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Finite groups all of whose minimal subgroups are *NE**-subgroups

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Abstract. Let G be a finite group. A subgroup H of G is called an *NE*-subgroup of G if it satisfies $H^G \cap N_G(H) = H$. A subgroup H of G is said to be a NE^*-subgroup of G if there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T$ is a *NE*-subgroup of G. In this article, we investigate the structure of G under the assumption that subgroups of prime order are NE^*-subgroups of G. The finite groups, all of whose minimal subgroups of the generalized Fitting subgroup are NE^*-subgroups are classified.

Keywords. *NE*-subgroup; NE^*-subgroup; the generalized fitting subgroup; saturated formation.

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1. Introduction

All groups considered will be finite. We use conventional notions and notation, as in Huppert [10]. Throughout this article, G stands for a finite group and $\pi(G)$ denotes the set of primes dividing $|G|$. Notation and basic results in the theory of formations are taken mainly from Doerk and Hawkes [6].

Recall that a subgroup H of a group G is called c-supplemented (c-normal, weakly c-normal, respectively) in G if there exists a subgroup (normal subgroup, subnormal subgroup, respectively) K of G such that $G = HK$ and $H \cap K \leq H_G$, where $H_G = \mathrm{Core}_G(H)$ is the largest normal subgroup of G contained in H (see [3, 15, 17]). Following Li [12], a subgroup R of G is called a *NE*-subgroup of G if $R^G \cap N_G(R) = R$. In the recent years, there has been much interest in investigating the influence of *NE*-subgroups of prime order and cyclic subgroups of order 4 on the structure of the groups. In [4], Bianchi *et al.* introduced the concept of a $\mathcal{H}$-subgroup and investigated the influence of $\mathcal{H}$-subgroups on the structure of a group G: a subgroup H of G is said to be a $\mathcal{H}$-subgroup of G if $N_G(H) \cap H^g \leq H$ for all $g \in G$. Asaad [1] described the groups, all of whose certain subgroups of prime power orders are $\mathcal{H}$-subgroups.

Clearly an *NE*-subgroup is an $\mathcal{H}$-subgroup in G. The converse is not true in general (see [13]).

The aim of this paper is threefold. First, we introduce a new concept called NE^*-subgroup which covers properly the notion of *NE*-subgroup (see Definition 1.1 and Example 1.2 below). Our second aim is to characterize the structure of a group G with the requirement that certain subgroups of G possess the NE^*-property. We state our results in the broader context of formation theory and only consider the conditions on minimal subgroups of G (dropping the assumption that every cyclic subgroup of order 4 is an NE^*-subgroup). Our final aim is to investigate the structure of groups G with the property that all the cyclic subgroups of prime order or order 4 of G satisfy the NE^*-property. We first introduce the following concept:

DEFINITION 1.1

A subgroup H of a finite group G is said to be an NE^*-subgroup of G if there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T$ is an *NE*-subgroup of G.

It is clear that *NE*-subgroups are NE^*-subgroups but the converse is not true in general.

Example 1.2. $G = S_4$, the symmetric group of degree 4, and $L = A_4$, the alternating group of degree 4. Clearly, $G = L \rtimes H$, where $H = \langle (13) \rangle$. Observe that $H \cap A_4 = 1$, this yields that H is an NE^*-subgroup of G by Definition 1.1. Now $H^{(12)(34)} = \langle (24) \rangle \leq N_G(H)$ and $(12)(34) \notin N_G(H)$ show that H is not an *NE*-subgroup of G.

Buckley [5] proved that a finite group of odd order, all of whose minimal subgroups are normal is supersolvable. We prove the following theorem which is an improvement of a recent result due to Asaad and Ramadan (see Theorem 1.1 of [2]). Hence, Q_8 will denote the quaternion group of order 8 and a group G is called Q_8-free if no quotient group of any subgroup of G is isomorphic to Q_8. Throughout this paper, $\mathcal{U}$ will denote the class of all supersolvable groups. Clearly, $\mathcal{U}$ is a formation. The $\mathcal{U}$-hypercentre $Z_{\mathcal{U}}(G)$ of G is the product of all normal subgroups H of G such that each chief factor of G below H has prime order.

