Proc. Indian Acad. Sci. (Math. Sci.) Vol. 123, No. 4, November 2013, pp. 499–513.

Global existence of solutions for a viscous Cahn–Hilliard equation with gradient dependent potentials and sources

CHENGYUAN QU[1] and YANG CAO[2,*]

[1]School of Science, Dalian Nationalities University, Dalian 116600, China
[2]School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China
*Corresponding author. E-mail: mathcy@dlut.edu.cn

MS received 25 April 2011; revised 11 July 2013

Abstract. We consider a class of nonlinear viscous Cahn–Hilliard equations with gradient dependent potentials and sources. By a Galerkin approximation scheme combined with the potential well method, we prove the global existence of weak solutions.

Keywords. Global solution; viscous Cahn–Hilliard equation; initial boundary value problem.

1. Introduction

In this paper, we investigate the global existence of weak solutions to the following initial boundary value problem for the viscous Cahn–Hilliard equation in one spatial dimension

$$\frac{\partial u}{\partial t} + k_1 D^4 u - k_2 \frac{\partial D^2 u}{\partial t} - D\phi(Du) + A(u) = 0, \quad (x,t) \in (0,1) \times (0,+\infty) \tag{1.1}$$

$$u(0,t) = u(1,t) = D^2u(0,t) = D^2u(1,t) = 0, \qquad t \in (0,+\infty), \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad x \in (0,1), \tag{1.3}$$

where the mobility k_1 and the viscosity k_2 are positive constants, $u_0(0) = u_0(1) = D^2u_0(0) = D^2u_0(1) = 0$ and $u_0(x) \not\equiv 0$. Throughout this paper, we assume that

$$\phi(s) = -\gamma_1|s|^{p-2}s + \gamma_2 s, \qquad A(s) = -\gamma_3|s|^{q-2}s + \gamma_4 s \tag{1.4}$$

with $p > 2$, $q > 2$ and γ_i, $i = 1, 2, 3, 4$ being constants.

Equation (1.1) includes many models. For example, if $k_1 = 0$, $k_2 > 0$, (1.1) reduces to the nonlinear Sobolev–Galpern equations which appear in the study of various problems of biodynamics, thermodynamics and filtration theory (see [4,16,18]). Besides, when $\gamma_3 = \gamma_4 = 0$, Liu and Wang in [14] proved that when $\phi'(s)$ has a lower bound, there exists a unique global generalized solution. Subsequently, Shang [17] investigated the case of $\phi(s) = a|s|^{p-1}s + bs$ and demonstrated that the solution would blow up in a finite time provided that $p > 1$ and $a < 0$.

If $k_1 > 0$, $k_2 = 0$, (1.1) becomes the Cahn–Hilliard type equation with gradient dependent potentials and sources for modelling the epitaxial growth of nanoscale thin films [13], in which, King *et al.* studied the following equation:

$$\frac{\partial u}{\partial t} + \Delta^2 u - \operatorname{div}(\phi(\nabla u)) = g(x, t),$$

where reasonable choice of $\phi(s)$ is $\phi(s) = |s|^{p-2} - s$. They proved the existence, uniqueness and regularity of solutions in an appropriate function space for initial-boundary value problem. Recently, problems (1.1)–(1.3) with $k_2 = 0$ have been considered in [12] and the global existence and uniqueness of classical solutions under some conditions were given. Further, Jin and Yin [12] pointed out that while the global existence conditions can not be satisfied, the solution will blow up in a finite time.

When $k_1 > 0$ and $k_2 > 0$, (1.1) reduces to a viscous Cahn–Hilliard equation which can be briefly derived from modeling cell growth with $u(x, t)$ being the concentration of density of the cell at point x and time t. Basic balance law gives

$$\frac{\partial u}{\partial t} = -\nabla \cdot j + g,$$

where j and g represent the diffusion flux and the reaction source, respectively. When the concentration or density is small (dilute system), then the flux is concentration and gradient dependent, i.e. $j = -m(u)\phi(\nabla u)$, namely diffusion is a local or short range effect. However, when the cell densities are relatively high, a nonlocal or long range diffusion should be included. One available choice is substituting the average density in some neighborhoods of the point x for $u(x, t)$ and taking the form of the flux as

$$j = -m(u)(\phi(\nabla u) - k_1 \nabla(\nabla^2 u)).$$

Further, in some special cases, it seems plausible that there should be microforces whose working accompanies changes in u. Gurtin in [11] describes this working through terms of the form $\partial u/\partial t$, thus the microforces are scalar quantities and the flux

$$j = -m(u)(\phi(\nabla u) - k_1 \nabla(\nabla^2 u)) - k_2 \nabla u_t.$$

Recently, a lot of attention has been paid to the viscous Canh–Hilliard equations. In [2] and [7], the authors proved the existence of the semigroups and the upper and lower semicontinuity of the global attractor for a viscous Cahn–Hilliard equation. However, the lower semicontinuity they obtained was proved under the assumption that all stationary solutions were hyperbolic and this assumption was relaxed in [6]. For more and deeper investigations of the stable analysis (as $t \to \infty$) and the asymptotic behavior of viscous Cahn–Hilliard models and perturbed viscous Cahn–Hilliard models, we refer readers to [3,5,9,10,15,22] and the references therein.

The purpose of the present paper is devoted to investigating the global existence of solutions for (1.1)–(1.3). By a Galerkin approximation scheme combined with the potential well method used in [19,20,21,8], it will be shown that, there exist global weak solutions if one of the following conditions hold:

(I) $\gamma_1 < 0$, $\gamma_3 > 0$ with $p > q$,

(II) $\gamma_1 < 0$, $\gamma_3 > 0$ with $p < q$ and $|\gamma_1|$ being appropriately large,

(III) $\gamma_1 > 0$, $\gamma_3 < 0$ with γ_1 being appropriately small,

$$\text{(IV)}\quad \gamma_1 > 0, \gamma_3 > 0 \begin{cases} \gamma_2 > 0, \gamma_4 > 0 \text{ with } \gamma_1 \text{ and } \gamma_3 \text{ being appropriately small,} \\ \gamma_2 < 0, \gamma_4 < 0 \text{ with } \gamma_1, |\gamma_2|, \gamma_3 \text{ and } |\gamma_4| \text{ being appropriately small.} \end{cases}$$

Actually, the results we obtained may further explain that there are some restricted relationships between the gradient dependent potential $\phi(s)$ and the reaction term $A(s)$. If only the reaction term is dominant, i.e. $\gamma_3 > 0$, then we need either $p > q$ or $|\gamma_1|$ being large enough to strengthen the effect of the gradient dependent potential, such that the solution will exist globally, e.g. (I) and (II); while if the gradient dependent potential is dominant, i.e. $\gamma_1 > 0$, then we can only set both γ_1 and γ_3 small enough to lower their effect to confirm the global existence, e.g. (III) and (IV).

