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Existence of positive solutions for systems of second order multi-point boundary value problems on time scales

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Abstract. In this paper, we establish the existence of positive solutions for systems of second order multi-point boundary value problems on time scales by applying Guo–Krasnosel'skii fixed point theorem.

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1. Introduction

The theory of time scales was introduced by Hilger [14] to unify not only continuous and discrete theory, but also to provide an accurate information of phenomena that manifest themselves partly in continuous time and partly in discrete time. The new methods developed in time scale calculus [1, 8, 9] are significant in the theoretical study of differential equations and difference equations. This theory is applicable to various real life situations like epidemic models, stock markets and mathematical modeling of physical and biological systems.

Multi-point boundary value problems (BVPs) for ordinary differential or difference equations arise in different areas of applied mathematics and physics such as the deflection of a curved beam having a constant or varying cross section, three layer beam, electromagnetic waves or gravity-driven flow and so on. For example, the vibrations of a guy wire of a uniform cross-section and composed of N parts of different densities can be set up as a multi-point BVP [18] and also many problems in the theory of elastic stability can be handled as multi-point problems [20]. The study of multi-point BVPs for second order differential equations was introduced by Il'in and Moiseev [15, 16]. Since then, such multi-point BVPs (continuous or discrete cases) have been studied by many authors using different methods such as fixed point theorems in cones.

There has been a lot of interest in establishing the existence of positive solutions of the boundary value problems on time scales, often using Guo–Krasnosel'skii fixed point theorem. To mention a few papers along these lines are [2–7, 10, 11, 13] and [19].

Till now, in the literature, the authors established results for the existence of positive solutions for the system of dynamic equations on time scales satisfying same type of boundary conditions. We wish to extend these results to system of dynamic equations on time scales satisfying general boundary conditions.

In this paper, we consider the system of nonlinear second order dynamic equations on time scales

$$\left.\begin{aligned} u^{\Delta\Delta}(t) + \lambda p(t) f(u(t), v(t)) = 0, \quad t \in [a, \sigma(b)]_{\mathbb{T}}, \\ v^{\Delta\Delta}(t) + \mu q(t) g(u(t), v(t)) = 0, \quad t \in [a, \sigma(b)]_{\mathbb{T}}, \end{aligned}\right\} \tag{1.1}$$

satisfying the multi-point boundary conditions,

$$\left.\begin{aligned} u(a) &= 0, \quad \alpha_1 u(\sigma(b)) + \beta_1 u^{\Delta}(\sigma(b)) = \sum_{k=2}^{m-1} u^{\Delta}(\xi_k),\ m \geq 3, \\ v(a) &= 0, \quad \alpha_2 v(\sigma(b)) + \beta_2 v^{\Delta}(\sigma(b)) = \sum_{k=2}^{n-1} v^{\Delta}(\eta_k),\ n \geq 3, \end{aligned}\right\} \tag{1.2}$$

where $\mathbb{T}$ is the time scale with $a, \sigma^2(b) \in \mathbb{T}$, $0 \leq a < \xi_2 < \cdots < \xi_{m-1} < \sigma(b)$, $0 \leq a < \eta_2 < \cdots < \eta_{n-1} < \sigma(b)$. We shall give sufficient conditions on λ, μ, f and g such that the BVP (1.1)–(1.2) has positive solutions. By a positive solution of the BVP (1.1)–(1.2), we mean a pair $(u, v) \in C^2([a, \sigma(b)]_{\mathbb{T}}) \times C^2([a, \sigma(b)]_{\mathbb{T}})$ satisfying (1.1) and (1.2) with $u(t) \geq 0$, $v(t) \geq 0$, for all $t \in [a, \sigma(b)]_{\mathbb{T}}$ and $(u, v) \neq (0, 0)$.

We assume the following conditions hold throughout the paper:

(A1) the functions $f, g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ are continuous,

(A2) the functions $p, q : [a, \sigma(b)]_{\mathbb{T}} \to \mathbb{R}^+$ are continuous and p, q do not vanish identically on any closed subinterval of $[a, \sigma(b)]_{\mathbb{T}}$,

(A3) $\alpha_1, \alpha_2, \beta_1$ and β_2 are positive constants such that $\alpha_1 \geq \dfrac{\beta_1}{\xi_2 - a}$, $\alpha_2 \geq \dfrac{\beta_2}{\eta_2 - a}$, $\beta_1 > m - 2$ and $\beta_2 > n - 2$,

(A4) each of these

$$\begin{aligned} f_0^s &= \lim_{(u,v)\to(0^+,0^+)} \sup \frac{f(u,v)}{u+v}, \quad g_0^s = \lim_{(u,v)\to(0^+,0^+)} \sup \frac{g(u,v)}{u+v}, \\ f_0^i &= \lim_{(u,v)\to(0^+,0^+)} \inf \frac{f(u,v)}{u+v}, \quad g_0^i = \lim_{(u,v)\to(0^+,0^+)} \inf \frac{g(u,v)}{u+v}, \\ f_\infty^s &= \lim_{(u,v)\to(\infty,\infty)} \sup \frac{f(u,v)}{u+v}, \quad g_\infty^s = \lim_{(u,v)\to(\infty,\infty)} \sup \frac{g(u,v)}{u+v}, \\ f_\infty^i &= \lim_{(u,v)\to(\infty,\infty)} \inf \frac{f(u,v)}{u+v}, \quad g_\infty^i = \lim_{(u,v)\to(\infty,\infty)} \inf \frac{g(u,v)}{u+v}, \end{aligned}$$

exist as positive real numbers.

The rest of the paper is organized as follows. In §2, we construct the Green's functions for the homogeneous problems corresponding to (1.1)–(1.2) and estimate bounds for the Green's functions. In §3, we establish the existence of positive solutions of the BVP (1.1)–

(1.2) by using Guo–Krasnosel'skii fixed point theorem for operators on a cone in a Banach space. Finally as an application, we give an example to illustrate our result.

2. Green's function and bounds

In this section, we construct the Green's functions for the homogeneous problems corresponding to (1.1)–(1.2) and estimate bounds for the Green's functions.

