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# Regularity criteria for the 3D magneto-micropolar fluid equations via the direction of the velocity

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**Abstract.** We consider sufficient conditions to ensure the smoothness of solutions to 3D magneto-micropolar fluid equations. It involves only the direction of the velocity and the magnetic field. Our result extends to the cases of Navier–Stokes and MHD equations.

**Keywords.** Magneto-micropolar fluid equations; regularity criteria; direction of velocity.

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## 1. Introduction and the main result

In this paper, we consider the 3D magneto-micropolar fluid equations studied by Galdi and Rionero [5]:

$$\begin{cases} \partial_t u + u\cdot\nabla u - (\mu+\chi)\triangle u - b\cdot\nabla b + \nabla(p+b^2) - \chi\nabla\times\omega = 0,\\ \partial_t\omega - \gamma\triangle\omega - \kappa\nabla\mathrm{div}\,\omega + 2\chi\omega + u\cdot\nabla\omega - \chi\nabla\times u = 0,\\ \partial_t b - \nu\triangle b + u\cdot\nabla b - b\cdot\nabla u = 0,\\ \nabla\cdot u = \nabla\cdot b = 0,\\ u(x,0) = u_0(x), \omega(0,x) = \omega_0(x), b(0,x) = b_0(x). \end{cases} \tag{1.1}$$

Here $u = u(x,t), b = b(x,t), \omega = \omega(x,t)$ represent the velocity field, the magnetic field and the micro-rotational velocity respectively; $p$ denotes the hydrodynamic pressure; $\mu > 0$ is the kinematic viscosity, $\chi > 0$ is the vortex viscosity, $\kappa > 0$ and $\gamma > 0$ are the spin viscosities, $1/\nu$ (with $\nu > 0$) is the magnetic Reynold; while $u_0, b_0, \omega_0$ are the corresponding initial data with $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$.

The global weak solution to system (1.1) is established by Rojas-Medar and Boldrini [10], while the local strong solutions are given by Rojas-Medar [9]. However, whether or not the local strong solutions can exist globally is still an open problem. Thus regularity criteria appears. Notice that:

(1) Yuan [14] first established the following fundamental regularity criterion in terms of the velocity or its gradient

$$u\in L^p(0,T;L^q(\mathbb{R}^3)),\quad \frac{2}{p}+\frac{3}{q}=1,\quad 3<q\leqslant\infty \tag{1.2}$$

and

$$\nabla u \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}=2, \quad \frac{3}{2}<q\leqslant\infty. \tag{1.3}$$

Then Gala [4] extended it to the Morrey–Campanato spaces, Zhang *et al* [16] improved it to some more general Triebel–Lizorkin spaces.

(2) When $\omega = b = 0$, system (1.1) is just the classical Naiver–Stokes equations. Serrin [11], Prodi [8] and Beirão da Veiga [1] proved regularity if some scaling-invariant norm of $u$ or $\nabla u$ is bounded.

(3) When $\omega = 0$, system (1.1) is then the 3D MHD equations, He and Xin [6], and Zhou [19] gave criteria similar to the case for Navier–Stokes equations.

(4) When $b = 0$, system (1.1) is reduced to the micropolar fluid equations, Yuan [13] gave some criteria in Lorentz spaces.

For later developments, see [2, 3, 15, 18] and references cited therein. Recently, Vasseur [12] proved that if

$$\operatorname{div}\frac{u}{|u|} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}\leqslant\frac{1}{2}, \quad p\geqslant 4, \quad q\geqslant 6, \tag{1.4}$$

then the solutions to the Navier–Stokes equations are smooth. Later, Luo [7] extended (1.4) to the MHD equations, but involves the magnetic field also. We now extend the result of Vasseur [12] and Luo [7] to system (1.1). The main result is the following:

**Theorem 1.1.** *Let $u_0, \omega_0, b_0 \in H^1(\mathbb{R}^3)$ with $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$ in the sense of distributions. Suppose that $(u,\omega,b)$ is a strong solution to (1.1) in $(0,T)$ such that*

$$u,\ \omega,\ b \in C((0,T);H^1(\mathbb{R}^3)) \cap C((0,T);H^2(\mathbb{R}^3))$$

