lambda_{11} + \lambda_{12} = 4\cos\frac{2\pi r}{n}$. This contradicts the fact that $\mathrm{Cay}(D_{2n}, S)$ is integral. Now let $(r, n) \neq 1$. Also let λ_{11} and λ_{12} be eigenvalues of $\mathrm{Cay}(D_{2n}, S)$ corresponding to irreducible character χ_1. Since $3 \nmid n$, with similar arguments we have a contradiction. So $S \subseteq \overline{b}$ and hence -5 is an eigenvalue of $\mathrm{Cay}(D_{2n}, S)$. Therefore $\mathrm{Cay}(D_{2n}, S)$ is bipartite.

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