Theorem 1.3. *Let G be Q_8-free and let P be a nontrivial normal p-subgroup of G. If all minimal subgroups of P are NE^*-subgroups of G, then $P \leq Z_{\mathcal{U}}(G)$, where $\mathcal{U}$ is the formation of all supersolvable groups.*

Theorem 1.3 may be false if we drop the first condition. The following example shows the necessity of the 'Q_8-free' hypothesis in Theorem 1.3.

Example 1.4. Let G be the semidirect product of the quaternion group P of order 8 and the cyclic group $\langle c \rangle$ of order 9, where $P = \langle a, b | a^4 = 1, b^2 = a^2, a^b = a^{-1} \rangle$, which is isomorphic to Q_8 and c acts on P or equivalently, c is the automorphism of order 3 of G given by $a^c = b, b^c = ab$. Thus $G = P \rtimes \langle c \rangle$ is a group of order $2^3 \cdot 3^2$. We obtain that there are only two minimal subgroups, i.e., $\langle a^2 \rangle$ and $\langle c^3 \rangle$ in G, and the centre of G is $Z(G) = \langle a^2 \rangle \times \langle c^3 \rangle$ (see p. 292 of [14]). Thus all minimal subgroups of G are normal and hence are certainly NE^*-subgroups of G. Note that the chief series of G containing P is $1 \lhd Z(P) \lhd P \lhd (P \times \langle c^3 \rangle) \lhd G$. This yields that $P \not\leq Z_{\mathcal{U}}(G)$.

Li showed that if every minimal subgroup of G is an *NE*-subgroup of G, then G is solvable (see Theorem 1(b) of [12]). The following theorem shows that this result remains true if, in Theorem 1 of [12], we consider NE^*-subgroups instead of *NE*-subgroups.

Theorem 1.5. *If all minimal subgroups of a group G are NE^*-subgroups of G, then G is solvable.*

Recall that a p-group G is called ultra-special if $G' = \Phi(G) = Z(G) = \Omega_1(G)$. For any group G, $F^*(G)$ denotes the generalized fitting subgroup: the set of all elements g of G which induce inner automorphisms on every chief factor of G. The following new characterizations of groups involve the requirement that certain minimal subgroups of $F^*(H)$ possess the NE^*-property.

Theorem 1.6. *Let $\mathcal{F}$ be a saturated formation containing all supersolvable groups and let H be a normal subgroup of G such that $G/H \in \mathcal{F}$. If all minimal subgroups of $F^*(H)$ are NE^*-subgroups of G, then either $G \in \mathcal{F}$ or G contains a minimal non-nilpotent subgroup K with the following properties*:

(1) *K has a nontrivial normal Sylow 2-subgroup, say K_2*;
(2) *$K_2 \leq O_2(H)$, $|K_2| = 2^{3s}$, and $|\Phi(K_2)| = 2^s$, where $s \geq 1$*;
(3) *K_2 is ultra-special, that is, $K_2' = \Phi(K_2) = Z(K_2) = \Omega_1(K_2)$*;
(4) *If p is the odd prime dividing $|K|$, then p divides $2^s + 1$.*

As an easy consequence of Theorem 1.6, we obtain the following result.

COROLLARY 1.7

Let $\mathcal{F}$ be a saturated formation containing all supersolvable groups and let H be a normal subgroup of a group G such that $G/H \in \mathcal{F}$. Let $F^*(H)$ be of even order and let S be a Sylow 2-subgroup of $F^*(H)$. Further, assume that every minimal subgroup of $F^*(H)$ is an NE^*-subgroup of G. Then $G \in \mathcal{F}$ if one of the following conditions holds:

(1) $\Omega_2(S) \leq Z(S)$;
(2) *For all primes p dividing $|G|$ and all $s \geq 1$, we have that p does not divide $2^s + 1$.*

Theorems 1.5 and 1.6 are not true if the hypothesis of the NE^*-condition on minimal subgroups of G (respectively of $F^*(H)$) is replaced by just the condition on minimal subgroups of noncyclic Sylow subgroups of G (respectively of $F^*(H)$). For example, the group $G := SL(2, 5)$ shows these facts: the only Sylow subgroups of G which are noncyclic are the Sylow 2-subgroups, which are quaternion groups. Then the only minimal subgroup under consideration would be the centre $Z(G)$ of the group, which is normal. It is clear that $Z(G)$ satisfies the NE^*-condition in G.