*and $\nabla\cdot u = \nabla\cdot b = 0$. If additionally,*

$$\operatorname{div}\frac{u}{|u|} \in L^p(0,T;L^q(\mathbb{R}^3)), \quad \frac{2}{p}+\frac{3}{q}\leqslant\frac{1}{2}, \quad 4\leqslant p<\infty, \quad 6\leqslant q\leqslant\infty \tag{1.5}$$

*and*

$$b \in L^r(0,T;L^s(\mathbb{R}^3)), \quad \frac{2}{r}+\frac{3}{s}\leqslant 1, \quad 2\leqslant r<\infty, \quad 3\leqslant s\leqslant\infty, \tag{1.6}$$

*then the solution can be extended smoothly beyond $t = T$.*

*Remark* 1.1. Theorem 1.1 shows that it is enough to control the rate of change in the direction of the velocity and the norm of $b$ to get full regularity of the solutions. Notice that we add no conditions on the micro-rotational velocity $\omega$.

*Remark* 1.2. Our theorem covers the results of Vasseur [12] and Luo [7] for Navier–Stokes and MHD equations, respectively. Observe that the condition (1.6) is a scaling-invariant, but (1.5) is not. Whether or not the $1/2$ in (1.5) can increase to 1 is our future work.

Before giving a proof, let us first recall the definition of weak solutions to system (1.1).

DEFINITION 1.1

Let $u_0, \omega_0, b_0 \in L^2(\mathbb{R}^3)$ with $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. A triple $(u, \omega, b)$ of measurable functions on $\mathbb{R}^3 \times (0, T)$ is said to be a weak solution of system (1.1) if

(1) $u, \omega, b \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$ with $\nabla \cdot u = \nabla \cdot b = 0$;
(2) System (1.1) holds in the sense of distributions.

*Remark* 1.3. Testing $(1.1)_1, (1.1)_2, (1.1)_3$ by $u, \omega, b$ respectively, after suitable integration by parts, one has the energy inequality:

$$\begin{aligned}
&\|(u(t), \omega(t), b(t))\|_{L^2}^2 + 2(\mu + \chi)\int_0^t \|\nabla u(s)\|_{L^2}^2\, \mathrm{d}s \\
&\quad + 2\gamma \int_0^t \|\nabla \omega(s)\|_{L^2}^2\, \mathrm{d}s + 2\nu \int_0^t \|\nabla b(s)\|_{L^2}^2\, \mathrm{d}s \\
&\quad + 2\chi \int_0^t \|\omega(s)\|_{L^2}^2\, \mathrm{d}s \leqslant \|(u_0, \omega_0, b_0)\|_{L^2}^2 .
\end{aligned} \tag{1.7}$$

Throughout the proof in the next section, we shall frequently use the following interpolation inequality (see [17]):

$$\|u\|_{p,q} \leqslant C\, \|u\|_{\infty,2}^{\frac{3}{q}-\frac{1}{2}}\, \|\nabla u\|_{2,2}^{\frac{3}{2}-\frac{3}{q}} \leqslant C\left(\|u\|_{\infty,2} + \|\nabla u\|_{2,2}\right), \tag{1.8}$$

for $(p, q)$ satisfying

$$\frac{2}{p} + \frac{3}{q} \geqslant \frac{3}{2}, \quad 2 \leqslant q \leqslant 6.$$

In this paper, we shall use standard notations for Lebesgue space $L^q(\mathbb{R}^3)$ endowed with the norm $\|\cdot\|_q$, and anisotropic Lebesgue space $L^p(I; L^q(\mathbb{R}^3))$ endowed with the norm $\|\cdot\|_{p,q}$. Here $I \subset \mathbb{R}^+$ is an interval. A constant $C$ ($C = C(*, *, \ldots)$ which depends on the parameters) may differ from line to line.

## 2. Proof of Theorem 1.1

By decreasing $p$ or $r$ if necessary, we may assume that

$$\frac{2}{p} + \frac{3}{q} = \frac{1}{2}, \quad \frac{2}{r} + \frac{3}{s} = 1.$$

For an $\varepsilon > 0$ to be chosen sufficiently small (see (2.12)), choose $t_1 \in (0, T)$ such that

$$\left\| \mathrm{div} \frac{u}{|u|} \right\|_{p,q} < \varepsilon \tag{2.1}$$

and

$$\|b\|_{r,s} < \varepsilon. \tag{2.2}$$

Hereafter, the integrals are over $\mathbb{R}^3 \times (t_1, T)$.