Let $G(t, s)$ be the Green's function for the homogeneous BVP,

$$-u^{\Delta\Delta}(t) = 0, \quad t \in [a, \sigma(b)]_{\mathbb{T}}, \tag{2.1}$$

$$u(a) = 0, \quad \alpha_1 u(\sigma(b)) + \beta_1 u^{\Delta}(\sigma(b)) = \sum_{k=2}^{m-1} u^{\Delta}(\xi_k), \quad m \geq 3. \tag{2.2}$$

Lemma 2.1. *Let* $d_1 = \alpha_1(\sigma(b) - a) + \beta_1 - m + 2 \neq 0$. *Then the Green's function* $G(t, s)$ *for the homogeneous BVP* (2.1)–(2.2) *is given by*

$$G(t,s) = \begin{cases} G_1(t,s), & a \leq s \leq \sigma(s) \leq \xi_2, \\ G_2(t,s), & \xi_2 \leq s \leq \sigma(s) \leq \xi_3, \\ \cdot & \\ \cdot & \\ \cdot & \\ G_{m-2}(t,s), & \xi_{m-2} \leq s \leq \sigma(s) \leq \xi_{m-1}, \\ G_{m-1}(t,s), & \xi_{m-1} \leq s \leq \sigma(s) \leq \sigma(b), \end{cases} \tag{2.3}$$

where

$$G_j(t,s) = \begin{cases} \dfrac{1}{d_1}[(\alpha_1(\sigma(b)-t)+\beta_1-m+j+1)(\sigma(s)-a)+(j-1)(t-\sigma(s))], \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \sigma(s) \leq t, \\ \dfrac{1}{d_1}(t-a)[\alpha_1(\sigma(b)-\sigma(s)) + \beta_1 - m + j + 1], \quad t \leq s, \end{cases}$$

for all $j = 1, 2, \ldots, m-1$.

Proof. It is easy to see that, if $h(t) \in C\left([a, \sigma(b)]_{\mathbb{T}}, \mathbb{R}^+\right)$, then the following problem

$$-u^{\Delta\Delta}(t) = h(t), \quad t \in [a, \sigma(b)]_{\mathbb{T}},$$

satisfying the boundary conditions (2.2) has a unique solution

$$u(t) = \frac{1}{d_1}(t-a)\left[\int_a^{\sigma(b)} (\alpha_1(\sigma(b)-\sigma(s))+\beta_1)h(s)\Delta s - \sum_{k=2}^{m-1}\int_a^{\xi_k} h(s)\Delta s\right] - \int_a^t (t-\sigma(s))h(s)\Delta s.$$

Rearranging the terms, it can be written as

$$u(t) = \frac{1}{d_1}(t-a)\left[\int_a^{\sigma(b)}(\alpha_1(\sigma(b)-\sigma(s))+\beta_1)h(s)\Delta s-(m-2)\int_a^{\xi_2}h(s)\Delta s\right.$$

$$\left.-\sum_{j=2}^{m-2}(m-j-1)\int_{\xi_j}^{\xi_{j+1}}h(s)\Delta s\right]+\int_a^t(\sigma(s)-t)h(s)\Delta s.$$

Case 1. Let $a \le s \le \sigma(s) \le \xi_2$ and $\sigma(s) \le t$. Then we have

$$\begin{aligned} G(t,s) &= \frac{1}{d_1}(t-a)[\alpha_1(\sigma(b)-\sigma(s))+\beta_1-(m-2)]+\sigma(s)-t \\ &= \frac{1}{d_1}(\alpha_1(\sigma(b)-t)+\beta_1-m+2)(\sigma(s)-a). \end{aligned}$$

Case 2. Let $a \le s \le \sigma(s) \le \xi_2$ and $t \le s$. Then we have

$$G(t,s) = \frac{1}{d_1}(t-a)[\alpha_1(\sigma(b)-\sigma(s))+\beta_1-m+2].$$

Case 3. Let $\xi_j \le s \le \sigma(s) \le \xi_{j+1}$, for $j = 2, 3, \ldots, m-2$ and $\sigma(s) \le t$. Then we have

$$\begin{aligned} G(t,s) &= \frac{1}{d_1}(t-a)[\alpha_1(\sigma(b)-\sigma(s))+\beta_1-(m-j-1)]+\sigma(s)-t \\ &= \frac{1}{d_1}[(\alpha_1(\sigma(b)-t)+\beta_1-m+j+1)(\sigma(s)-a)+(j-1)(t-\sigma(s))]. \end{aligned}$$

Case 4. Let $\xi_j \le s \le \sigma(s) \le \xi_{j+1}$, for $j = 2, 3, \ldots, m-2$ and $t \le s$. Then we have

$$G(t,s) = \frac{1}{d_1}(t-a)[\alpha_1(\sigma(b)-\sigma(s))+\beta_1-m+j+1].$$

Case 5. Let $\xi_{m-1} \le s \le \sigma(s) \le \sigma(b)$ and $\sigma(s) \le t$. Then we have

$$\begin{aligned} G(t,s) &= \frac{1}{d_1}(t-a)[\alpha_1(\sigma(b)-\sigma(s))+\beta_1]+\sigma(s)-t \\ &= \frac{1}{d_1}[(\alpha_1(\sigma(b)-t)+\beta_1)(\sigma(s)-a)+(m-2)(t-\sigma(s))]. \end{aligned}$$

Case 6. Let $\xi_{m-1} \le s \le \sigma(s) \le \sigma(b)$ and $t \le s$. Then we have

$$G(t,s) = \frac{1}{d_1}(t-a)[\alpha_1(\sigma(b)-\sigma(s))+\beta_1]. \qquad \square$$

Lemma 2.2. *Assume that the condition* (*A*3) *is satisfied. Then the Green's function* $G(t,s)$ *of* (2.1)–(2.2) *is positive, for all* $(t,s) \in (a,\sigma(b))_{\mathbb{T}} \times (a,b)_{\mathbb{T}}$.

Proof. By simple algebraic calculations, we can easily establish the positivity of the Green's function. □

Lemma 2.3. *Assume that the condition* $(A3)$ *is satisfied. Then the Green's function* $G(t,s)$ *in* (2.3) *satisfies the following inequality*:

$$g(t)G(\sigma(s),s) \le G(t,s) \le G(\sigma(s),s), \ for\ all\ (t,s) \in [a,\sigma(b)]_{\mathbb{T}} \times [a,b]_{\mathbb{T}}, \tag{2.4}$$

where

$$g(t) = \min\left\{\frac{\sigma(b)-t}{\sigma(b)-a}, \frac{t-a}{\sigma(b)-a}\right\}. \tag{2.5}$$

Proof. The Green's function $G(t,s)$ is given in (2.3). In each case, we prove the inequality as in (2.4).

