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Rigidity theorem for Willmore surfaces in a sphere

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Abstract. Let M^2 be a compact Willmore surface in the $(2+p)$-dimensional unit sphere S^{2+p}. Denote by H and S the mean curvature and the squared length of the second fundamental form of M^2, respectively. Set $\rho^2 = S - 2H^2$. In this note, we proved that there exists a universal positive constant C, such that if $\|\rho^2\|_2 < C$, then $\rho^2 = 0$ and M^2 is a totally umbilical sphere.

Keywords. Willmore functional; Sobolev inequality; mean curvature; totally umbilical surface.

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1. Introduction

Let M be a compact surface in the $(2+p)$-dimensional unit sphere S^{2+p}. Choose a local orthonormal frame field $\{e_1, e_2, \ldots, e_{2+p}\}$ in S^{2+p} such that, restricted to M, the $\{e_1, e_2\}$ are tangent to M. The following convention of indices are used throughout.

$$1 \le i, j, k \le 2; \qquad 3 \le \alpha, \beta, \gamma \le 2+p.$$

Denote by H and S the mean curvature and the squared length of the second fundamental form of M, respectively. Then, we have

$$S = \sum_{\alpha,i,j} (h_{ij}^{\alpha})^2, \quad \mathbf{H} = \sum_{\alpha} H^{\alpha} e_{\alpha}, \quad H^{\alpha} = \frac{1}{2}\sum_k h_{kk}^{\alpha}, \quad H = |\mathbf{H}|,$$

where h_{ij}^{α} is the component of the second fundamental tensor of M.

Let $\rho^2 = S - 2H^2$. In fact, if we set $\tilde{h}_{ij}^{\alpha} = h_{ij}^{\alpha} - \delta_{ij} H^{\alpha}$, by a direct computation, one has

$$\rho^2 = \sum_{\alpha,i,j} (\tilde{h}_{ij}^{\alpha})^2.$$

So, $\rho^2 \ge 0$, and ρ vanishes at the umbilical points of M.

The Willmore functional is defined by

$$W(x) = \int_M \rho^2 \mathrm{d}v = \int_M (S - 2H^2)\mathrm{d}v.$$

Here the integration is with respect to the area measure of M. In [3], Chen proved that this functional is invariant under conformal transformations of S^{2+p}.

DEFINITION

$x : M \to S^{2+p}$ is called a Willmore surface if it is a critical surface of the Willmore functional $W(x)$.

It was proved by Bryant [1] and Weiner [7] that M is a Willmore surface if and only if

$$\Delta^{\perp} H^{\alpha} + \sum_{\beta,i,j} h_{ij}^{\alpha} h_{ij}^{\beta} H^{\beta} - 2H^2 H^{\alpha} = 0, \tag{1.1}$$

i.e.,

$$\Delta^{\perp} H^{\alpha} + \sum_{\beta,i,j} \tilde{h}_{ij}^{\alpha} \tilde{h}_{ij}^{\beta} H^{\beta} = 0,$$

where $\Delta^{\perp} H^{\alpha} = \sum_k H_{kk}^{\alpha}$.

From (1.1), we know that all minimal surfaces in S^{2+p} are Willmore surfaces. So, the Veronese surface must be the Willmore surface. Moreover, Pinkall [4] constructed many compact non-minimal flat Willmore surfaces in S^3, and Castro and Urbano [2] constructed many compact non-minimal Willmore surfaces in S^4.

In [6], Li obtained the following rigidity theorem for Willmore surfaces in a unit sphere.

Theorem A. *Let M be a compact Willmore surface in S^{2+p}. Then*

$$\int_M \rho^2 \left(2 - \frac{2}{B}\rho^2\right) \mathrm{d}v \leq 0,$$

where

$$B = \begin{cases} 2, & p = 1\,, \\ \frac{4}{3}, & p \geq 2\,. \end{cases}$$

In particular, if

$$\rho^2 \leq B,$$

then either $\rho^2 = 0$ and M is totally umbilical, or $\rho^2 = B$. In the latter case, $p = 2$ and M is the Veronese surface or $p = 1$ and $M = S^1\left(\frac{1}{\sqrt{2}}\right) \times S^1\left(\frac{1}{\sqrt{2}}\right)$.