Using Theorems 1.5 and 1.6, we can derive the following results.

Theorem 1.8. *Let $\mathcal{F}$ be a saturated formation containing all supersolvable groups and let H be a normal subgroup of G such that $G/H \in \mathcal{F}$. If all minimal subgroups and cyclic subgroups of order 4 of $F^*(H)$ are NE^*-subgroups of G, then $G \in \mathcal{F}$.*

Theorem 1.9. *Assume that every minimal subgroup of a group G is an NE^*-subgroup of G. Then either G is supersolvable or G is solvable, $G_{2'}$ is supersolvable and $G_{2'}/C_{G_{2'}}(O_2(G))$ is abelian for any Hall $2'$-subgroup $G_{2'}$ of G.*

Theorem 1.8 is an improvement of Theorem 4.2 of Li in [13]. We do not know whether Theorem 1.8 can be extended by considering minimal subgroups and cyclic subgroups of order 4 of only noncyclic Sylow subgroups of $F^*(H)$.

It is natural to ask the question: are there differences between both NE^*-subgroups and weakly c-normal subgroups? For any subgroup H of a finite group G, it follows that every weakly c-normal subgroup is an NE^*-subgroup. Here we give an explanation. Let H be a weakly c-normal subgroup of G. Then there exists a subnormal subgroup K of G such that $G = HK$ and $H \cap K \leq H_G$. Let $K_1 = H_G K$, applying Wielandt's results (see [16]), we have $K_1 = \langle H_G, K\rangle$ is a subnormal subgroup of G. Since $G = HK_1$ and $H \cap K_1 = H_G(H \cap K) = H_G$, so $H \cap K_1$ is normal in G. Thus H is an NE^*-subgroup of G. In general, if R is an NE^*-subgroup of G, R is not necessary to be weakly c-normal in G (see Example 1.10 below). But if H is an NE^*-subgroup contained in a normal nilpotent subgroup K of G, then it is true that H is weakly c-normal in G (see Lemma 2.2).

Example 1.10. Let $G = A_5$ and $H = A_4$, the alternating subgroups with degree 5 and 4, respectively. Then $G = HG$ and $H^G \cap N_G(H) = H$. Thus H is a NE^*-subgroup of G but not weakly c-normal in G.

Example 1.11. Let $G = A_5$, the alternating group of degree 5, and H a Sylow 5-subgroup. Noting that G is a nonabelian simple group, we get that $H^G \cap N_G(H) = N_G(H)$ of order 10. Hence H is neither an NE^*-subgroup nor a weakly c-normal subgroup of G.

2. Preliminaries

In this section, we state some lemmas which are useful.

Lemma 2.1. *Let K and H be subgroups of a group G.*

(1) *If $H \leq K$ and H is an NE-subgroup of G, then H is an NE-subgroup of K.*
(2) *If H is a subnormal subgroup of K and H is an NE-subgroup of G, then H is normal in K.*

Proof. See Lemmas 1 and 4 of [12]. □

Lemma 2.2. *Let K and H be subgroups of a group G.*

(1) *If $H \leq K$ and H is a NE^*-subgroup of G, then H is a NE^*-subgroup of K.*
(2) *If H is a NE^*-subgroup that is contained in a normal nilpotent subgroup K, then H is weakly c-normal in G.*

Proof.

(1) Since H is an NE^*-subgroup of G, there exists a subnormal subgroup L of G such that $G = HL$ and $H \cap L$ is an NE-subgroup of G. It follows that $K = K \cap HL =$

$H(K \cap L)$ and $K \cap L$ is subnormal in K (see [16]). This implies that $H \cap L$ is an *NE*-subgroup of K by Lemma 2.1(1). So claim (1) holds.
(2) Clearly H is subnormal in G. Since H is an NE^*-subgroup of G, there exists a subnormal subgroup L of G such that $G = HL$ and $H \cap L$ is a *NE*-subgroup of G. It follows that the intersection $H \cap L$ is a subnormal subgroup of G. Therefore it follows immediately from Lemma 2.1(2) that $H \cap L$ is normal in G. Thus $H \cap L \leq H_G$ holds. □

Lemma 2.3. *Let S be a nontrivial* 2*-group and let H be a nontrivial group of automorphisms of S fixing the involutions of S. If H is cyclic of odd order and H acts irreducibly on $S/\Phi(S)$, then $|S| = 2^{3s}$, $|\Phi(S)| = 2^s$, where $s \geq 1$, S is ultra-special and $|H|$ divides* $2^s + 1$.