This paper is arranged as follows: Section 2 is devoted to some preliminaries and the main results, and subsequently, the global existence of weak solutions is studied in §3.

2. Statement of main results

Before going further, we first introduce some notations which will be used throughout this paper,

$$\begin{aligned} &L^p = L^p(0,1), \qquad W^{m,p} = W^{m,p}(0,1), \qquad W_0^{m,p} = W_0^{m,p}(0,1), \\ &H^m = W^{m,2}, \qquad H_0^m = W_0^{m,2}, \qquad \|\cdot\|_p = \|\cdot\|_{L^p}, \qquad \|\cdot\|_2 = \|\cdot\|_{L^2}, \end{aligned}$$

where $p \geq 1$, $m \in \mathbb{R}$ are real numbers. The symbol $(\cdot,\cdot)$ stands for the L^2-inner product. Denote by E the reasonable weak solutions space, namely

$$\begin{aligned} E = \{&u \in L^\infty([0,T]; W_0^{1,p}) \cap L^\infty([0,T]; W_0^{2,2}) \cap L^\infty([0,T]; L^q); \\ &u_t \in L^2([0,T]; H_0^1)\}. \end{aligned}$$

Now we state the main results of this paper.

Theorem 2.1. *Assume $u_0 \in H_0^2$. Let $\gamma_1 < 0$, $\gamma_3 > 0$. Then for any $T > 0$, the problem (1.1)–(1.3) admits a weak solution $u \in E$, provided that $p < q$ and $|\gamma_1|$ is large enough, or $p > q$.*

Theorem 2.2. *Assume $u_0 \in H_0^2$. Let $\gamma_1 > 0$, $\gamma_3 > 0$. If further $\gamma_2 > 0$, $\gamma_4 > 0$, then for any $T > 0$, the problem (1.1)–(1.3) admits a weak solution $u \in E$, provided that γ_1, γ_3 are small enough; while if $\gamma_2 < 0$, $\gamma_4 < 0$, then for any $T > 0$, the problem (1.1)–(1.3) admits a weak solution, provided that $\gamma_1, |\gamma_2|, \gamma_3, |\gamma_4|$ are small enough.*

Theorem 2.3. *Assume $u_0 \in H_0^2$. Let $\gamma_1 > 0$, $\gamma_3 < 0$. Then for any $T > 0$, the problem (1.1)–(1.3) admits a weak solution $u \in E$, provided that γ_1 is small enough.*

3. Global existence

In this section, we establish the global existence of weak solutions of the problem (1.1)–(1.3) following the Galerkin's method and the potential well method used in [19,20,21].

Set $u_0(x) = \sum_{i=1}^{\infty} c_i \omega_i(x)$ and let

$$u^m(x,t) = \sum_{i=1}^{m} c_i^m(t)\omega_i(x), \qquad u_0^m = \sum_{i=1}^{m} c_i\omega_i(x),$$

where $\{\omega_j(x)\}_{i=1}^{\infty}$ is the standard orthogonal basis in H_0^2, also in L^2, and the coefficients $\{c_i^m(t)\}_{i=1}^m$ satisfy $c_i^m(t) = (u^m, \omega_i)$ with

$$\begin{aligned}(u_t^m, \omega_i) \ &+ \ k_1(D^2u^m, D^2\omega_i) + k_2(Du_t^m, D\omega_i) + (\phi(Du^m), D\omega_i) \\ &+ \ (A(u^m), \omega_i) = 0, \qquad t > 0, \quad i = 1, 2, \ldots, m, \end{aligned} \tag{3.1}$$

$$u^m(x, 0) = u_0^m \to u_0 \text{ strongly in } H_0^2 \text{ as } m \to \infty. \tag{3.2}$$

From the standard theory, the problem (3.1), (3.2) admits a solution on some interval $(0, T_m)$ for each m. And the estimates we show allow taking $T_m = T$ for all m. Multiplying (3.1) by $\mathrm{d}/\mathrm{d}t(c_i^m(t))$ and then summing on i from 1 up to m yields

$$\|u_t^m\|_2^2 + k_2\|Du_t^m\|_2^2 + \frac{\mathrm{d}}{\mathrm{d}t}B_m(t) = 0, \tag{3.3}$$

where

$$B_m(t) = \frac{k_1}{2}\|D^2u^m\|_2^2 + \int_0^1\int_0^{Du^m}\phi(s)\mathrm{d}s\mathrm{d}x + \int_0^1\int_0^{u^m}A(s)\mathrm{d}s\mathrm{d}x, \tag{3.4}$$

with

$$B_m(0) = \frac{k_1}{2}\|D^2u_0^m\|_2^2 + \int_0^1\int_0^{Du_0^m}\phi(s)\mathrm{d}s\mathrm{d}x + \int_0^1\int_0^{u_0^m}A(s)\mathrm{d}s\mathrm{d}x. \tag{3.5}$$

Proof of Theorem 2.1. We first consider the case $p < q$ and $|\gamma_1|$ is large enough. For $\phi(s) = -\gamma_1|s|^{p-2}s + \gamma_2 s$, it is obvious that

$$\phi(s) \geq -\frac{\gamma_1}{2}|s|^{p-1} \text{ with } s > 0 \text{ and } \gamma_2 > 0, \text{ or } |s| \geq \left|\frac{2\gamma_2}{\gamma_1}\right|^{1/(p-2)} \text{ and } \gamma_2 < 0,$$

$$-\phi(s) \geq -\frac{\gamma_1}{2}|s|^{p-1} \text{ with } s < 0 \text{ and } \gamma_2 > 0, \text{ or } |s| \geq \left|\frac{2\gamma_2}{\gamma_1}\right|^{1/(p-2)} \text{ and } \gamma_2 < 0,$$

which leads to

$$|s|^{p-1} \leq \frac{2\phi(s)}{-\gamma_1} + \left|\frac{2\gamma_2}{\gamma_1}\right|^{(p-1)/(p-2)} \quad \text{with } s \geq 0;$$

$$|s|^{p-1} \leq \frac{2\phi(s)}{\gamma_1} + \left|\frac{2\gamma_2}{\gamma_1}\right|^{(p-1)/(p-2)} \quad \text{with } s < 0.$$