Applying Theorem A and the Sobolev inequality, we proved the following result (see [9]).

Theorem B. *Let M be a compact Willmore surface in S^{2+p}. There exists a positive constant $\tilde{C}(H_0)$, defined by*

$$\tilde{C}(H_0) = \frac{B\left(\sqrt{9+H_0^2} - \sqrt{1+H_0^2}\right)\sqrt{\pi}}{48\sqrt{3}},$$

such that if

$$\|\rho^2\|_2 < \tilde{C}(H_0),$$

then M is a totally umbilical surface, where $H_0 = \max_{x\in M} H$ and B is defined in Theorem A.

We shall improve the constant of Theorem B and obtain the following global pinching theorem for compact Willmore surfaces in S^{2+p}.

Main theorem. *Let M be a compact Willmore surface in the unit sphere S^{2+p}. There exists an explicit positive constant*

$$C = \frac{(\sqrt{2}-1)\sqrt{\pi}}{12\sqrt{3}} B,$$

such that if

$$\|\rho^2\|_2 < C, \qquad \text{None}$$

then $\rho^2 = 0$ and M is a totally umbilical sphere, where B is defined in Theorem A.

Remark 1. By a simple calculation, we know that the pinching constant in Theorem B $\tilde{C}(H_0) \to 0$ as $H \to \infty$. But the pinching constant C in our main theorem is independent of mean curvature H. So C is superior to $\tilde{C}(H_0)$.

2. Basic lemmas

In this section, we introduce several useful lemmas

Lemma 2.1. *Let $x : M^2 \to S^{2+p}$ be a surface in a unit sphere. We have the following inequality*:

$$|\nabla\rho|^2 \le \sum_{\alpha,i,j,k} (\tilde{h}^{\alpha}_{ijk})^2. \tag{2.1}$$

Proof. We can see from $\rho^2 = \sum_{\alpha,i,j} (\tilde{h}^{\alpha}_{ij})^2$ and the Cauchy–Schwarz inequality that

$$|\nabla\rho|^2 \le \sum_{\alpha,i,j,k} (\tilde{h}^{\alpha}_{ijk})^2 \tag{2.2}$$

at all points where $\rho \neq 0$ and hence by analyticity at all the points. □

Lemma 2.2 [10]. *Let $x : M \to S^{2+p}$ be a surface. Then*

$$\Sigma R_{\alpha\beta 12}^2 \leq \frac{2-B}{B}\rho^4, \tag{2.3}$$

where equality holds if and only if $p = 1$ or $p \geq 2$, $\sum_\alpha (\tilde{h}_{11}^\alpha)^2 = \sum_\alpha (\tilde{h}_{12}^\alpha)^2$ and $\sum_\alpha \tilde{h}_{11}^\alpha \tilde{h}_{12}^\alpha = 0$. Here B is defined in Theorem A.

Lemma 2.3. *Let $x : M \to S^{2+p}$ be a surface.*

$$|\nabla \tilde{h}|^2 \geq |\nabla^\perp \mathbf{H}|^2, \tag{2.4}$$

where $|\nabla^\perp \mathbf{H}|^2 = \sum_{\alpha,i} (H_i^\alpha)^2$.

Proof. By a simple calculation, we have

$$|\nabla \tilde{h}|^2 = \sum_{\alpha,i,j,k} (\tilde{h}_{ijk}^\alpha)^2 = \sum_{\alpha,i,j,k} (h_{ijk}^\alpha)^2 - 2|\nabla^\perp \mathbf{H}|^2. \tag{2.5}$$

In [6], Li proved

$$\sum_{\alpha,i,j,k} (h_{ijk}^\alpha)^2 \geq 3|\nabla^\perp \mathbf{H}|^2. \tag{2.6}$$

Substituting (2.6) into (2.5), we obtain (2.4). □

Lemma 2.4 [10]. *Let $x : M^2 \to S^{2+p}$ be a compact Willmore surface in a unit sphere. Then*

$$\int_M |\nabla^\perp \mathbf{H}|^2 \mathrm{d}v \leq \int_M \rho^2 H^2 \mathrm{d}v. \tag{2.7}$$