Proof. See Theorems 1.3 and 2.2 of [9].

Lemma 2.4. *Let S be a nontrivial* 2*-group and let H be a nontrivial group of automorphisms of S fixing the involutions of S. If* 2 *does not divide $|H|$, then H is abelian.*

Proof. See Theorem 4.4 of [9].

3. Proofs

Proof of Theorem 1.3. We prove the theorem by induction on $|G| + |P|$. Suppose, first, that $p > 2$. Then by Lemma 2.2(2), the condition that every minimal subgroup of P is NE^*-subgroup of G implies that every minimal subgroup of P is weakly c-normal in G. In particular, every minimal subgroup of P is c-supplemented in G. Hence we conclude that $P \leq Z_{\mathcal{U}}(G)$ by Theorem 1.1 of [2]. Thus we may assume that $p = 2$. If every minimal subgroup H of P is normal in G, then $HQ = H \times Q$ for any Sylow q-subgroup Q of G, where q is an odd prime. This implies that $\Omega_1(P) \leq C_G(Q)$. Since G is Q_8-free, by Lemma 2.15 of [7], we obtain that $Q \leq C_G(P)$, yielding that $G/C_G(P)$ is a p-group. This means that $P \leq Z_{\mathcal{U}}(G)$, as claimed. Then we may assume that P has a minimal subgroup H such that H is not normal in G, which implies that H is a NE^*-subgroup of G. It follows that there exists a subnormal subgroup K of G such that $G = HK$ and $H \cap K$ is a *NE*-subgroup of G. Assume $H \cap K \neq 1$, $G = K$ and so H is a *NE*-subgroup and, of course, a $\mathcal{H}$-subgroup. Since H is a subnormal subgroup of G, it follows that $H \lhd G$ by Lemma 2.1, a contradiction. Hence we conclude that $H \cap K$ must be 1. Let $L = P \cap K$. Since K is a maximal subgroup of G, we conclude that the subnormal subgroup K of G is normal in G. Thus $L = P \cap K \lhd G$. Because $H \leq P$, Dedekind's law implies $P = HL$. By our hypothesis, every minimal subgroup of L is a NE^*-subgroup of G. Therefore, $L \leq Z_{\mathcal{U}}(G)$ by induction. Observe that if P/L is normal in G/L with order p then $P/L \leq Z_{\mathcal{U}}(G/L)$. So $L \leq Z_{\mathcal{U}}(G)$, yielding $Z_{\mathcal{U}}(G/L) = Z_{\mathcal{U}}(G)/L$, which implies that $P \leq Z_{\mathcal{U}}(G)$ and the proof is complete. □

Proof of Theorem 1.5. Assume that the theorem is false and let G be a counterexample of minimal order. Then:

(1) *Every proper subgroup of G is solvable*. Let T be a proper subgroup of G. By Lemma 2.2(1), every minimal subgroup of T is a NE^*-subgroup of T, and so T satisfies the hypothesis of G. The minimal choice of G yields that T is solvable.

(2) *$G/\Phi(G)$ is a minimal simple group*. By (1), G has a nontrivial maximal normal solvable subgroup, say M. Clearly, $\Phi(G)$ is a subgroup of M. We shall show that $\Phi(G) = M$. Because otherwise we have $M \nleq \Phi(G)$, and we can conclude that there exists a maximal subgroup N of G such that $G = MN$ and consequently G is solvable by (1), a contradiction. Thus the subgroup $\Phi(G)$ is the unique maximal subgroup of G, and so $G/\Phi(G)$ is a minimal simple group.
(3) *$\Phi(G)$ is a 2-group*. By (2), applying Thompson's classification of minimal simple groups, we obtain that $G/\Phi(G)$ is isomorphic to one of the following groups:

(i) $PSL(3, 3)$;
(ii) the Suzuki group $S_z(2^q)$, where q is an odd prime;
(iii) $PSL(2, p)$, where p is an odd prime with $p^2 \equiv 1 (\mathrm{mod}\ 5)$;
(iv) $PSL(2, 2^q)$, where q is a prime;
(v) $PSL(2, 3^q)$, where q is an odd prime.