Hence for any $s \in \mathbb{R}$, by using the Young inequality, we have

$$\begin{aligned}\frac{|s|^p}{p} = \left|\int_0^s |\tau|^{p-1}\mathrm{d}\tau\right| \ &\leq \ -\frac{2}{\gamma_1}\int_0^s \phi(\tau)\mathrm{d}\tau + \left|\frac{2\gamma_2}{\gamma_1}\right|^{(p-1)/(p-2)}|s| \\ &\leq \ -\frac{2}{\gamma_1}\int_0^s \phi(\tau)\mathrm{d}\tau + \frac{|s|^p}{2p} \\ &\quad + \frac{2^{1/(p-1)}(p-1)}{p}\left|\frac{2\gamma_2}{\gamma_1}\right|^{p/(p-2)},\end{aligned}$$

which implies that

$$\int_0^1 \int_0^{Du^m} \phi(s)\mathrm{d}s \geq \frac{-\gamma_1}{4p} \|Du^m\|_p^p - C_1, \tag{3.6}$$

where $C_1 = \dfrac{-\gamma_1 2^{1/(p-1)}(p-1)}{2p} \left|\dfrac{2\gamma_2}{\gamma_1}\right|^{p/(p-2)}$. As for $A(s) = -\gamma_3|s|^{q-2}s + \gamma_4 s$, by the Young inequality, there holds

$$\begin{aligned} \int_0^1 \int_0^{u^m} A(s)\mathrm{d}s\mathrm{d}x &= -\frac{\gamma_3}{q}\int_0^1 |u^m|^q \mathrm{d}x + \frac{\gamma_4}{2}\int_0^1 |u^m|^2 \mathrm{d}x \\ &\geq -\left(\frac{\gamma_3}{q} + \frac{|\gamma_4|}{2}\right)\|u^m\|_q^q - C_2, \end{aligned} \tag{3.7}$$

where $C_2 = \dfrac{(q-2)|\gamma_4|}{2q} 2^{2/(q-2)}$. Substituting (3.6) and (3.7) into (3.4), we get

$$B_m(t) \geq \frac{k_1}{2}\int_0^1 |D^2u^m|^2 \mathrm{d}x + J(u^m) - C_1 - C_2, \tag{3.8}$$

where

$$\begin{aligned} J(u^m) &= \frac{\sqrt{|\gamma_1|}}{4p}\sqrt{|\gamma_1|}\|Du^m\|_p^p - \left(\frac{1}{q} + \frac{|\gamma_4|}{2q\gamma_3}\right)\gamma_3\|u^m\|_q^q \\ &= E_2 I(u^m) + (E_1 - E_2)\sqrt{|\gamma_1|}\|Du^m\|_p^p, \end{aligned}$$

with

$$\begin{aligned} I(u^m) &= \sqrt{|\gamma_1|}\|Du^m\|_p^p - \gamma_3\|u^m\|_q^q, \qquad E_1 = \frac{\sqrt{|\gamma_1|}}{4p}, \\ E_2 &= \left(\frac{1}{q} + \frac{|\gamma_4|}{2q\gamma_3}\right). \end{aligned}$$

By the assumption that $|\gamma_1|$ is large enough, we have that $d = E_1 - E_2 = \dfrac{\sqrt{|\gamma_1|}}{4p} - \left(\dfrac{1}{q} + \dfrac{|\gamma_4|}{2q\gamma_3}\right) > 0$.

Integrating (3.3) from 0 to t and substituting (3.8) into the resulting expression, we arrive at

$$\begin{aligned} &\int_0^t \int_0^1 |u_t^m|^2 \mathrm{d}x\mathrm{d}t + k_2\int_0^t \int_0^1 |Du_t^m|^2 \mathrm{d}x\mathrm{d}t \\ &\quad + \frac{k_1}{2}\int_0^1 |D^2u^m|^2 \mathrm{d}x + J(u^m) - C_1 - C_2 \leq B_m(0). \end{aligned} \tag{3.9}$$

By the integral mean value theorem, the Hölder inequality and (1.4), we have

$$\begin{aligned} &\left|\int_0^1 \int_0^{Du_0^m} \phi(s)\mathrm{d}s\mathrm{d}x - \int_0^1 \int_0^{Du_0} \phi(s)\mathrm{d}s\mathrm{d}x\right| = \left|\int_0^1 \phi(\xi)(Du_0^m - Du_0)\mathrm{d}x\right| \\ &\quad \leq \left(\int_0^1 (|\gamma_1| + |\gamma_2|)^2(1 + |\xi|^{p-1})\mathrm{d}x\right)^{1/2} \|Du_0^m - Du_0\|_2, \end{aligned} \tag{3.10}$$

$$\left|\int_0^1\int_0^{u_0^m}A(s)\mathrm{d}s\mathrm{d}x-\int_0^1\int_0^{u_0}A(s)\mathrm{d}s\mathrm{d}x\right|=\left|\int_0^1 A(\eta)(u_0^m-u_0)\mathrm{d}x\right|$$
$$\leq\left(\int_0^1(|\gamma_3|+|\gamma_4|)^2(1+|\eta|^{q-1})\mathrm{d}x\right)^{1/2}\|u_0^m-u_0\|_2, \tag{3.11}$$

where $\xi = Du_0+\theta(Du_0^m-Du_0)$, $0<\theta<1$, $\eta=u_0+\bar\theta(u_0^m-u_0)$, $0<\bar\theta<1$. Therefore, combining (3.10) and (3.11) with (3.2) leads to $B_m(0)\to B(0)$ as $m\to\infty$, where

$$\begin{aligned}B(0) &= \frac{k_1}{2}\int_0^1|D^2u_0|^2\mathrm{d}x+\int_0^1\int_0^{Du_0}\phi(s)\mathrm{d}s\mathrm{d}x+\int_0^1\int_0^{u_0}A(s)\mathrm{d}s\mathrm{d}x\\ &= \frac{k_1}{2}\int_0^1|D^2u_0|^2\mathrm{d}x+\int_0^1\frac{-\gamma_1}{p}|Du_0|^p\mathrm{d}x+\int_0^1\frac{\gamma_2}{2}|Du_0|^2\mathrm{d}x\\ &\quad-\int_0^1\frac{\gamma_3}{q}|u_0|^q\mathrm{d}x+\int_0^1\frac{\gamma_4}{2}|u_0|^2\mathrm{d}x.\end{aligned}$$