Lemma 2.5. *Let $x : M \to S^{2+p}$ be a compact surface.*

$$\int_M \sum_{\alpha,i,j} \tilde{h}_{ij}^\alpha H_{ij}^\alpha \mathrm{d}v = -\int_M |\nabla^\perp \mathbf{H}|^2 \mathrm{d}v. \tag{2.8}$$

Proof. By Stoke formula, we have

$$\int_M \sum_{\alpha,i,j} \tilde{h}_{ij}^\alpha H_{ij}^\alpha \mathrm{d}v = -\int_M \sum_{\alpha,i,j} \tilde{h}_{ijj}^\alpha H_i^\alpha \mathrm{d}v. \tag{2.9}$$

$$\begin{aligned} \sum_{\alpha,i,j} \tilde{h}_{ijj}^\alpha H_i^\alpha &= \sum_{\alpha,i,j} (h_{ijj}^\alpha - \delta_{ij} H_j^\alpha) H_i^\alpha \\ &= \sum_{\alpha,i,j} h_{jji}^\alpha H_i^\alpha - \sum_{\alpha,i} H_i^\alpha H_i^\alpha \\ &= |\nabla^\perp \mathbf{H}|^2. \end{aligned} \tag{2.10}$$

We obtain (2.8) by putting (2.10) into (2.9). □

Lemma 2.6. *Let M be a compact 2-dimensional surface in S^{2+p}. Then for any $g \in C^1(M)$, $g \geq 0$, $t > 0$, g satisfies*

$$\int_M |\nabla g|^2 \mathrm{d}v \geq \frac{t}{A}\left(\int_M g^4 \mathrm{d}v\right)^{\frac{1}{2}} - t^2 \int_M g^2 \mathrm{d}v - t\int_M \left(1+\frac{H^2}{2}\right) g^2 \mathrm{d}v, \tag{2.11}$$

where $A = \frac{12\sqrt{3}}{\sqrt{\pi}}$.

Proof. From [5] and [8], we have

$$\left(\int_M g^2 \mathrm{d}v\right)^{\frac{1}{2}} \leq A\int_M (|\nabla g| + \sqrt{1+H^2}g)\mathrm{d}v.$$

Replacing g by g^2, we get

$$\begin{aligned}
\left(\int_M g^4 \mathrm{d}v\right)^{\frac{1}{2}} &\leq A\int_M (|\nabla g^2| + \sqrt{1+H^2}g^2)\mathrm{d}v \\
&= A\int_M (g|\nabla g| + \sqrt{1+H^2}g^2)\mathrm{d}v \\
&\leq A\left(\int_M f^2 \mathrm{d}v\right)^{\frac{1}{2}}\left(\int_M |\nabla g|^2 \mathrm{d}v\right)^{\frac{1}{2}} + A\int_M \left(1+\frac{H^2}{2}\right) g^2 \mathrm{d}v \\
&\leq At\int_M g^2 \mathrm{d}v + \frac{A}{t}\int_M |\nabla g|^2 \mathrm{d}v + A\int_M \left(1+\frac{H^2}{2}\right) g^2 \mathrm{d}v,
\end{aligned}$$

where $t \in \mathbf{R}^+$. So, we have

$$\int_M |\nabla g|^2 \mathrm{d}v \geq \frac{t}{A}\left(\int_M g^4 \mathrm{d}v\right)^{\frac{1}{2}} - t^2 \int_M g^2 \mathrm{d}v - t\int_M \left(1+\frac{H^2}{2}\right) g^2 \mathrm{d}v,$$

i.e.,

$$\|\nabla g\|_2^2 \geq \frac{t}{A}\|g^2\|_2 - (t^2+t)\|g^2\|_1 - \frac{t}{2}\|H^2 g^2\|_1.$$

This proves Lemma 2.6. □

3. Proof of the main theorem

In this section, we give the proof of our main theorem. From Lemma 2.1 in [10] and (2.3), we have

$$\begin{aligned}
\frac{1}{2}\Delta\rho^2 &= \sum_{\alpha,i,j,k} (\tilde{h}^\alpha_{ijk})^2 + 2\sum_{\alpha,i,j} \tilde{h}^\alpha_{ij} H^\alpha_{ij} + \rho^2(2-\rho^2+2H^2) - \sum_{\alpha,\beta} R^2_{\alpha\beta 12} \\
&\geq \sum_{\alpha,i,j,k} (\tilde{h}^\alpha_{ijk})^2 + 2\sum_{\alpha,i,j} \tilde{h}^\alpha_{ij} H^\alpha_{ij} + \rho^2(2-\rho^2+2H^2) - \frac{2-B}{B}\rho^4.
\end{aligned}$$