Case 1. Let $s \in [a,b]_{\mathbb{T}}$ and $\sigma(s) \le t$. Then

$$\begin{aligned}\frac{G(t,s)}{G(\sigma(s),s)} &= \frac{(\alpha_1(\sigma(b)-t)+\beta_1-m+j+1)(\sigma(s)-a)+(j-1)(t-\sigma(s))}{(\alpha_1(\sigma(b)-\sigma(s))+\beta_1-m+j+1)(\sigma(s)-a)}\\ &\le \frac{\alpha_1(\sigma(b)-t)+\beta_1-m+j+1+\alpha_1(t-\sigma(s))}{\alpha_1(\sigma(b)-\sigma(s))+\beta_1-m+j+1} = 1\end{aligned}$$

and also

$$\begin{aligned}\frac{G(t,s)}{G(\sigma(s),s)} &= \frac{(\alpha_1(\sigma(b)-t)+\beta_1-m+j+1)(\sigma(s)-a)+(j-1)(t-\sigma(s))}{(\alpha_1(\sigma(b)-\sigma(s))+\beta_1-m+j+1)(\sigma(s)-a)}\\ &\ge \frac{\sigma(b)-t}{\sigma(b)-a}.\end{aligned}$$

Case 2. Let $s \in [a,b]_{\mathbb{T}}$ and $t \le s$. Then

$$\frac{G(t,s)}{G(\sigma(s),s)} = \frac{t-a}{\sigma(s)-a} \le 1$$

and also

$$\frac{G(t,s)}{G(\sigma(s),s)} = \frac{t-a}{\sigma(s)-a} \ge \frac{t-a}{\sigma(b)-a}.$$

Hence the result. □

Lemma 2.4. *Assume that the condition* $(A3)$ *is satisfied and* $s \in [a,b]_{\mathbb{T}}$. *Then the Green's function* $G(t,s)$ *in* (2.3) *satisfies*

$$\min_{t\in[\xi_{m-1},\sigma(b)]_{\mathbb{T}}} G(t,s) \ge k_1 G(\sigma(s),s),$$

where

$$k_1 = \frac{\beta_1 - m + 2}{\alpha_1(\sigma(b) - a) + \beta_1 - m + 2} < 1. \tag{2.6}$$

Proof. By Lemma 2.3, we can easily establish the result. □

We can also formulate the same results as Lemmas 2.1–2.4 above for the following BVP,

$$-v^{\Delta\Delta}(t) = 0, \quad t \in [a, \sigma(b)]_{\mathbb{T}}, \tag{2.7}$$

$$v(a) = 0, \quad \alpha_2 v(\sigma(b)) + \beta_2 v^{\Delta}(\sigma(b)) = \sum_{k=2}^{n-1} v^{\Delta}(\eta_k),\ n \geq 3, \tag{2.8}$$

where $0 \leq a < \eta_2 < \cdots < \eta_{n-1} < \sigma(b)$.

If $d_2 = \alpha_2(\sigma(b) - a) + \beta_2 - n + 2 \neq 0$, we denote by $H(t, s)$, the Green's function for the homogeneous BVP (2.7)–(2.8) and define in a similar manner as $G(t, s)$.

Under similar assumptions as those from Lemmas 2.2–2.4, we have

(B1) the Green's function $H(t, s)$ is positive, for all $(t, s) \in (a, \sigma(b))_{\mathbb{T}} \times (a, b)_{\mathbb{T}}$,
(B2) $g(t)H(\sigma(s), s) \leq H(t, s) \leq H(\sigma(s), s)$, for all $(t, s) \in [a, \sigma(b)]_{\mathbb{T}} \times [a, b]_{\mathbb{T}}$, where $g(t)$ is given in (2.5),
(B3)

$$\min_{t \in [\eta_{n-1}, \sigma(b)]_{\mathbb{T}}} H(t, s) \geq k_2 H(\sigma(s), s), \quad s \in [a, b]_{\mathbb{T}},$$

where

$$k_2 = \frac{\beta_2 - n + 2}{\alpha_2(\sigma(b) - a) + \beta_2 - n + 2} < 1. \tag{2.9}$$

To establish criteria for the existence of positive solutions for the BVP (1.1)–(1.2), we will employ the following Guo–Krasnosel'skii fixed point theorem [12, 17].

Theorem 2.5. *Let X be a Banach space, $\kappa \subseteq X$ be a cone and suppose that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$. Suppose further that $T : \kappa \cap (\bar{\Omega}_2 \backslash \Omega_1) \to \kappa$ is a completely continuous operator such that either*

(i) $\|Tu\| \leq \|u\|,\ u \in \kappa \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|,\ u \in \kappa \cap \partial\Omega_2$, *or*
(ii) $\|Tu\| \geq \|u\|,\ u \in \kappa \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|,\ u \in \kappa \cap \partial\Omega_2$ *holds.*

Then T has a fixed point in $\kappa \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

3. Existence of positive solutions

In this section, we shall give sufficient conditions on λ, μ, f and g such that the BVP (1.1)–(1.2) has positive solutions in a cone.

We consider the Banach space $X = \{x \mid x \in C[a, \sigma(b)]_{\mathbb{T}}\}$ with the supremum norm $\|\cdot\|$, and the Banach space $Y = X \times X$ with the norm $\|(u, v)\|_Y = \|u\| + \|v\|$, where

$$\|u\| = \sup_{t \in [a, \sigma(b)]_{\mathbb{T}}} |u(t)|.$$

Define a cone $P \subset Y$ by

$$P = \left\{ \begin{array}{c} (u, v) \in Y \mid u(t) \geq 0, v(t) \geq 0 \text{ on } [a, \sigma(b)]_{\mathbb{T}} \text{ and} \\ \min_{t \in [r, \sigma(b)]_{\mathbb{T}}} (u(t) + v(t)) \geq k\|(u, v)\|_Y \end{array} \right\},$$

where $r = \max\{\xi_{m-1}, \eta_{n-1}\}$, $k = \min\{k_1, k_2\}$ and k_1, k_2 are defined in (2.6) and (2.9) respectively.

We shall present some existence results for the positive solutions of the BVP (1.1)–(1.2) under various assumptions on f_0^s, g_0^s, f_0^i, g_0^i, f_∞^s, g_∞^s, f_∞^i and g_∞^i.