Since $|\gamma_1|$ is large enough, there holds $B(0)>0$. Without loss of generality, we suppose $B_m(0)<2B(0)$ for all m. Thus (3.9) implies that, for all m,

$$\int_0^t\int_0^1(|u_t^m|^2+k_2|Du_t^m|^2)\mathrm{d}x\mathrm{d}t+\frac{k_1}{2}\int_0^1|D^2u^m|^2\mathrm{d}x$$
$$+E_2I(u^m)+d\sqrt{|\gamma_1|}\|Du^m\|_p^p<2B(0)+C_1+C_2. \tag{3.12}$$

Define the potential well

$$W=\{u\in W_0^{1,p}\cap H_0^2|I(u)=\sqrt{|\gamma_1|}\|Du\|_p^p-\gamma_3\|u\|_q^q>0\}\cup\{0\},$$

and choose $|\gamma_1|$ large enough such that $u_0\in W$ and $I(u_0)>0$. By (3.2), we have $I(u_0^m)\to I(u_0)$ as $m\to\infty$. Without loss of generality, let $I(u_0^m)>0$ and $u_0^m\in W$ for all m. We deduce that for all m, $u^m(t)\in W$ and $I(u^m(t))>0$, $t>0$. If there exists a $T>0$ such that $u^m(t)\in W$, $t\in[0,T)$, while $u^m(T)\in\partial W$, namely $I(u^m(T))=0$ for some m, then by (3.12), we have

$$d\sqrt{|\gamma_1|}\|Du^m\|_p^p<2B(0)+C_1+C_2,\quad t\in[0,T],$$
$$\frac{d}{2}(\sqrt{|\gamma_1|}\|Du^m\|_p^p+\gamma_3\|u^m\|_q^q)<2B(0)+C_1+C_2,\quad t\in[0,T].$$

If we choose $|\gamma_1|$ large enough such that

$$B(0)<\left(\frac{\sqrt{|\gamma_1|}}{\gamma_3C_*}\right)^{p/(q-p)}\frac{d\sqrt{|\gamma_1|}}{4}-\frac{C_1}{2}-\frac{C_2}{2},\quad t\in[0,T],$$

where C_* is the constant in the Sobolev imbedding theorem, then from the Sobolev imbedding theorem, we have

$$\gamma_3\|u^m\|_q^q\leq\gamma_3C_*\|Du^m\|_p^{q-p}\|Du^m\|_p^p<\frac{2C_*}{d\sqrt{|\gamma_1|}}$$
$$\times(2B(0)+C_1+C_2)^{(q-p)/p}\|Du^m\|_p^p<\sqrt{|\gamma_1|}\|Du^m\|_p^p,\quad t\in[0,T]$$

which indicates that

$$I(u^m(T)) > 0. \tag{3.13}$$

That is a contradiction.

Hence, it follows from (3.12) and (3.13) that

$$\int_0^t \int_0^1 (|u_t^m|^2 + k_2|Du_t^m|^2)\mathrm{d}x\mathrm{d}t + \frac{k_1}{2}\int_0^1 |D^2u^m|\mathrm{d}x$$
$$+ \frac{d}{2}\left(\sqrt{|\gamma_1|}\|Du^m\|_p^p + \gamma_3\|u^m\|_q^q\right) < 2B(0) + C_1 + C_2,$$

from which we can deduce that the solution u^m of (3.1), (3.2) exists on $[0, T]$ for each m and we can extract a subsequence from $\{u^m\}$, supposed to be $\{u^m\}$ itself, and a function $u \in E$, such that for any $T > 0$,

$$\begin{aligned}
&u^m \to u \quad \text{weakly * in } L^\infty([0,T]; W_0^{1,p}) \cap L^\infty([0,T]; W_0^{2,2}),\\
&u^m \to u \quad \text{weakly * in } L^\infty([0,T]; L^q),\\
&u_t^m \to u_t \quad \text{weakly in } L^2([0,T]; H_0^1),\\
&D^2u^m \to D^2u \quad \text{weakly * in } L^\infty([0,T]; L^2),\\
&u^m \to u \quad \text{weakly * in } L^\infty([0,T]; L^p),\\
&Du^m \to Du \quad \text{weakly * in } L^\infty([0,T]; L^p),\\
&u^m \to u \quad \text{strongly in } L^\infty([0,T]; L^2),\\
&Du^m \to Du \quad \text{strongly in } L^\infty([0,T]; L^2).
\end{aligned}$$

By the Sobolev imbedding theorem (see Theorem 4.12 of [1]), we know that u^m and u belong to L^∞ $([0, T]; C_B^1(0, 1))$, where $C_B^j(0, 1)$ is defined to consist of those functions $f \in C^j(0, 1)$ for which $D^\alpha f$ is bounded on $(0, 1)$ for $0 \le |\alpha| \le j$. $C_B^j(0, 1)$ is a Banach space with norm given by

$$\|f\|_{C_B^j} = \max_{0\le|\alpha|\le j} \sup_{x\in\Omega} |D^\alpha f|.$$

From the continuity of $\phi(s)$ and $A(s)$, we further have

$$\begin{aligned}
&\int_0^1 |\phi(Du^m) - \phi(Du)|^2\mathrm{d}x\\
&\quad = \int_0^1 |-\gamma_1(|Du^m|^{p-2}Du^m - |Du|^{p-2}Du) + \gamma_2(Du^m - Du)|^2\mathrm{d}x\\
&\quad \le 2\gamma_1^2 \int_0^1 ||Du^m|^{p-2}Du^m - |Du|^{p-2}Du|^2\mathrm{d}x + 2\gamma_2^2\int_0^1 |Du^m - Du|^2\mathrm{d}x\\
&\quad = 2\gamma_1^2\int_0^1 (p-1)^2|\rho_1 Du^m + (1-\rho_1)Du|^{2(p-2)}|Du^m - Du|^2\mathrm{d}x\\
&\qquad + 2\gamma_2^2\int_0^1 |Du^m - Du|^2\mathrm{d}x\\
&\quad \le C\int_0^1 |Du^m - Du|^2\mathrm{d}x
\end{aligned}$$

