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The classification of exact travelling wave solutions to two-component Dullin–Gottwald–Holm system

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Abstract. In this study, by making use of the direct integral method and the complete discrimination system for the polynomial method, all the travelling wave solutions to the two-component Dullin–Gottwald–Holm (DGH2) system are obtained, including solitary wave solutions, singular periodic solutions and Jacobian elliptic function double periodic solutions. Some of them are initially given. Moreover, concrete examples are presented to make sure that several solutions can be realised, and the corresponding figures are also given to show their nature. This means every solution in the paper may reflect the corresponding natural phenomenon, such as tidal waves and tsunami waves.

Keywords. Two-component Dullin–Gottwald–Holm system; complete discrimination system for polynomial; direct integral method; travelling wave solution.

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1. Introduction

The shallow water wave equation is a meaningful model that is used to describe the storm tide, tidal waves etc. [1–3]. Scientists found that many other real-world models could also be described by it. Thus, a growing academic interest has been drawn in the extension of this kind of equation [4–6].

Here, we study the two-component Dullin–Gottwald–Holm (DGH2) system

$$\begin{cases} u_t - u_{xxt} - Au_x + 3uu_x - uu_{xxx} - 2u_x u_{xx} \\ \quad + \gamma u_{xxx} + \rho\rho_x = 0, \\ \rho_t + (u\rho)_x = 0, \end{cases} \tag{1}$$

where $u(x,t)$ is the fluid velocity in the x direction (or equivalently the height of the water's free surface above a flat bottom), $\rho(x,t)$ is related to the free surface elevation from equilibrium (or scalar density), the parameter $A (A > 0)$ characterises a linear underlying shear flow propagating in the positive direction of the x-coordinate (or the critical shallow-water speed) and the parameter γ is a constant determining the dispersion effect. The above system is an extension of the DGH equation developed by Dullin, Gottwald and Holm in 2001 [7]. Related results such as well-posedness and stability of this system can be seen in [8–10]. Furthermore, system (1) contains many famous models as specific examples. For example, if $\gamma = 0$ and $\rho = 0$, system (1) becomes the noted Camassa–Holm (CH) equation [11–13]. If $\gamma = 0$ and $\rho \neq 0$, system (1) turns into the two-component CH system [14,15].

System (1) can be used to describe shallow water waves with curl zero. It is applied in ocean exploitation, disaster prevention etc. [16–23]. Thus, constructing exact solutions to it would shed light on the related area. Zhu and Xu gave sufficient conditions for the existence of a strong global solution to system (1) in [24,25]. Cheung [26] constructed some blow-up solutions of system (1) using the perturbation method.

The travelling wave solution mainly describes wave propagations with constant velocity, and so has wide applications in various areas. Different methods have been proposed to obtain such types of solutions [27–29], such as the F-expansion method [30], trial equation method [31–34] and the complete discrimination system for polynomial method (CDSPM) [35–43]. Among these, the complete discrimination system for the polynomials by Liu is more powerful, because it not only can construct all the travelling wave solutions if the original model is reduced to an integral form, but also can be applied to conduct qualitative analysis [44–48].

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So in this paper, we use the CDSPM to system (1), and all the travelling wave solutions, i.e., the classification of travelling wave solutions are obtained. Some solutions, such as Jacobian elliptic function double periodic solutions are obtained, which is difficult to obtain by other methods. This also shows the effectiveness of the method adopted in this paper.

2. Simplify system

By taking the following travelling wave transformation

$$u(x,t)=u(\eta),\quad \rho\rho(x,t)=\rho(\eta),\quad \rho\eta=x-kt, \tag{2}$$

where $k\neq 0$ is a real constant, and then substituting eq. (2) into system (1), we have

$$\begin{cases}(k+\gamma)u'''-uu'''-2u'u''-(A+k)u'\\ \quad +3uu'+\rho\rho'=0,\\ (u\rho)'-k\rho'=0.\end{cases} \tag{3}$$

Integrating (3), once yields

$$\begin{cases}-(u-k-\gamma)u''-\frac{1}{2}(u')^2+\frac{3}{2}u^2-(A+k)u\\ \quad +\frac{1}{2}\rho^2=M,\\ \rho=\frac{N}{u-k},\end{cases} \tag{4}$$

where M and $N\neq 0$ are integral constants. From system (4), we have

$$-(u-k-\gamma)u''-\frac{1}{2}(u')^2+\frac{3}{2}u^2-(A+k)u +\frac{1}{2}\left(\frac{N}{u-k}\right)^2=M. \tag{5}$$

Thus, the following equation can be obtained:

$$u''+\frac{1}{2(u-k-\gamma)}(u')^2 +\frac{\frac{3}{2}u^2-(A+k)u+\frac{1}{2}(\frac{N}{u-k})^2-M}{u-k-\gamma}=0, \tag{6}$$

whose general solution is shown as follows:

$$\pm(\eta-\eta_0)=\int\sqrt{\frac{(u-k)(u-k-\gamma)}{(u-k)^4+B_3(u-k)^3+B_2(u-k)^2+B_1(u-k)+B_0}}\mathrm{d}u, \tag{7}$$

where

$$\begin{aligned}&B_3=2k-A,\\ &B_2=k^2-2kA-2M,\\ &B_1=3k^2A+2kM-2K^3+c_0,\\ &B_0=2k^4-2k^3A-2kc_0-N^2,\end{aligned} \tag{8}$$

and η_0, c_0 are arbitrary constants.