Now, we define the positive numbers L_1, L_2, L_3 and L_4 by

$$L_1 = \alpha \left[kk_1 f_\infty^i \int_r^{\sigma(b)} G(\sigma(s), s) p(s) \Delta s \right]^{-1},$$

$$L_2 = \alpha \left[f_0^s \int_a^{\sigma(b)} G(\sigma(s), s) p(s) \Delta s \right]^{-1},$$

$$L_3 = \beta \left[kk_2 g_\infty^i \int_r^{\sigma(b)} H(\sigma(s), s) q(s) \Delta s \right]^{-1},$$

$$L_4 = \beta \left[g_0^s \int_a^{\sigma(b)} H(\sigma(s), s) q(s) \Delta s \right]^{-1},$$

where $\alpha > 0$ and $\beta > 0$ are two positive real numbers such that $\alpha + \beta = 1$.

Theorem 3.1. *Assume that the conditions* $(A1)$–$(A4)$ *hold.*

- (i) *If* $f_0^s, g_0^s, f_\infty^i, g_\infty^i \in (0, \infty)$, $L_1 < L_2$ *and* $L_3 < L_4$, *then for each* $\lambda \in (L_1, L_2)$ *and* $\mu \in (L_3, L_4)$ *there exists a positive solution* $(u(t), v(t))$ *on* $[a, \sigma(b)]_{\mathbb{T}}$ *for* (1.1)–(1.2).
- (ii) *If* $f_0^s = g_0^s = 0$, $f_\infty^i, g_\infty^i \in (0, \infty)$, *then for each* $\lambda \in (L_1, \infty)$ *and* $\mu \in (L_3, \infty)$ *there exists a positive solution* $(u(t), v(t))$ *on* $[a, \sigma(b)]_{\mathbb{T}}$ *for* (1.1)–(1.2).
- (iii) *If* $f_0^s, g_0^s \in (0, \infty)$, $f_\infty^i = g_\infty^i = \infty$, *then for each* $\lambda \in (0, L_2)$ *and* $\mu \in (0, L_4)$ *there exists a positive solution* $(u(t), v(t))$ *on* $[a, \sigma(b)]_{\mathbb{T}}$ *for* (1.1)–(1.2).
- (iv) *If* $f_0^s = g_0^s = 0$, $f_\infty^i = g_\infty^i = \infty$, *then for each* $\lambda \in (0, \infty)$ *and* $\mu \in (0, \infty)$ *there exists a positive solution* $(u(t), v(t))$ *on* $[a, \sigma(b)]_{\mathbb{T}}$ *for* (1.1)–(1.2).

Proof.

(i) Let $T_1, T_2 : P \to X$ and $T : P \to Y$ be the operators defined by

$$T_1(u, v)(t) = \lambda \int_a^{\sigma(b)} G(t, s) p(s) f(u(s), v(s)) \Delta s,$$

$$T_2(u,v)(t) = \mu \int_a^{\sigma(b)} H(t,s)q(s)g(u(s),v(s))\Delta s$$

and

$$T(u,v)(t) = (T_1(u,v)(t), T_2(u,v)(t)), \qquad \text{for } (u,v) \in Y,$$

where $G(t,s)$ and $H(t,s)$ are the Green's functions for the homogeneous BVPs (2.1)–(2.2) and (2.7)–(2.8) respectively. It is obvious that a fixed point of T is a solution of the BVP (1.1)–(1.2). We now show that $T : P \to P$. Let $(u,v) \in P$. From Lemma 2.2 and (B1), $T_1(u,v)(t) \geq 0$ and $T_2(u,v)(t) \geq 0$ on $[a,\sigma(b)]_{\mathbb{T}}$. Also, for $(u,v) \in P$, by Lemma 2.3, we have

$$\begin{aligned} T_1(u,v)(t) &= \lambda \int_a^{\sigma(b)} G(t,s)p(s)f(u(s),v(s))\Delta s \\ &\leq \lambda \int_a^{\sigma(b)} G(\sigma(s),s)p(s)f(u(s),v(s))\Delta s \end{aligned}$$

so that

$$\|T_1(u,v)\| \leq \lambda \int_a^{\sigma(b)} G(\sigma(s),s)p(s)f(u(s),v(s))\Delta s.$$

Next, if $(u,v) \in P$, then by Lemma 2.4, we have

$$\begin{aligned} \min_{t\in[r,\sigma(b)]_{\mathbb{T}}} T_1(u,v)(t) &\geq \min_{t\in[\xi_{m-1},\sigma(b)]_{\mathbb{T}}} T_1(u,v)(t) \\ &= \min_{t\in[\xi_{m-1},\sigma(b)]_{\mathbb{T}}} \lambda \int_a^{\sigma(b)} G(t,s)p(s)f(u(s),v(s))\Delta s \\ &\geq \lambda k_1 \int_a^{\sigma(b)} G(\sigma(s),s)p(s)f(u(s),v(s))\Delta s \\ &\geq k_1\|T_1(u,v)\|. \end{aligned}$$

In a similar manner, we conclude that

$$\min_{t\in[r,\sigma(b)]_{\mathbb{T}}} T_2(u,v)(t) \geq k_2\|T_2(u,v)\|.$$

Therefore,

$$\begin{aligned} \min_{t\in[r,\sigma(b)]_{\mathbb{T}}} (T_1(u,v)(t) + T_2(u,v)(t)) &\geq \min_{t\in[r,\sigma(b)]_{\mathbb{T}}} T_1(u,v)(t) \\ &\quad + \min_{t\in[r,\sigma(b)]_{\mathbb{T}}} T_2(u,v)(t) \\ &\geq k_1\|T_1(u,v)\| + k_2\|T_2(u,v)\| \\ &\geq k\|(T_1(u,v), T_2(u,v))\|_Y \\ &= k\|T(u,v)\|_Y. \end{aligned}$$

Hence $T(u,v) \in P$ and so $T : P \to P$. By standard arguments, we can easily show that T_1 and T_2 are completely continuous and so, T is a completely continuous operator.