and

$$
\begin{aligned}
&\int_0^1 |A(u^m)-A(u)|^2\mathrm{d}x \\
&\quad = \int_0^1 |-\gamma_3(|u^m|^{q-2}u^m-|u|^{q-2}u)+\gamma_4(u^m-u)|^2\mathrm{d}x \\
&\quad \le 2\gamma_3^2\int_0^1 ||u^m|^{q-2}u^m-|u|^{q-2}u|^2\mathrm{d}x+2\gamma_4^2\int_0^1 |u^m-u|^2\mathrm{d}x \\
&\quad = 2\gamma_3^2\int_0^1 (q-1)^2|\rho_2u^m+(1-\rho_2)u|^{2(q-2)}|u^m-u|^2\mathrm{d}x \\
&\qquad +2\gamma_4^2\int_0^1 |u^m-u|^2\mathrm{d}x \\
&\quad \le C\int_0^1 |u^m-u|^2\mathrm{d}x,
\end{aligned}
$$

where ρ_1 and ρ_2 are constants in [0, 1], C is a constant which depends on γ_1, γ_2, γ_3, γ_4, p, q and $\|u^m\|_{L^\infty([0,T];C_B^1)}$, $\|u\|_{L^\infty([0,T];C_B^1)}$. Thus we can deduce that

$$
\begin{aligned}
&\phi(Du^m)\to\phi(Du) \quad \text{strongly in } L^\infty([0,T];L^2), \\
&A(u^m)\to A(u) \quad \text{strongly in } L^\infty([0,T];L^2).
\end{aligned}
$$

Letting $m\to\infty$, we deduce from the density of $\{\omega_i(x)\}_{i=1}^k$ that u is a global weak solution of the problem (1.1)–(1.3).

In the following, we prove the case $p>q$, for which we consider the equivalent equation

$$
\frac{\partial u}{\partial t}+k_1D^4u-k_2\frac{\partial D^2u}{\partial t}-D\tilde{\phi}(Du)-K_0D^2u+A(u)=0, \tag{3.14}
$$

where $\tilde{\phi}(s)=\phi(s)-K_0s$ with $K_0\le\gamma_2$ such that $\tilde{\phi}'(s)=-\gamma_1(p-1)|s|^{p-2}+\gamma_2-K_0\ge 0$. Similar as above, here we need to do some *a priori* estimates for the following equation:

$$
\|u_t^m\|_2^2+k_2\|Du_t^m\|_2^2+\frac{\mathrm{d}}{\mathrm{d}t}\tilde{B}_m(t)=0, \tag{3.15}
$$

where

$$
\begin{aligned}
\tilde{B}_m(t) \;=\; & \frac{k_1}{2}\|D^2u^m\|_2^2+K_0\|Du^m\|_2^2+\int_0^1\int_0^{Du^m}\tilde{\phi}(s)\mathrm{d}s\mathrm{d}x \\
& +\int_0^1\int_0^{u^m}A(s)\mathrm{d}s\mathrm{d}x.
\end{aligned}
$$

For $\tilde{\phi}(s)=-\gamma_1|s|^{p-2}s+\gamma_2s-K_0s$, after some simple calculations, we find

$$
|\tilde{\phi}(s)|\le C_3(1+|s|^{p-1}), \qquad \text{where } C_3=|\gamma_1|+|\gamma_2|+|K_0|, \tag{3.16}
$$

$$
|\tilde{\phi}(s)|\ge-\frac{\gamma_1}{4}|s|^{p-1}, \qquad \text{if } |s|\ge M_1=\max\left\{\left|\frac{2\gamma_2}{\gamma_1}\right|^{1/(p-2)},\left|\frac{4K_0}{\gamma_1}\right|^{1/(p-2)}\right\}. \tag{3.17}
$$

Therefore $|s|^{p-1} \leq -\frac{4}{\gamma_1}|\tilde{\phi}(s)| + M_1^{p-1}$ for $s \in \mathbb{R}$, which implies that when $s < 0$, $|s|^{p-1} \leq -\frac{4}{\gamma_1}\tilde{\phi}(s) + M_1^{p-1}$ and

$$\int_s^0 |\tau|^{p-1}\mathrm{d}\tau \leq \frac{4}{\gamma_1}\int_0^s \tilde{\phi}(s)\mathrm{d}\tau + M_1^{p-1}|s|.$$

Using the Young inequality, we have, for any $s \in \mathbb{R}$,

$$\frac{|s|^p}{p} \leq \frac{4}{\gamma_1}\int_0^s \tilde{\phi}(\tau)\mathrm{d}\tau + \frac{|s|^p}{2p} + \frac{2^{1/(p-1)(p-1)}}{p}M_1^{p-1}. \tag{3.18}$$

Using the Poincaré inequality and the Young inequality, we get

$$\begin{aligned}\int_0^1\int_0^{Du^m} \tilde{\phi}(s)\mathrm{d}s\mathrm{d}x &\geq \int_0^1 \frac{\gamma_1}{8p}|Du^m|^p\mathrm{d}x - \frac{2^{1/(p-1)(p-1)}}{4p}\gamma_1 M_1^{p-1}\\ &\geq C_4\|u^m\|_{1,p}^p - C_5,\end{aligned} \tag{3.19}$$

$$\begin{aligned}\int_0^1\int_0^{Du_0^m} \tilde{\phi}(s)\mathrm{d}s\mathrm{d}x &\leq \int_0^1\int_0^{Du_0^m} C_3(1+|s|^{p-1})\mathrm{d}s\mathrm{d}x\\ &\leq \int_0^1\Bigg(\frac{C_3|Du_0^m|^p}{2p} + \frac{2^{1/(p-1)}(p-1)C_3}{p}\\ &\qquad + \frac{C_3}{p}|Du_0^m|^p\Bigg)\mathrm{d}x,\\ &\leq C_6(1+\|u_0^m\|_{1,p}^p),\end{aligned} \tag{3.20}$$