Utilizing the regularity criteria (1.2), we complete the proof of Theorem 1.1 provided

$$u \in L^8(t_1, T; L^4(\mathbb{R}^3)). \tag{2.3}$$

To this end, denote by

$$I = \||u|^2\|_{\infty,2} + \|\nabla|u|^2\|_{2,2} + \||b|^2\|_{\infty,2} + \|\nabla|b|^2\|_{2,2}. \tag{2.4}$$

Multiplying $(1.1)_1, (1.1)_3$ by $|u|^2 u, |b|^2 b$ respectively and integrating over $\mathbb{R}^3 \times (t_1, T)$, we find that

$$\begin{aligned}
&\frac{1}{4}\||u|^2\|_{\infty,2}^2 + \frac{\mu+\chi}{2}\|\nabla|u|^2\|_{2,2}^2 + (\mu+\chi)\,\||u|\,|\nabla u|\|_{2,2}^2 \\
&\quad = \int_{t_1}^{T}\int_{\mathbb{R}^3} - b\cdot\nabla(|u|^2 u)\cdot b + (p+|b|^2)u\cdot\nabla|u|^2 \\
&\quad + \chi\omega\cdot\nabla\times(|u|^2 u)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{2.5}$$

as well as

$$\begin{aligned}
&\frac{1}{4}\||b|^2\|_{\infty,2}^2 + \frac{\nu}{2}\|\nabla|b|^2\|_{2,2}^2 + \nu\,\||b|\,|\nabla b|\|_{2,2}^2 \\
&\quad = \int_{t_1}^{T}\int_{\mathbb{R}^3} -b\cdot\nabla(|b|^2 b)\cdot u\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{2.6}$$

Using (2.5) and (2.6), notice that (see [12])

$$\mathrm{div}\frac{u}{|u|} = -\frac{u}{|u|^2}\nabla u$$

and (see [13])

$$|\nabla|u|| \leqslant |\nabla u|.$$

Thus we have

$$\begin{aligned}
&I^2 + \||u|\,|\nabla u|\|_{2,2}^2 + \||b|\,|\nabla b|\|_{2,2}^2 \\
&\quad \leqslant C\int_{t_1}^{T}\int_{\mathbb{R}^3} |b|^2\,|u|\,(|u|\,|\nabla u| + |b|\,|\nabla b|)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + C\int_{t_1}^{T}\int_{\mathbb{R}^3} (p+|b|^2)\,|u|^3\left|\mathrm{div}\frac{u}{|u|}\right|\mathrm{d}x\,\mathrm{d}t \\
&\quad + C\int_{t_1}^{T}\int_{\mathbb{R}^3} |\omega|\,|u|^2\,|\nabla u|\,\mathrm{d}x\,\mathrm{d}t \qquad (2.7)\\
&\quad \equiv I_1 + I_2 + I_3. \qquad (2.8)
\end{aligned}$$

Here $C$ is a constant depending only on $\mu, \chi, \nu$.

Using Cauchy–Schwartz inequality, $I_1$ can be bounded as

$$I_1 \leqslant \frac{1}{4}\||u|\,|\nabla u|\|_{2,2}^2 + \frac{1}{2}\||b|\,|\nabla b|\|_{2,2}^2 + C\||b|^2|u|\|_{2,2}^2.$$

By generalized Hölder inequality and (1.8), it follows that

$$\begin{aligned}
\||b|^2|u|\|_{2,2}^2 &\leqslant \|b\|_{r,s}^2\,\|b\|_{a,b}^2\,\|u\|_{c,d}^2 \\
&= \|b\|_{r,s}^2\,\||b|^2\|_{\frac{a}{2},\frac{b}{2}}\||u|^2\|_{\frac{c}{2},\frac{d}{2}} \\
&\leqslant C\varepsilon^2 I^2,
\end{aligned}$$

where

$$\begin{cases} \frac{1}{r}+\frac{1}{a}+\frac{1}{c}=\frac{1}{2}=\frac{1}{s}+\frac{1}{b}+\frac{1}{d}, \\ \frac{2}{a/2}+\frac{3}{b/2}=\frac{3}{2}, \\ \frac{2}{c/2}+\frac{3}{d/2}=\frac{3}{2}, \end{cases}$$

and we have used (1.8). In fact, we can choose

$$a=c=\frac{4r}{r-2}, \quad b=d=\frac{4s}{s-2},$$

where $r$, $s$ are as in (1.6). Thus

$$I_1 \leqslant \frac{1}{4}\,\||u|\,|\nabla u|\|_{2,2}^2+\frac{1}{2}\,\||b|\,|\nabla b|\|_{2,2}^2+C\varepsilon I^2. \tag{2.9}$$