Now, let $\lambda \in (L_1, L_2)$, $\mu \in (L_3, L_4)$ and let $\epsilon > 0$ be a positive number such that $\epsilon < f_\infty^i$, $\epsilon < g_\infty^i$ and

$$\alpha \left[kk_1(f_\infty^i - \epsilon) \int_r^{\sigma(b)} G(\sigma(s), s)p(s)\Delta s \right]^{-1} \leq \lambda,$$

$$\alpha \left[(f_0^s + \epsilon) \int_a^{\sigma(b)} G(\sigma(s), s)p(s)\Delta s \right]^{-1} \geq \lambda,$$

$$\beta \left[kk_2(g_\infty^i - \epsilon) \int_r^{\sigma(b)} H(\sigma(s), s)q(s)\Delta s \right]^{-1} \leq \mu,$$

$$\beta \left[(g_0^s + \epsilon) \int_a^{\sigma(b)} H(\sigma(s), s)q(s)\Delta s \right]^{-1} \geq \mu.$$

By the definitions of f_0^s and g_0^s, there exists $J_1 > 0$ such that

$$f(u, v) \leq (f_0^s + \epsilon)(u + v) \quad \text{and} \quad g(u, v) \leq (g_0^s + \epsilon)(u + v), \ 0 < u + v \leq J_1.$$

By (A1), the above inequalities are also valid for $u = v = 0$.

Let $(u, v) \in P$ with $\|(u, v)\|_Y = J_1$. i.e., $\|u\| + \|v\| = J_1$. Then, from Lemma 2.3, for $a \leq t \leq \sigma(b)$, we have

$$\begin{aligned} T_1(u, v)(t) &= \lambda \int_a^{\sigma(b)} G(t, s)p(s)f(u(s), v(s))\Delta s \\ &\leq \lambda \int_a^{\sigma(b)} G(\sigma(s), s)p(s)f(u(s), v(s))\Delta s \\ &\leq \lambda \int_a^{\sigma(b)} G(\sigma(s), s)p(s)(f_0^s + \epsilon)(u(s) + v(s))\Delta s \\ &\leq \lambda(f_0^s + \epsilon) \int_a^{\sigma(b)} G(\sigma(s), s)p(s)(\|u\| + \|v\|)\Delta s \\ &\leq \alpha(\|u\| + \|v\|) \\ &= \alpha\|(u, v)\|_Y. \end{aligned}$$

Hence,

$$\|T_1(u, v)\| \leq \alpha\|(u, v)\|_Y.$$

In a similar manner, we conclude that

$$\|T_2(u, v)\| \leq \beta\|(u, v)\|_Y.$$

Therefore,

$$\begin{aligned} \|T(u, v)\|_Y &= \|(T_1(u, v), T_2(u, v))\|_Y \\ &= \|T_1(u, v)\| + \|T_2(u, v)\| \\ &\leq \alpha\|(u, v)\|_Y + \beta\|(u, v)\|_Y \\ &= (\alpha + \beta)\|(u, v)\|_Y \\ &= \|(u, v)\|_Y. \end{aligned}$$

Hence, $\|T(u,v)\|_Y \leq \|(u,v)\|_Y$. If we set

$$\Omega_1 = \{(u,v) \in Y \mid \|(u,v)\|_Y < J_1\},$$

then

$$\|T(u,v)\|_Y \leq \|(u,v)\|_Y, \qquad \text{for } (u,v) \in P \cap \partial\Omega_1. \tag{3.1}$$

By the definitions of f_∞^i and g_∞^i, there exists $\bar{J}_2 > 0$ such that

$$f(u,v) \geq (f_\infty^i - \epsilon)(u+v) \quad \text{and} \quad g(u,v) \geq (g_\infty^i - \epsilon)(u+v),\ u+v \geq \bar{J}_2.$$

Let

$$J_2 = \max\left\{2J_1, \frac{\bar{J}_2}{k}\right\}.$$

Choose $(u,v) \in P$ with $\|(u,v)\|_Y = J_2$. Then

$$\min_{t \in [r,\sigma(b)]_{\mathbb{T}}} (u(t) + v(t)) \geq k\|(u,v)\|_Y \geq \bar{J}_2.$$

From Lemma 2.4, we have

$$\begin{aligned}
T_1(u,v)(t) &= \lambda \int_a^{\sigma(b)} G(t,s)p(s)f(u(s),v(s))\Delta s \\
&\geq \lambda k_1 \int_a^{\sigma(b)} G(\sigma(s),s)p(s)f(u(s),v(s))\Delta s \\
&\geq \lambda k_1 \int_r^{\sigma(b)} G(\sigma(s),s)p(s)f(u(s),v(s))\Delta s \\
&\geq \lambda k_1 \int_r^{\sigma(b)} G(\sigma(s),s)p(s)(f_\infty^i - \epsilon)(u(s)+v(s))\Delta s \\
&\geq \lambda k_1 (f_\infty^i - \epsilon) \int_r^{\sigma(b)} G(\sigma(s),s)p(s)k\|(u,v)\|_Y \Delta s \\
&\geq \alpha\|(u,v)\|_Y.
\end{aligned}$$

Hence,

$$\|T_1(u,v)\| \geq \alpha\|(u,v)\|_Y.$$

In a similar manner, we conclude that

$$\|T_2(u,v)\| \geq \beta\|(u,v)\|_Y.$$

Therefore,

$$\begin{aligned}
\|T(u,v)\|_Y &= \|(T_1(u,v), T_2(u,v))\|_Y \\
&= \|T_1(u,v)\| + \|T_2(u,v)\| \\
&\geq \alpha\|(u,v)\|_Y + \beta\|(u,v)\|_Y \\
&= (\alpha+\beta)\|(u,v)\|_Y \\
&= \|(u,v)\|_Y.
\end{aligned}$$

Hence, $\|T(u,v)\|_Y \geq \|(u,v)\|_Y$. If we set

$$\Omega_2 = \{(u,v) \in Y \mid \|(u,v)\|_Y < J_2\},$$

then

$$\|T(u,v)\|_Y \geq \|(u,v)\|_Y, \text{ for } (u,v) \in P \cap \partial\Omega_2. \tag{3.2}$$

Applying Theorem 2.5 to (3.1) and (3.2), we obtain that T has a fixed point (u,v) in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$ and hence the BVP (1.1)–(1.2) has a positive solution such that $J_1 \leq \|u\| + \|v\| \leq J_2$.