where $C_4 = \frac{\gamma_1}{8p}$, $C_5 = \frac{2^{1/(p-1)(p-1)}}{4p}\gamma_1 M_1^{p-1}$, $C_6 = \max\left\{\frac{3C_3}{2p}, \frac{2^{p/(p-1)}(p-1)C_3}{p}\right\}$.
For $A(s) = -\gamma_3|s|^{q-2}s + \gamma_4 s$, since $A(s) \geq -C_7(1+|s|^{q-1})$, where $C_7 = |\gamma_3| + |\gamma_4|$, then by the Young inequality, we have

$$\begin{aligned}\int_0^1\int_0^{u^m} A(s)\mathrm{d}s\mathrm{d}x &\geq -\int_0^1\int_0^{u^m} C_7(1+|s|^{q-1})\mathrm{d}s\mathrm{d}x\\ &\geq -C_7\int_0^1\Bigg[\frac{2\varepsilon_1}{p}|u^m|^p + \frac{p-1}{p}\varepsilon_1^{-1/(p-1)}\\ &\qquad + \frac{p-q}{pq}\varepsilon_1^{-q/(p-q)}\Bigg]\mathrm{d}x\\ &\geq -\frac{C_4}{2}\|u^m\|_{1,p}^p - C_8,\end{aligned} \tag{3.21}$$

where $C_8 = C_7\left[\frac{p-1}{p}\varepsilon_1^{-1/(p-1)} + \frac{p-q}{pq}\varepsilon_1^{-q/(p-q)}\right]$ and ε_1 is small enough such that $\frac{2C_7\varepsilon_1}{p} < \frac{C_4}{2}$. Further, the Young inequality also leads to

$$\begin{aligned}\int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{-K_0}{2}|Du^m|^2\mathrm{d}x\mathrm{d}t &= \frac{-K_0}{2}\int_0^1 |Du^m|^2\mathrm{d}x + \frac{K_0}{2}\int_0^1 |Du_0^m|^2\mathrm{d}x\\ &\le \left|\frac{K_0\varepsilon_2}{p}\right| \|u^m\|_{1,p}^p + \frac{|K_0|(p-2)}{4}\varepsilon_2^{-4/(p(p-2))}\\ &\quad + \frac{|K_0|}{2}\|Du_0^n\|_2^2.\end{aligned}$$

Set $C_9 = \left|\frac{K_0\varepsilon_2}{p}\right|$, $C_{10} = \frac{|K_0|(p-2)}{4}\varepsilon_1^{-4/(p(p-2))} + \frac{|K_0|}{2}\|Du_0^n\|_2^2$. Then integrating (3.15) over $(0,t)$ and substituting (3.19)–(3.21) into the resulting expression, we have

$$\begin{aligned}&\int_0^t\int_0^1 |u_t^m|^2\mathrm{d}x\mathrm{d}t + k_2\int_0^t\int_0^1 |Du_t^m|^2\mathrm{d}x\mathrm{d}t\\ &\quad + \frac{k_1}{2}\int_0^1 |D^2u^m|^2\mathrm{d}x + C_4\|u^m\|_{1,p}^p - C_5 - \frac{C_4}{2}\|u^m\|_{1,p}^p - C_8\\ &\le C_6(1+\|u^m\|_{1,p}^p) + \int_0^1\int_0^{u_0^m} A(s)\mathrm{d}s\mathrm{d}x + C_9\|u^m\|_{1,p}^p + C_{10}\\ &\quad + \frac{k_1}{2}\|D^2u_0^m\|_2^2,\end{aligned}$$

which indicates the following if choosing ε_2 small enough such that $C_9 = \frac{C_4}{4}$,

$$\begin{aligned}&\int_0^t\int_0^1 |u_t^m|^2\mathrm{d}x\mathrm{d}t + k_2\int_0^t\int_0^1 |Du_t^m|^2\mathrm{d}x\mathrm{d}t + \frac{k_1}{2}\int_0^1 |D^2u^m|^2\mathrm{d}x + \frac{C_4}{8}\|u^m\|_{1,p}^p\\ &\le C_6(1+\|u\|_{1,p}^p) + \int_0^1\int_0^{u_0^m} A(s)\mathrm{d}s\mathrm{d}x + C_{10} + \frac{k_1}{2}\|D^2u_0^m\|_2^2 + C_5 + C_8.\end{aligned}$$

Actually, since $p > q$, we can choose ε_1 small enough such that $\int_0^1\int_0^{u_0^m} A(s)\mathrm{d}s\mathrm{d}x + C_8 > 0$. Thus, there exist a subsequence of $\{u^m\}$, supposed to be $\{u^m\}$ itself, and a function $u \in E$ such that

$$\begin{aligned}&u^m \to u \quad \text{weakly * in } L^\infty([0,T]; W_0^{1,p}) \cap L^\infty([0,T]; W_0^{2,2}),\\ &u_t^m \to u_t \quad \text{weakly in } L^2([0,T]; H_0^1),\\ &D^2u^m \to D^2u \quad \text{weakly * in } L^\infty([0,T]; L^2),\\ &u^m \to u \quad \text{weakly * in } L^\infty([0,T]; L^p),\\ &Du^m \to Du \quad \text{weakly * in } L^\infty([0,T]; L^p),\\ &u^m \to u \quad \text{strongly in } L^\infty([0,T]; L^2),\\ &Du^m \to Du \quad \text{strongly in } L^\infty([0,T]; L^2),\\ &\tilde{\phi}(Du^m) \to \tilde{\phi}(Du) \quad \text{strongly in } L^\infty([0,T]; L^2),\\ &A(u^m) \to A(u) \quad \text{strongly in } L^\infty([0,T]; L^2).\end{aligned}$$

Let $m \to \infty$ and use the density of $\{\omega_i(x)\}_{i=1}^{k}$. Then u is a global weak solution of the problem (1.1)–(1.3). □