Using this result, we shall show that $\Phi(G)$ is a 2-group. Let K be the 2-complement of $\Phi(G)$, then $K \lhd G$ and K is nilpotent. We wish to show, first, that $K \leq Z(G)$. Let p be a prime dividing $|K|$ and let $P \in \mathrm{Syl}_p(K)$. It is clear that P is normal in G. By hypothesis, every subgroup L of order p in P is a NE^*-subgroup of G. It follows that there exists a subnormal subgroup T of G such that $G = LT$ and $L \cap T$ is a *NE*-subgroup of G. If $L \cap T = 1$, then G has a subgroup T of index p. Since T is a maximal subgroup of G and T is a subnormal subgroup of G, we have $T \lhd G$. Thus G is solvable by (1), a contradiction. So $L \cap T = L$ and so $T = G$. This implies that L is a *NE*-subgroup of G, and is normal in G by Lemma 2.1(2). Assume that $L \nleq Z(G)$. Then $C_G(L)$ is a proper subgroup and so $C_G(L) \leq \Phi(G)$ by simplicity of $G/\Phi(G)$. This implies that $G/C_G(L)$ is cyclic and so G is solvable, a contradiction. Consequently, each subgroup of P of order p lies in the centre $Z(G)$. Consider the group $D = SP$, where S is a Sylow 2-subgroup of G. It follows from Itô's lemma (see Chapter IV, Satz 5.5 of [10]) that D is p-nilpotent, and hence, D is nilpotent. Thus $S \leq C_G(P) \lhd G$. Applying the simplicity of $G/\Phi(G)$ again, we can conclude that $P \leq Z(G)$. Next, we denote by S_0 a Sylow 2-subgroup of $\Phi(G)$, and consider the group $\bar{G} = G/S_0$. Since $K \leq Z(G)$, we have that $G/Z(G) \cong G/\Phi(G)$ and $\bar{G}$ is a quasisimple group with the centre of odd order. By checking the table on Schur multipliers of the known simple groups (see p. 302 of [8]), we can conclude that the Schur multiplier of each of the minimal simple groups is a 2-group. It follows that $Z(\bar{G})$ must be 1, and therefore $\Phi(G)$ is a 2-group.
(4) *Let R be a Sylow r-subgroup of G, where $r > 2$. Then there exists a subgroup L of order r such that L is not normal in G*. Obviously, $C_G(\Omega_1(R)) < G$, because otherwise we would have $\Omega_1(R) \leq Z(G)$ and so G would be r-nilpotent by Chapter IV, Satz 5.5 of [10]. It follows that G is solvable by (1), a contradiction. Then $\Omega_1(R)$ is solvable by (1). Assume that every minimal subgroup of R is normal in G. Then we can conclude that $\Omega_1(R)$ is an elementary abelian normal subgroup of G and every chief factor of G which lies below $\Omega_1(R)$ is cyclic of order r, which implies that $\Omega_1(R) \leq Z_{\mathcal{U}}(G)$. It follows that $G/C_G(\Omega_1(R))$ is supersolvable by Chapter IV, Theorem 6.10 of [6] and so G is solvable, a contradiction. Thus there exists a subgroup L of order r such that L is not normal in G.
(5) *Let $\bar{G} = G/\Phi(G)$. Then 3 does not divide $|\bar{G}|$*. Assume that 3 divides $|\bar{G}|$. Then G has a subgroup L of order 3 such that L is not normal in G by (4). By hypothesis, L is a NE^*-subgroup of G. It follows that there exists a subnormal subgroup T of G such

that $G = LT$ and $L \cap T$ is a *NE*-subgroup of G. If $L \cap T = 1$, consequently G has a subgroup T of index 3. Since T is a maximal subgroup of G, we conclude that the subnormal subgroup T of G is normal in G. Thus G is solvable by (1), a contradiction. So assume for the rest of this paragraph that $L \leq T$. Clearly L is a NE-subgroup of G, which implies that $L^G \cap N_G(L) = L$ and so L is a Sylow subgroup of L^G. By the Frattini argument, we can conclude that $G = N_G(L)L^G$. Moreover L^G is a Frobenius group with complement L. Let N be the kernel of L^G. Then N is nilpotent and so normal in G. Hence $G = N_G(L)N$, which means that G is solvable, a contradiction. Thus 3 does not divide $|\bar{G}|$.