(ii) Let $\lambda \in (L_1, \infty)$, $\mu \in (L_3, \infty)$ and let $\epsilon > 0$ be a positive number such that $\epsilon < f_\infty^i$, $\epsilon < g_\infty^i$ and

$$\alpha \left[kk_1(f_\infty^i - \epsilon) \int_r^{\sigma(b)} G(\sigma(s), s)p(s)\Delta s \right]^{-1} \leq \lambda,$$

$$\beta \left[kk_2(g_\infty^i - \epsilon) \int_r^{\sigma(b)} H(\sigma(s), s)q(s)\Delta s \right]^{-1} \leq \mu,$$

$$\epsilon \leq \frac{\alpha}{\lambda} \left[\int_a^{\sigma(b)} G(\sigma(s), s)p(s)\Delta s \right]^{-1},$$

$$\epsilon \leq \frac{\beta}{\mu} \left[\int_a^{\sigma(b)} H(\sigma(s), s)q(s)\Delta s \right]^{-1}.$$

By the definitions of $f_0^s = 0$ and $g_0^s = 0$, there exists $J_1 > 0$ such that

$$f(u,v) \leq \epsilon(u+v) \quad \text{and} \quad g(u,v) \leq \epsilon(u+v),\ 0 \leq u+v \leq J_1.$$

Let $(u,v) \in P$ with $\|(u,v)\|_Y = J_1$. i.e., $\|u\| + \|v\| = J_1$. Then, from Lemma 2.3, for $a \leq t \leq \sigma(b)$, we have

$$\begin{aligned}
T_1(u,v)(t) &= \lambda \int_a^{\sigma(b)} G(t,s)p(s)f(u(s), v(s))\Delta s \\
&\leq \lambda \int_a^{\sigma(b)} G(\sigma(s), s)p(s)f(u(s), v(s))\Delta s \\
&\leq \lambda \int_a^{\sigma(b)} G(\sigma(s), s)p(s)\epsilon(u(s) + v(s))\Delta s \\
&\leq \lambda\epsilon \int_a^{\sigma(b)} G(\sigma(s), s)p(s)(\|u\| + \|v\|)\Delta s \\
&\leq \alpha(\|u\| + \|v\|) \\
&= \alpha\|(u,v)\|_Y.
\end{aligned}$$

Hence,

$$\|T_1(u,v)\| \leq \alpha\|(u,v)\|_Y.$$

In a similar manner, we conclude that

$$\|T_2(u,v)\| \leq \beta\|(u,v)\|_Y.$$

Therefore,

$$\begin{aligned}\|T(u,v)\|_Y &= \|(T_1(u,v), T_2(u,v))\|_Y \\ &= \|T_1(u,v)\| + \|T_2(u,v)\| \\ &\le \alpha\|(u,v)\|_Y + \beta\|(u,v)\|_Y \\ &= (\alpha+\beta)\|(u,v)\|_Y \\ &= \|(u,v)\|_Y.\end{aligned}$$

Hence, $\|T(u,v)\|_Y \le \|(u,v)\|_Y$. Define the set

$$\Omega_1 = \{(u,v) \in Y \mid \|(u,v)\|_Y < J_1\},$$

then

$$\|T(u,v)\|_Y \le \|(u,v)\|_Y, \quad \text{for } (u,v) \in P \cap \partial\Omega_1. \tag{3.3}$$

By the definitions of $f_\infty^i, g_\infty^i \in (0,\infty)$, there exists $\bar{J}_2 > 0$ such that

$$f(u,v) \ge (f_\infty^i - \epsilon)(u+v) \quad \text{and} \quad g(u,v) \ge (g_\infty^i - \epsilon)(u+v),\ u+v \ge \bar{J}_2.$$

Define the set

$$\Omega_2 = \{(u,v) \in Y \mid \|(u,v)\|_Y < J_2\}$$

and proceeding in a similar manner of proof (i), we get

$$\|T(u,v)\|_Y \ge \|(u,v)\|_Y, \quad \text{for } (u,v) \in P \cap \partial\Omega_2. \tag{3.4}$$

Applying Theorem 2.5 to (3.3) and (3.4), we obtain that T has a fixed point (u,v) in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$ and hence the BVP (1.1)–(1.2) has a positive solution such that $J_1 \le \|u\| + \|v\| \le J_2$. Similarly, we can prove the remaining. □

Prior to our next result, we define the positive numbers M_1, M_2, M_3 and M_4 by

$$M_1 = \gamma\left[kk_1 f_0^i \int_r^{\sigma(b)} G(\sigma(s),s)p(s)\Delta s\right]^{-1},$$

$$M_2 = \gamma\left[f_\infty^s \int_a^{\sigma(b)} G(\sigma(s),s)p(s)\Delta s\right]^{-1},$$

$$M_3 = \delta\left[kk_2 g_0^i \int_r^{\sigma(b)} H(\sigma(s),s)q(s)\Delta s\right]^{-1},$$

$$M_4 = \delta\left[g_\infty^s \int_a^{\sigma(b)} H(\sigma(s),s)q(s)\Delta s\right]^{-1},$$

where $\gamma > 0$ and $\delta > 0$ are two positive real numbers such that $\gamma + \delta = 1$.

Theorem 3.2. *Assume that the conditions $(A1)$–$(A4)$ hold.*

- (i) *If $f_0^i, g_0^i, f_\infty^s, g_\infty^s \in (0,\infty)$, $M_1 < M_2$ and $M_3 < M_4$, then for each $\lambda \in (M_1, M_2)$ and $\mu \in (M_3, M_4)$ there exists a positive solution $(u(t), v(t))$ on $[a, \sigma(b)]_{\mathbb{T}}$ for* (1.1)–(1.2).
- (ii) *If $f_\infty^s = g_\infty^s = 0$, $f_0^i, g_0^i \in (0,\infty)$, then for each $\lambda \in (M_1, \infty)$ and $\mu \in (M_3, \infty)$ there exists a positive solution $(u(t), v(t))$ on $[a, \sigma(b)]_{\mathbb{T}}$ for* (1.1)–(1.2).
- (iii) *If $f_\infty^s, g_\infty^s \in (0,\infty)$, $f_0^i = g_0^i = \infty$, then for each $\lambda \in (0, M_2)$ and $\mu \in (0, M_4)$ there exists a positive solution $(u(t), v(t))$ on $[a, \sigma(b)]_{\mathbb{T}}$ for* (1.1)–(1.2).
- (iv) *If $f_\infty^s = g_\infty^s = 0$, $f_0^i = g_0^i = \infty$, then for each $\lambda \in (0,\infty)$ and $\mu \in (0,\infty)$ there exists a positive solution $(u(t), v(t))$ on $[a, \sigma(b)]_{\mathbb{T}}$ for* (1.1)–(1.2).

Proof.