Integrating the above inequality and using Lemma 2.5, we get

$$\begin{aligned}
0 &\geq \int_M \sum_{\alpha,i,j,k} (\tilde{h}^\alpha_{ijk})^2 \mathrm{d}v + 2\int_M \sum_{\alpha,i,j} \tilde{h}^\alpha_{ij} H^\alpha_{ij} \mathrm{d}v + \int_M \rho^2 \left[2(1+H^2) - \frac{2}{B}\rho^2\right] \mathrm{d}v \\
&= \int_M |\nabla \tilde{h}|^2 \mathrm{d}v - 2\int_M |\nabla^\perp \mathbf{H}|^2 \mathrm{d}v + \int_M \rho^2 \left[2(1+H^2) - \frac{2}{B}\rho^2\right] \mathrm{d}v \\
&= \eta \int_M |\nabla \tilde{h}|^2 \mathrm{d}v + (1-\eta) \int_M |\nabla \tilde{h}|^2 \mathrm{d}v - 2\int_M |\nabla^\perp \mathbf{H}|^2 \mathrm{d}v \\
&\quad + \int_M \rho^2 \left[2(1+H^2) - \frac{2}{B}\rho^2\right] \mathrm{d}v,
\end{aligned} \tag{3.1}$$

where $0 < \eta < 1$. From (2.1), (2.7) and (3.1), we have

$$\begin{aligned}
0 &\geq \eta \int_M |\nabla f_\varepsilon|^2 \mathrm{d}v + (1-\eta) \int_M |\nabla^\perp \mathbf{H}|^2 \mathrm{d}v - 2\int_M |\nabla^\perp \mathbf{H}|^2 \mathrm{d}v \\
&\quad + \int_M \rho^2 \left[2(1+H^2) - \frac{2}{B}\rho^2\right] \mathrm{d}v \\
&= \eta \int_M |\nabla f_\varepsilon|^2 \mathrm{d}v - (1+\eta) \int_M |\nabla^\perp \mathbf{H}|^2 \mathrm{d}v + \int_M \rho^2 \left[2(1+H^2) - \frac{2}{B}\rho^2\right] \mathrm{d}v \\
&\geq \eta \int_M |\nabla f_\varepsilon|^2 \mathrm{d}v - (1+\eta) \int_M \rho^2 H^2 \mathrm{d}v + \int_M \rho^2 \left[2(1+H^2) - \frac{2}{B}\rho^2\right] \mathrm{d}v \\
&= \eta \int_M |\nabla f_\varepsilon|^2 \mathrm{d}v + (1-\eta) \int_M \rho^2 H^2 \mathrm{d}v + 2\int_M \rho^2 \mathrm{d}v - \int_M \frac{2}{B}\rho^4 \mathrm{d}v.
\end{aligned} \tag{3.2}$$

Substituting (2.11) into (3.2), we get

$$\begin{aligned}
0 &\geq \eta \frac{t}{A} \left(\int_M f_\varepsilon^4\right)^{\frac{1}{2}} \mathrm{d}v - \eta t^2 \int_M f_\varepsilon^2 \mathrm{d}v - \eta t \int_M \left(1 + \frac{H^2}{2}\right) f_\varepsilon^2 \mathrm{d}v \\
&\quad + (1-\eta) \int_M \rho^2 H^2 \mathrm{d}v + 2\int_M \rho^2 \mathrm{d}v - \frac{2}{B} \int_M \rho^4 \mathrm{d}v.
\end{aligned}$$

As $\varepsilon \to 0$, this implies

$$\begin{aligned}
0 &\geq \eta \frac{t}{A} \left(\int_M \rho^4 \mathrm{d}v\right)^{\frac{1}{2}} - (\eta t^2 + \eta t - 2) \int_M \rho^2 \mathrm{d}v \\
&\quad + \left(1 - \eta - \frac{\eta t}{2}\right) \int_M \rho^2 H^2 \mathrm{d}v - \frac{2}{B} \int_M \rho^4 \mathrm{d}v.
\end{aligned}$$