Proof of Theorem 2.2. First we consider the case $\gamma_1 > 0$, $\gamma_3 > 0$, $\gamma_2 < 0$ and $\gamma_4 < 0$. After a simple calculation, we have that

$$B_m(t) \geq \frac{k_1}{2}\int_0^1 |D^2u^m|^2 \mathrm{d}x - M_2(\|Du^m\|_p^p + 2\|Du^m\|_2^2 + \|Du^m\|_q^q),$$

where $M_2 = \max\left\{\frac{\gamma_1}{p}, \frac{|\gamma_2|}{2}, \frac{\gamma_3}{q}, \frac{|\gamma_4|}{2}\right\}$. Set

$$\begin{aligned} J(u^m) &= \frac{k_1}{4}\|D^2u^m\|_2^2 - M_2^{1-a}M_2^a(\|Du^m\|_p^p + 2\|Du^m\|_2^2 + \|Du^m\|_q^q) \\ &= E_4 I(u^m) + (E_3 - E_4)\|D^2u^m\|_2^2 \end{aligned}$$

with

$$I(u^m) = \|D^2u^m\|_2^2 - M_2^a(\|Du^m\|_p^p + 2\|Du^m\|_2^2 + \|Du^m\|_q^q),$$

where $E_3 = \frac{k_1}{4}$, $E_4 = M_2^{1-a}$, $a \in (0, 1)$. Since now $\gamma_1, |\gamma_2|, \gamma_3, |\gamma_4|$ are small enough, then we can suppose that $B(0) > 0$ and $d = E_3 - E_4 > 0$. As mentioned in the proof of Theorem 2.1, here u satisfies

$$\begin{aligned} &\int_0^t \int_0^1 (|u_t^m|^2 + k_2|Du_t^m|^2)\mathrm{d}x\mathrm{d}t + \frac{k_1}{2}\int_0^1 |D^2u^m|^2 \mathrm{d}x \\ &\quad + E_4 I(u^m) + d\|D^2u^m\|_2^2 < 2B(0). \end{aligned} \tag{3.22}$$

The main purpose of the following is to prove that for all $t > 0$, u^m belongs to the set

$$\begin{aligned} W &= \big\{u \in W_0^{1,p} \cap H_0^2 | I(u) = \|D^2u\|_2^2 \\ &\qquad - M_2^a(\|Du\|_p^p + 2\|Du\|_2^2 + \|Du\|_q^q) > 0\} \cup \{0\big\}, \end{aligned}$$

which already contains u_0 and u_0^m if $\gamma_1, |\gamma_2|, \gamma_3, |\gamma_4|$ are small enough. If not, namely there exists a $T > 0$ such that for some m, $I(u^m(T)) = 0$. Then from (3.22), we have

$$M_2^a(\|Du^m\|_p^p + 2\|Du^m\|_2^2 + \|Du^m\|_q^q) \leq \|D^2u^m\|_2^2 < \frac{2}{d}B(0), \ \ t \in [0, T],$$

from which and the Sobolev imbedding theorem, we have

$$\begin{aligned} M_2^a\|Du^m\|_p^p &\leq CM_2^a(\|D^2u^m\|_2^2 + \|Du^m\|_2^2)^{p/2} \\ &\leq CM_2^a(1 + M_2^{-a})^{p/2}\left(\frac{2}{d}B(0)\right)^{(p-2)/2}\|D^2u^m\|_2^2. \end{aligned}$$

In the same way, we have

$$M_2^a\|Du^m\|_q^q \leq CM_2^a(1 + M_2^{-a})^{q/2}\left(\frac{2}{d}B(0)\right)^{(q-2)/2}\|D^2u^m\|_2^2.$$

Then choosing $\gamma_1, |\gamma_2|, \gamma_3, |\gamma_4|$ small enough, such that

$$CM_2^a(1+M_2^{-a})^{p/2}\left(\frac{2}{d}B(0)\right)^{(p-2)/2} + CM_2^a(1+M_2^{-a})^{q/2}\left(\frac{2}{d}B(0)\right)^{(q-2)/2} + 2M_2^a < 1,$$

we have that $I(u^m(T)) > 0$, which is a contradiction. Via the same process of the proof of Theorem 2.1, there exists a global weak solution.

When $\gamma_1 > 0$, $\gamma_3 > 0$, $\gamma_2 > 0$ and $\gamma_4 > 0$, then set

$$\begin{aligned} J(u^m) &= M_3(\|D^2u^m\|_2^2 + \|Du^m\|_2^2 + \|u^m\|_2^2) \\ &\quad - M_4^{1-a}M_4^a(\|Du^m\|_p^p + \|Du^m\|_q^q) \\ &= E_5 I(u^m) + (E_5 - E_6)(\|D^2u^m\|_2^2 + \|Du^m\|_2^2 + \|u^m\|_2^2), \end{aligned}$$

where $M_3 = \min\left\{\frac{k_1}{4}, \frac{\gamma_2}{2}, \frac{\gamma_4}{2}\right\}$, $M_4 = \max\left\{\frac{\gamma_1}{p}, \frac{\gamma_3}{q}\right\}$, $E_5 = \min\left\{\frac{k_1}{4}, \frac{\gamma_2}{2}, \frac{\gamma_4}{2}\right\}$, $E_6 = M_4^{1-a}$, $0 < a < 1$ and

$$I(u^m) = \|D^2u^m\|_2^2 + \|Du^m\|_2^2 + \|u^m\|_2^2 - M_4^a(\|Du^m\|_p^p + \|Du^m\|_q^q).$$

Since now γ_1 and γ_3 are small enough, then $B(0) > 0$ and $d = E_5 - E_6 > 0$. At this time, u satisfies

$$\begin{aligned} &\int_0^t\int_0^1(|u_t^m|^2 + k_2|Du_t^m|^2)\mathrm{d}x\mathrm{d}t + \frac{k_1}{2}\int_0^1|D^2u^m|^2\mathrm{d}x \\ &\quad + E_5I(u^m) + d(\|D^2u^m\|_2^2 + \|Du^m\|_2^2 + \|u^m\|_2^2) < 2B(0). \end{aligned} \tag{3.23}$$

We deduce that for all $t > 0$, u^m belongs to the set

$$\begin{aligned} W &= \big\{u \in W_0^{1,p} \cap H_0^2 | I(u) = \|D^2u\|_2^2 + \|Du\|_2^2 + \|u\|_2^2 \\ &\quad - M_4^a(\|Du\|_p^p + \|Du\|_q^q) > 0\big\} \cup \{0\}, \end{aligned}$$

which already contains u_0 and u_0^m if γ_1 and γ_3 are small enough. If not, namely there exists a $T > 0$ such that for some m, $I(u^m(T)) = 0$. Then from (3.23), we have

$$\|D^2u^m\|_2^2 + \|Du^m\|_2^2 + \|u^m\|_2^2 < \frac{2}{d}B(0),$$

which together with the Sobolev imbedding theorem leads to

$$M_4^a\|D^2u^m\|_p^p \le CM_4^a\left(\frac{2}{d}B(0)\right)^{(p-2)/2}(\|D^2u^m\|_2^2 + \|Du^m\|_2^2),$$

$$M_4^a\|D^2u^m\|_q^q \le CM_4^a\left(\frac{2}{d}B(0)\right)^{(q-2)/2}(\|D^2u^m\|_2^2 + \|Du^m\|_2^2).$$