(6) *Completing the proof*. By (1), (2), and (5), $\bar{G}$ is a minimal simple group, $(3, |\bar{G}|) = 1$. It follows by Chapter II, Bemerkung 7.5 of [10], that $\bar{G}$ is isomorphic to the Suzuki group $S_z(2^q)$, where q is odd. However $|S_z(2^q)| \equiv 0 \pmod 5$ by Chapter XI, Remarks 3.7(b) of [11] and so 5 divides $|\bar{G}|$. Therefore $\bar{G}$ has a subgroup of index 5 by a discussion similar to (5) above. This implies that $\bar{G}$ is isomorphic to a subgroup of S_5, the symmetric group on five letters. Since 3 does not divide $|\bar{G}|$ by (5), it implies that $|\pi(G)| = 2$ because $\Phi(G)$ is a 2-group by (3) and so G is solvable, a final contradiction. □

Proof of Theorem 1.6. Suppose that the result is false and let G be a counterexample of a minimal order. Then:

(1) $F^*(H) = F(H)$. By Lemma 2.2, every minimal subgroup of $F^*(H)$ is a NE^*-subgroup of $F^*(H)$. Then $F^*(H)$ is solvable by Theorem 1.5. It follows that $F^*(H) = F(H)$ by Chapter X, Theorem 13.13 of [11].

(2) $F(H)$ *is of even order*. Otherwise, $F(H)$ is of odd order. Theorem 1.3 implies that $F(H) \leq Z_{\mathcal{U}}(G)$. Since $\mathcal{U} \subseteq \mathcal{F}$ and $\mathcal{U}$ and $\mathcal{F}$ are saturated formations, it follows that $Z_{\mathcal{U}}(G) \leq Z_{\mathcal{F}}(G)$ by Proposition 3.11 of [6]. Then we can conclude that $F(H) \leq Z_{\mathcal{F}}(G)$ and hence $G/C_G(F(H)) \in \mathcal{U}$ by Chapter IV, Theorem 6.10 of [6]. Moreover, since $G/H \in \mathcal{U}$ by our hypothesis, it follows that $G/C_H(F(H)) \in \mathcal{U}$. By Chapter X, Theorem 13.12 of [11], we get that $C_H(F^*(H)) \leq F(H)$ and $C_H(F(H)) \leq F(H)$ since $F^*(H) = F(H)$ by (1). Then $G/F(H) \in \mathcal{F}$ and since $F(H) \leq Z_{\mathcal{F}}(G)$, it follows that $G \in \mathcal{F}$, a contradiction. This proves (2).

(3) *There exists a Sylow subgroup P of G such that $O_2(H)P$ is not 2-nilpotent, where $|O_2(H)|$ and $|P|$ are co-prime*. If not, $O_2(H) \leq Z_\infty(G)$, where $Z_\infty(G)$ is the hypercentre of G. Since $Z_\infty(G) \leq Z_{\mathcal{U}}(G)$, it follows that $O_2(H) \leq Z_{\mathcal{U}}(G)$. Applying Theorem 1.3, we get that every Sylow subgroup of $F(H)$ of odd order lies in $Z_{\mathcal{U}}(G)$ and hence $F(H) \leq Z_{\mathcal{U}}(G)$. By a discussion similar to Step (2), noting that $Z_{\mathcal{U}}(G) \leq Z_{\mathcal{F}}(G)$, it follows that $G \in \mathcal{F}$, a contradiction. Therfore there exists a Sylow subgroup P of G such that $O_2(H)P$ is not 2-nilpotent, where $(|O_2(H)|, |P|) = 1$.