For $I_2$, let us first take divergence of $(1.1)_1$ to see

$$-\triangle p=\sum_{i,j=1}^{3}\partial_{ij}(u_iu_j-b_ib_j+\delta_{ij}\,|b|^2),$$

thus classical Calderón–Zygmund estimates imply

$$\|p\|_{a,b}\leqslant C(\||u|^2\|_{a,b}+\||b|^2\|_{a,b}),$$

invoking again the generalized Hölder inequality and (1.8),

$$\begin{aligned} I_2 &\leqslant C\|p+|b|^2\|_{a,b}\||u|^3\|_{c,d}\|\mathrm{div}\frac{u}{|u|}\|_{p,q} \\ &\leqslant C\varepsilon\left(\|u\|_{a_1,b_1}\,\|u\|_{3c,3d}+\|b\|_{a_1,b_1}\,\|b\|_{3c,3d}\right)\|u\|_{3c,3d}^3 \\ &= C\varepsilon\,\|u\|_{a_1,b_1}\,\||u|^2\|_{\frac{3c}{2},\frac{3d}{2}}^2+C\varepsilon\,\|b\|_{a_1,b_1}\,\||b|^2\|_{\frac{3c}{2},\frac{3d}{2}}^{\frac{1}{2}}\||u|^2\|_{\frac{3c}{2},\frac{3d}{2}}^{\frac{3}{2}} \\ &\leqslant C\varepsilon I^2, \end{aligned} \tag{2.10}$$

where

$$\begin{cases} \frac{1}{a}+\frac{1}{c}+\frac{1}{p}=1=\frac{1}{b}+\frac{1}{d}+\frac{1}{q}, \\ \frac{1}{a}=\frac{1}{a_1}+\frac{1}{3c}, \quad \frac{1}{b}=\frac{1}{b_1}+\frac{1}{3d}, \\ \frac{2}{a_1}+\frac{3}{b_1}=\frac{3}{2}, \quad \frac{2}{3c/2}+\frac{3}{3d/2}=\frac{3}{2}. \end{cases}$$

In fact, we can choose

$$a=c=\frac{2}{3}a_1=\frac{2p}{p-1}, \quad b=d=\frac{2}{3}b_1=\frac{2q}{q-1},$$

where $p$, $q$ are as in (1.5).

Finally, using (1.8), $I_3$ is treated as

$$\begin{aligned} I_3 &\leqslant \frac{1}{4}\,\||u|\,|\nabla u|\|_{2,2}^2 + C\,\||\omega|\,|u|\|_{2,2}^2 \\ &\leqslant \frac{1}{4}\,\||u|\,|\nabla u|\|_{2,2}^2 + C\,\|\omega\|_{2,4}^2\,\|u\|_{\infty,4}^2 \\ &\leqslant \frac{1}{4}\,\||u|\,|\nabla u|\|_{2,2}^2 + CI \\ &\leqslant \frac{1}{4}\,\||u|\,|\nabla u|\|_{2,2}^2 + C\varepsilon I^2 + C. \end{aligned} \tag{2.11}$$

Combining the estimates for $I_1, I_2, I_3$, i.e. (2.9), (2.10), (2.11), and substituting into (2.7), we find

$$I^2 + \frac{1}{2}\,\||u|\,|\nabla u|\|_{2,2}^2 + \frac{1}{2}\,\||b|\,|\nabla b|\|_{2,2}^2 \leqslant 3C\varepsilon I^2 + C,$$

where $C$ is the generic constant appearing in (2.9), (2.10) and (2.11). Thus, we see that

$$I \leqslant \sqrt{2C} < \infty,$$

provided

$$\varepsilon = \frac{1}{6C}. \tag{2.12}$$

Consequently, by (2.4), we have

$$u \in L^\infty(t_1, T, L^4(\mathbb{R}^3)) \subset L^8(t_1, T, L^4(\mathbb{R}^3)).$$

The proof is completed.

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