Then choosing γ_1 and γ_3 small enough, such that

$$CM_4^a\left(\frac{2}{d}B(0)\right)^{(p-2)/2} + CM_4^a\left(\frac{2}{d}B(0)\right)^{(q-2)/2} < 1,$$

we have that $I(u^m(T)) > 0$, which is a contradiction. Via the same process of the proof of Theorem 2.1, there exists a global weak solution. □

Proof of Theorem 2.3. Integrating (3.3) from 0 to t, we arrive at

$$\int_0^t\int_0^1 |u_t^m|^2 \mathrm{d}x\mathrm{d}t + k_2\int_0^t\int_0^1 |Du_t^m|^2 \mathrm{d}x\mathrm{d}t + B_m(t) = B_m(0), \tag{3.24}$$

where $B_m(t)$ and $B_m(0)$ are as in (3.4) and (3.5). In what follows, we estimate $B_m(t)$. First we have

$$\begin{aligned}\int_0^1\int_0^{u^m} A(s)\mathrm{d}s\mathrm{d}x &\ge \int_0^1 \frac{-\gamma_3}{q}|u^m|^q\mathrm{d}x - \int_0^1 \frac{|\gamma_4|}{2}|u^m|^2\mathrm{d}x\\ &\ge \int_0^1\left(\frac{-\gamma_3}{q} - \frac{\varepsilon_3|\gamma_4|}{q}\right)|u^m|^q\mathrm{d}x - \frac{(q-2)|\gamma_4|}{2q\varepsilon_3^{2/(q-2)}}\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}\int_0^1\int_0^{Du^m} \phi(s)\mathrm{d}s &\ge \int_0^1\left(\frac{-\gamma_1}{p}|Du^m|^p - \left|\frac{\gamma_2}{2}\right||Du^m|^2\right)\mathrm{d}x\\ &\ge \int_0^1\left(\frac{-\gamma_1}{p} - \frac{2\varepsilon_4}{p}\left|\frac{\gamma_2}{2}\right|\right)|Du^m|^p\mathrm{d}x - \frac{|\gamma_2|(p-2)}{2p\varepsilon_4^{2/(p-2)}},\end{aligned} \tag{3.26}$$

where ε_3 and ε_4 are constants small enough. By the Sobolev imbedding theorem and the Young inequality, we have

$$\begin{aligned}\int_0^1 |Du^m|^p\mathrm{d}x &\le C\int_0^1 (|D^2u^m|^2 + |Du^m|^2)\mathrm{d}x\\ &\le C\left(\int_0^1 |D^2u^m|^2\mathrm{d}x + \frac{2\varepsilon_4}{p}|Du^m|^p + \frac{p-2}{p\varepsilon_4^{2/(p-2)}}\right),\end{aligned}$$

which leads to

$$\int_0^1 |Du^m|^p\mathrm{d}x \le \frac{Cp}{p-2C\varepsilon_4}\left(\int_0^1 |D^2u^m|^2\mathrm{d}x + \frac{p-2}{p\varepsilon_4^{2/(p-2)}}\right). \tag{3.27}$$

Combining (3.26) with (3.27), we arrive at

$$\begin{aligned}\int_0^1\int_0^{Du^m} \phi(s)\mathrm{d}s &\ge \frac{C(-\gamma_1 - \varepsilon_4|\gamma_2|)}{p-2C\varepsilon_4}\int_0^1 |D^2u^m|^2\mathrm{d}x\\ &\quad + \frac{C(-\gamma_1 - \varepsilon_4|\gamma_2|)}{p-2C\varepsilon_4}\frac{p-2}{p\varepsilon_4^{2/(p-2)}} - \frac{|\gamma_2|(p-2)}{2p\varepsilon_4^{2/(p-2)}}.\end{aligned} \tag{3.28}$$

Since γ_1 is small enough, $B(0) > 0$ and we suppose $B_m(0) < 2B(0)$. Then it follows from (3.24), (3.25) and (3.28) that

$$\int_0^t \int_0^1 |u_t^m|^2 \mathrm{d}x\mathrm{d}t + k_2 \int_0^t \int_0^1 |Du_t^m|^2 \mathrm{d}x\mathrm{d}t + \left[\frac{k_1}{2} - \frac{C(\gamma_1 + \varepsilon_4 |\gamma_2|)}{p - 2C\varepsilon_4}\right] \int_0^1 |D^2 u^m|^2 \mathrm{d}x + \frac{C(-\gamma_1 - \varepsilon_4 |\gamma_2|)}{p - 2C\varepsilon_4} \frac{p-2}{p\varepsilon_4^{2/(p-2)}} - \frac{|\gamma_2|(p-2)}{2p\varepsilon_4^{2/(p-2)}} + \int_0^1 \left(\frac{-\gamma_3}{q} - \frac{\varepsilon_3|\gamma_4|}{q}\right) |u^m|^q \mathrm{d}x - \frac{(q-2)|\gamma_4|}{2q\varepsilon_3^{2/(q-2)}} < 2B(0).$$

Since $\gamma_1 > 0$, $\varepsilon_3 > 0$ and $\varepsilon_4 > 0$ are small enough and $\gamma_3 < 0$, we can choose appropriate coefficients such that

$$\frac{k_1}{2} - \frac{C(\gamma_1 + \varepsilon_4 |\gamma_2|)}{p - 2C\varepsilon_4} > 0, \qquad \frac{-\gamma_3}{q} - \frac{\varepsilon_3|\gamma_4|}{q} > 0.$$

Via the same process of the proof of Theorem 2.1, there exists a global weak solution. □

Acknowledgements

This work is supported by the National Natural Science Foundation of China (Grant Nos 11226179, 11201047), the Doctor Startup Foundation of Liaoning Province (Grant No. 20121025), the Doctor Startup Fund of Dalian Nationalities University, and the Fundamental Research Funds for the Central Universities (Grant No. DUT13LK08). The authors would like to thank Prof. Jingxue Yin of South China Normal University for his helpful suggestions. The authors would also like to express their sincere thanks to the referees for their valuable suggestions which contributed greatly to this work.

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