(i) let $\lambda \in (M_1, M_2)$, $\mu \in (M_3, M_4)$ and let $\epsilon > 0$ be a positive number such that $\epsilon < f_0^i$, $\epsilon < g_0^i$ and

$$\gamma\left[kk_1(f_0^i - \epsilon)\int_r^{\sigma(b)} G(\sigma(s), s)p(s)\Delta s\right]^{-1} \le \lambda,$$

$$\gamma\left[(f_\infty^s + \epsilon)\int_a^{\sigma(b)} G(\sigma(s), s)p(s)\Delta s\right]^{-1} \ge \lambda,$$

$$\delta\left[kk_2(g_0^i - \epsilon)\int_r^{\sigma(b)} H(\sigma(s), s)q(s)\Delta s\right]^{-1} \le \mu,$$

$$\delta\left[(g_\infty^s + \epsilon)\int_a^{\sigma(b)} H(\sigma(s), s)q(s)\Delta s\right]^{-1} \ge \mu.$$

By the definitions of $f_0^i, g_0^i \in (0,\infty)$, there exists $J_3 > 0$ such that

$$f(u,v) \ge (f_0^i - \epsilon)(u+v) \ \text{ and } g(u,v) \ge (g_0^i - \epsilon)(u+v),\ 0 < u+v \le J_3.$$

By (A1), the above inequalities are also valid for $u = v = 0$.

Let $(u,v) \in P$ with $\|(u,v)\|_Y = J_3$. i.e., $\|u\| + \|v\| = J_3$. Then, from Lemma 2.4, for $a \le t \le \sigma(b)$, we have

$$\begin{aligned}
T_1(u,v)(t) &= \lambda \int_a^{\sigma(b)} G(t,s)p(s)f(u(s), v(s))\Delta s \\
&\ge \lambda k_1 \int_r^{\sigma(b)} G(\sigma(s), s)p(s)f(u(s), v(s))\Delta s \\
&\ge \lambda k_1 \int_r^{\sigma(b)} G(\sigma(s), s)p(s)(f_0^i - \epsilon)(u(s) + v(s))\Delta s \\
&\ge \lambda k_1 (f_0^i - \epsilon)\int_r^{\sigma(b)} G(\sigma(s), s)p(s)k\|(u,v)\|_Y\Delta s \\
&\ge \gamma\|(u,v)\|_Y.
\end{aligned}$$

Hence,

$$\|T_1(u,v)\| \ge \gamma\|(u,v)\|_Y.$$

In a similar manner, we conclude that

$$\|T_2(u, v)\| \geq \delta \|(u, v)\|_Y.$$

Therefore,

$$\begin{aligned}
\|T(u, v)\|_Y &= \|(T_1(u, v), T_2(u, v))\|_Y \\
&= \|T_1(u, v)\| + \|T_2(u, v)\| \\
&\geq \gamma \|(u, v)\|_Y + \delta \|(u, v)\|_Y \\
&= (\gamma + \delta)\|(u, v)\|_Y \\
&= \|(u, v)\|_Y.
\end{aligned}$$

Hence, $\|T(u, v)\|_Y \geq \|(u, v)\|_Y$. If we set

$$\Omega_3 = \{(u, v) \in Y \mid \|(u, v)\| < J_3\},$$

then

$$\|T(u, v)\|_Y \geq \|(u, v)\|_Y, \text{ for } (u, v) \in P \cap \partial\Omega_3. \tag{3.5}$$

Now, we define the functions $f^*, g^* : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$f^*(x) = \max_{0 \leq u+v \leq x} f(u, v) \quad \text{and} \quad g^*(x) = \max_{0 \leq u+v \leq x} g(u, v), \quad \text{for all } x \in \mathbb{R}^+.$$

Then

$$f(u, v) \leq f^*(x) \quad \text{and} \quad g(u, v) \leq g^*(x), \ u + v \leq x.$$

It follows that the functions $f^*, \ g^*$ are nondecreasing and satisfy the conditions

$$\lim_{x\to\infty} \sup \frac{f^*(x)}{x} = f_\infty^s, \quad \lim_{x\to\infty} \sup \frac{g^*(x)}{x} = g_\infty^s.$$

Next, by the definitions of $f_\infty^s, g_\infty^s \in (0, \infty)$, there exists $\bar{J}_4 > 0$ such that

$$f^*(x) \leq (f_\infty^s + \epsilon)x \quad \text{and} \quad g^*(x) \leq (g_\infty^s + \epsilon)x, \ x \geq \bar{J}_4.$$

Let

$$J_4 = \max\left\{2J_3, \bar{J}_4\right\}.$$

Choose $(u, v) \in P$ with $\|(u, v)\|_Y = J_4$. Then, by the definitions of f^* and g^*, we have

$$f(u(t), v(t)) \leq f^*(u(t) + v(t)) \leq f^*(\|u\| + \|v\|) = f^*(\|(u, v)\|_Y)$$

and

$$g(u(t), v(t)) \leq g^*(u(t) + v(t)) \leq g^*(\|u\| + \|v\|) = g^*(\|(u, v)\|_Y).$$

From Lemma 2.3, for $a \leq t \leq \sigma(b)$, we have

$$T_1(u, v)(t) = \lambda \int_a^{\sigma(b)} G(t, s)p(s)f(u(s), v(s))\Delta s$$

$$
\begin{aligned}
&\le \lambda \int_a^{\sigma(b)} G(\sigma(s), s)p(s)f(u(s), v(s))\Delta s \\
&\le \lambda \int_a^{\sigma(b)} G(\sigma(s), s)p(s)f^*(\|(u, v)\|_Y)\Delta s \\
&\le \lambda \int_a^{\sigma(b)} G(\sigma(s), s)p(s)(f_\infty^s + \epsilon)\|(u, v)\|_Y \Delta s \\
&\le \gamma\|(u, v)\|_Y.
\end{aligned}
$$

Hence,

$$\|T_1(u, v)\| \le \gamma\|(u, v)\|_Y.$$

In a similar manner, we conclude that

$$\|T_2(u, v)\| \le \delta\|(u, v)\|_Y.$$

Therefore,

$$
\begin{aligned}
\|T(u, v)\|_Y &= \|(T_1(u, v), T_2(u, v))\|_Y \\
&= \|T_1(u, v)\| + \|T_2(u, v)\| \\
&\le \gamma\|(u, v)\|_Y + \delta\|(u, v)\|_Y \\
&= (\gamma + \delta)\|(u, v)\|_Y \\
&= \|(u, v)\|_Y.
\end{aligned}
$$

Hence, $\|T(u, v)\| \le \|(u, v)\|_Y$. If we set

$$\Omega_4 = \{(u, v) \in Y \mid \|(u, v)\| < J_4\},$$

then

$$\|T(u, v)\|_Y \le \|(u, v)\|_Y, \text{ for } (u, v) \in P \cap \partial\Omega_4. \tag{3.6}$$

Applying Theorem 2.5 to (3.5) and (3.6), we obtain that T has a fixed point (u, v) in $P \cap (\bar{\Omega}_4 \backslash \Omega_3)$ and hence the BVP (1.1)–(1.2) has a positive solution such that $J_3 \le \|(u, v)\|_Y \le J_4$.