Choose $t = \dfrac{2(1-\eta)}{\eta}$, then $1 - \eta - \dfrac{\eta t}{2} = 0$. So we have

$$\begin{aligned}
0 &\geq \frac{2(1-\eta)}{A} \left(\int_M \rho^4 \mathrm{d}v\right)^{\frac{1}{2}} - \left[\frac{4(1-\eta)^2}{\eta} + 2(1-\eta) - 2\right] \int_M \rho^2 \mathrm{d}v \\
&\quad - \frac{2}{B} \int_M \rho^4 \mathrm{d}v,
\end{aligned}$$

i.e.,

$$\begin{aligned} 0 \;\geq\; & \frac{(1-\eta)}{A}\left(\int_M \rho^4 \mathrm{d}v\right)^{\frac{1}{2}} - \left[\frac{2(1-\eta)^2}{\eta} - \eta\right]\int_M \rho^2 \mathrm{d}v - \frac{1}{B}\int_M \rho^4 \mathrm{d}v \\ = \; & \frac{1-\eta}{A}\|\rho^2\|_2 - \left[\frac{2(1-\eta)^2}{\eta} - \eta\right]\|\rho^2\| - \frac{1}{B}\|\rho^2\|_2^2 \\ = \; & \left[\frac{1-\eta}{A} - \frac{1}{B}\|\rho^2\|_2\right]\|\rho^2\|_2 - \left[\frac{2(1-\eta)^2}{\eta} - \eta\right]\|\rho^2\|. \end{aligned} \tag{3.3}$$

We take $\eta = 2 - \sqrt{2}$. This together with (3.3) yields

$$0 \geq \left\{\frac{\sqrt{2}-1}{A} - \frac{1}{B}\|\rho^2\|_2\right\}\|\rho^2\|_2,$$

which implies $\|\rho^2\|_2 = 0$ for

$$\|\rho^2\|_2 < C = \frac{(\sqrt{2}-1)B}{A} = \frac{(\sqrt{2}-1)\sqrt{\pi}}{12\sqrt{3}}B,$$

i.e., $S = 2H^2$ and M is a totally umbilical Willmore surface. This completes the proof of the main theorem.

As we all know, minimal surfaces must be Willmore surfaces, so we obtain the following corollary.

COROLLARY

Let M be a compact minimal surface in the unit sphere S^{2+p}. There exists a positive constant

$$C = \frac{(\sqrt{2}-1)\sqrt{\pi}}{12\sqrt{3}}B,$$

such that if

$$\|S\|_2 < C,$$

then $S = 0$ and M is a totally geodesic, *where B is defined in Theorem A.*

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References

[1] Bryant R, A duality theorem for Willmore surfaces, *J. Differential Geom.* **20** (1984) 23–53

[2] Castro I and Urbano F, Willmore surfaces of R^4 and the Whitney sphere, *Ann. Global Anal. Geom.* **19** (2001) 153–175

[3] Chen B Y, Some conformal invariants of submanifolds and their application, *Boll. Un. Mat. Ital.* **10** (1974) 380–385

[4] Chen B Y, Geometry of submanifolds (1973) (New York: Marcel Dekker)

[5] Hoffman D and Spruck J, Sobolev and isoperimetric inequalities for Riemannian submanifold, *Comm. Pure Appl. Math.* **27** (1974) 715–727

[6] Li H Z, Willmore surfaces in S^n, *Ann. Global Anal. Geom.* **21** (2002) 203–213

[7] Weiner J, On a problem of Chen, Willmore *et al.*, *Indiana Univ. Math. J.* **27** (1978) 19–35

[8] Xu H W, $L^{\frac{n}{2}}$-pinching theorems for submanifolds with parallel mean curvature in a sphere, *J. Math. Soc. Japan* **46** (1994) 503–515

[9] Xu H W and Yang D Y, A global pinching theorem for Willmore submanifolds in a sphere, Preprint (2008)

[10] Yu C C and Hsu Y J, Willmore surfaces in the unit *n*-sphere, *Taiwanese J. Math.* **8** (2004) 467–476

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