(4) *Completing the proof*. By (3), it is clear that $O_2(H)P$ contains a minimal non-2-nilpotent subgroup, say K. By Chapter IV, Satz 5.4 of [10], we have that K is a minimal non-nilpotent subgroup of G. Applying Chapter III, Satz 5.2 of [10] we can conclude that K has a normal Sylow 2-subgroup K_2 and a cyclic Sylow p-subgroup K_p, for a prime $p \neq 2$. Clearly K_p fixes the involutions of K_2, because otherwise we get that K_p is normal in K, a contradiction. Moreover K_p acts irreducibly on $K_2/\Phi(K_2)$. It follows by Lemma 2.3 that $|K_2| = 2^{3s}$ and $\Phi(|K_2|) = 2^s$, where $s \geq 1$, K_2 is ultra-special and $K_p/C_{K_p}(K_2)$ divides $2^s + 1$. This is a final contradiction and the proof is complete. □

Proof of Theorem 1.8. Suppose that the result is false. By Theorem 1.6, there exists a minimal non-nilpotent subgroup K of G satisfying the properties (1), (2) and (3). For every cyclic subgroup L of K_2 of order 4, since K_2 is ultra-special, it follows that $L \nleq Z(K_2) = \Omega_1(K_2)$ and consequently $K_2 \nleq C_K(L)$. If $C_K(L)$ is normal in K, it follows that K_p is normal in K, where K_p is a Sylow p-subgroup of K and $p > 2$, a contradiction. Thus $C_K(L)$ is not normal in K and so L is not normal in K. We may conclude, by our assumptions, that L is a NE^*-subgroup of G and so L is a NE^*-subgroup of K. Then there exists a subnormal subgroup K_1 of K such that $K = LK_1$ and $L \cap K_1$ is a NE-subgroup of G. Since L is not normal and K is a minimal non-nilpotent group, it follows that if K_1 is a proper subgroup of K, then the fact that K_p char K_1 and $K_1 \lhd K$ would imply that K_p is subnormal in K. Since K_p is a subnormal Hall-subgroup of K, K_p is normal in K, a contradiction. This means that $K_1 = K$ and hence L is a NE-subgroup of G. Thus, by Theorem 4.2 of [13], we get that G belongs to $\mathcal{F}$. This is a final contradiction and the proof is complete. □

Proof of Theorem 1.9. Theorem 1.5 immediately yields the solvability of G. Let $G_{2'}$ be a Hall $2'$-subgroup of G. It follows by Lemma 2.2 and Theorem 1.6 that $G_{2'}$ is supersolvable. Hence, if G has odd order, then G is supersolvable and we are done. Assume that 2 divides the order of G and that $O_2(G)$ is nontrivial. Then if $G_{2'}$ centralizes the involutions of $O_2(G)$, then Lemma 2.4 implies that $G_{2'}/C_{G_{2'}}(O_2(G))$ is abelian. Hence we may assume that there exists an involution $x \in O_2(G)$ which is not centralized by G, which implies that $\langle x\rangle$ is not normal in G. Noting that $\langle x\rangle$ is a NE^*-subgroup of G, we deduce that $G = \langle x\rangle K$ for some subnormal subgroup K of G such that $\langle x\rangle \cap K$ is a NE-subgroup of G. If $\langle x\rangle \cap K = \langle x\rangle$ and so $K = G$. This implies that $\langle x\rangle$ is a NE-subgroup of G, and so normal in G by Lemma 2.1(2), a contradiction. Thus $\langle x\rangle \cap K$ must be 1, and so $K \lhd G$ since $|G : K| = 2$. It follows that $G_{2'}$ is a Hall 2-subgroup of K because $G_{2'} \leq K$. It follows that $[O_2(G), G_{2'}] \leq O_2(G) \cap K = O_2(K)$. If we argue by the induction on the order of G, we can deduce by inductive hypothesis that $[O_2(G), (G_{2'})', (G_{2'})'] \leq [O_2(K), (G_{2'})'] = 1$. This means that $[O_2(G), (G_{2'})'] = 1$ by co-prime action. So $G_{2'}/C_{G_{2'}}(O_2(G))$ is abelian and the proof is complete. □

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