The proofs of the remaining cases (ii)–(iv) are similar that of (i) and we shall omit them. □

4. Example

Let us consider an example to illustrate the above result.

Let $\mathbb{T} = \{(\frac{1}{2})^p : p \in \mathbb{N}_0\} \cup [1, 2]$. Take $m = 3, n = 4, a = \frac{1}{2}, b = 2, \xi_2 = \frac{3}{2}, \eta_2 = 1, \eta_3 = \frac{3}{2}, \alpha_1 = 4,\ \alpha_2 = 7, \beta_1 = 2, \beta_2 = 3$. Now, consider the BVP,

$$
\left.
\begin{aligned}
u^{\Delta\Delta}(t) + \lambda p(t)f(u(t), v(t)) &= 0, \quad t \in \left[\frac{1}{2}, \sigma(2)\right]_{\mathbb{T}}, \\
v^{\Delta\Delta}(t) + \mu q(t)g(u(t), v(t)) &= 0, \quad t \in \left[\frac{1}{2}, \sigma(2)\right]_{\mathbb{T}},
\end{aligned}
\right\} \tag{4.1}
$$

$$\left.\begin{aligned} u\left(\frac{1}{2}\right) &= 0, \quad 4u(\sigma(2)) + 2u^{\Delta}(\sigma(2)) = u^{\Delta}\left(\frac{3}{2}\right), \\ v\left(\frac{1}{2}\right) &= 0, \quad 7v(\sigma(2)) + 3v^{\Delta}(\sigma(2)) = v^{\Delta}(1) + v^{\Delta}\left(\frac{3}{2}\right), \end{aligned}\right\} \tag{4.2}$$

where

$$f(u, v) = \frac{(u+v)[1600(u+v)+1](5+\sin v)}{u+v+1},$$

$$g(u, v) = \frac{(u+v)[700(u+v)+1](10+\cos u)}{u+v+1},$$

and $p(t) = q(t) = 1$. The Green's function $G(t, s)$ is

$$G(t, s) = \begin{cases} G_1(t, s), & \frac{1}{2} \le s \le \sigma(s) \le \frac{3}{2}, \\ G_2(t, s), & \frac{3}{2} \le s \le \sigma(s) \le \sigma(2), \end{cases}$$

where

$$G_1(t, s) = \begin{cases} \frac{1}{7}[4(\sigma(2) - t) + 1]\left(\sigma(s) - \frac{1}{2}\right), & \sigma(s) \le t, \\ \frac{1}{7}\left(t - \frac{1}{2}\right)[4(\sigma(2) - \sigma(s)) + 1], & t \le s, \end{cases}$$

and

$$G_2(t, s) = \begin{cases} \frac{1}{7}\left[(4(\sigma(2) - t) + 2)\left(\sigma(s) - \frac{1}{2}\right) + t - \sigma(s)\right], & \sigma(s) \le t, \\ \frac{1}{7}\left(t - \frac{1}{2}\right)[4(\sigma(2) - \sigma(s)) + 2], & t \le s. \end{cases}$$

The Green's function $H(t, s)$ is

$$H(t, s) = \begin{cases} H_1(t, s) & \frac{1}{2} \le s \le \sigma(s) \le 1, \\ H_2(t, s), & 1 \le s \le \sigma(s) \le \frac{3}{2}, \\ H_3(t, s), & \frac{3}{2} \le s \le \sigma(s) \le \sigma(2), \end{cases}$$

where

$$H_1(t, s) = \begin{cases} \frac{2}{23}[7(\sigma(2) - t) + 1]\left(\sigma(s) - \frac{1}{2}\right), & \sigma(s) \le t, \\ \frac{2}{23}\left(t - \frac{1}{2}\right)[7(\sigma(2) - \sigma(s)) + 1], & t \le s, \end{cases}$$

$$H_2(t, s) = \begin{cases} \frac{2}{23}\left[(7(\sigma(2) - t) + 2)\left(\sigma(s) - \frac{1}{2}\right) + t - \sigma(s)\right], & \sigma(s) \le t, \\ \frac{2}{23}\left(t - \frac{1}{2}\right)[7(\sigma(2) - \sigma(s)) + 2], & t \le s, \end{cases}$$

and

$$H_3(t,s) = \begin{cases} \dfrac{2}{23}\left[(7(\sigma(2)-t)+3)\left(\sigma(s)-\dfrac{1}{2}\right)+2(t-\sigma(s))\right], & \sigma(s) \le t, \\ \dfrac{2}{23}\left(t-\dfrac{1}{2}\right)[7(\sigma(2)-\sigma(s))+3], & t \le s. \end{cases}$$

After simple calculations, we get

$$\begin{aligned} &k_1 = \frac{1}{7},\ k_2 = \frac{2}{23},\ k = \frac{2}{23},\ f_0^s = 5,\ f_0^i = 5,\ f_\infty^s = 9600, \\ &f_\infty^i = 6400,\ g_0^s = 11,\ g_0^i = 11,\ g_\infty^s = 7700,\ g_\infty^i = 6300, \\ &L_1 = 0.04802556818\alpha,\ L_2 = 0.1162629758\alpha, \\ &L_3 = 0.08336416581\beta,\ L_4 = 0.1095672886\beta, \end{aligned}$$

where $\alpha > 0$, $\beta > 0$ are two positive real numbers such that $\alpha + \beta = 1$.

Employing Theorem 3.1 of (i), for each $\lambda \in (L_1, L_2)$ and $\mu \in (L_3, L_4)$, there exists a positive solution $(u(t), v(t))$ of the BVP (4.1)–(4.2).

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