For brevity, by using the transformation $\psi=u-k$, (7) becomes

$$\pm(\eta-\eta_0)=\int\sqrt{\frac{\psi(\psi-\gamma)}{\psi^4+B_3\psi^3+B_2\psi^2+B_1\psi+B_0}}\mathrm{d}\psi. \tag{9}$$

In the following, we shall construct exact solutions to the original equation according to (9).

3. Travelling wave solutions of the system

Case 1. If $B_0=0$, eq. (9) turns into

$$\pm(\eta-\eta_0)=\int\sqrt{\frac{\psi-\gamma}{\psi^3+B_3\psi^2+B_2\psi+B_1}}\mathrm{d}\psi. \tag{10}$$

According to the complete discrimination system of third order

$$\begin{aligned}&\Delta=-27\left(\frac{2B_3^3}{27}+B_1-\frac{B_2B_3}{3}\right)^2-4\left(B_2-\frac{B_3^2}{3}\right)^3,\\ &D_1=B_2-\frac{B_3^2}{3},\end{aligned} \tag{11}$$

four cases can be discussed.

Case 1.1. If $\Delta=0$, $D_1<0$, then we get $F(\psi)=(\psi-\alpha)^2(\psi-\beta)$, where $\alpha\neq\beta$. By the substitution

$$\vartheta^2=\frac{\psi-\gamma}{\psi-\beta},$$

that is,

$$\vartheta^2=\frac{u-k-\gamma}{u-k-\beta},$$

we can obtain

$$\pm(\eta-\eta_0)=\ln\left|\frac{\vartheta+1}{\vartheta-1}\right|+\sqrt{\frac{\alpha-\gamma}{\alpha-\beta}}\ln\left|\frac{\vartheta-\sqrt{\frac{\alpha-\gamma}{\alpha-\beta}}}{\vartheta+\sqrt{\frac{\alpha-\gamma}{\alpha-\beta}}}\right|, \tag{12}$$

where

$$\frac{\alpha-\gamma}{\alpha-\beta}>0.$$

$$\pm(\eta-\eta_0)=\ln\left|\frac{\vartheta+1}{\vartheta-1}\right|-2\sqrt{\frac{\alpha-\gamma}{\beta-\alpha}}\arctan\left(\vartheta\sqrt{\frac{\beta-\alpha}{\alpha-\gamma}}\right), \tag{13}$$

where

$$\frac{\alpha-\gamma}{\alpha-\beta}<0.$$

Case 1.2. If $\Delta=0$, $D_1=0$, then we get $F(\psi)=(\psi-\alpha)^3$. Thus, the solution is as follows:

$$\pm\frac{1}{2}(\eta-\eta_0)=\pm\sqrt{\frac{u-k-\gamma}{u-k-\alpha}}+\frac{1}{2}\ln\left|\frac{\sqrt{\frac{u-k-\gamma}{u-k-\alpha}}\mp1}{\sqrt{\frac{u-k-\gamma}{u-k-\alpha}}\pm1}\right|. \tag{14}$$

Case 1.3. If $\Delta>0, D_1<0$, then $F(\psi)=(\psi-\alpha)(\psi-\beta)(\psi-\delta)$, where $\alpha>\beta>\delta$. Using the transformation

$$\vartheta^2=\frac{\psi-\gamma}{\psi-\alpha},$$

that is,

$$\vartheta^2=\frac{u-k-\gamma}{u-k-\alpha},$$

we deduce that

$$\pm\frac{\sqrt{(\alpha-\beta)(\alpha-\delta)}}{2(\alpha-\gamma)}(\eta-\eta_0)=\int\left\{1+\frac{1}{2}\left(\frac{1}{\vartheta-1}-\frac{1}{\vartheta+1}\right)\right\}\times\frac{1}{\sqrt{(\vartheta^2+a_0)(\vartheta^2+b_0)}}\mathrm{d}\vartheta, \tag{15}$$

where

$$a_0=\frac{\beta-\gamma}{\alpha-\beta},\quad b_0=\frac{\delta-\gamma}{\alpha-\delta}.$$

Case 1.4. If $\Delta<0$, we have $F(\psi)=(\psi-\alpha)(\psi^2+l_1\psi+s_1)$, where $l_1^2-4s_1<0$. Let

$$\psi=\frac{\alpha\vartheta-\gamma}{\vartheta-1}.$$

Then, eq. (9) becomes

$$\pm\frac{1}{\alpha-\gamma}(\eta-\eta_0)=\int\left(1+\frac{1}{\vartheta-1}\right)\times\frac{1}{\sqrt{\vartheta(a_0\vartheta^2+b_0\vartheta+d_0)}}\mathrm{d}\vartheta, \tag{16}$$

where $a_0=l_1\alpha+s_1+\alpha^2, b_0=-\gamma(2\alpha+l_1)-l_1\alpha-2s_1$ and $d_0=\gamma^2+l_1\gamma+s_1$.

Additionally, based on the above analysis, the solutions of Cases 1.3 and 1.4 can be represented by the elliptic function of the first and third types, respectively.

Case 2. If $B_0\neq0$, let $\psi_1=\psi+\frac{1}{4}B_3$. Then, eq. (9) becomes

$$\pm(\eta-\eta_0)=\int\sqrt{\frac{(\psi_1-\frac{1}{4}B_3)(\psi_1-\frac{1}{4}B_3-\gamma)}{\psi_1^4+P_2\psi_1^2+P_1\psi_1+P_0}}\mathrm{d}\psi_1, \tag{17}$$

where

$$P_2=B_2-\frac{3}{8}B_3^2,$$
$$P_1=B_1+\frac{1}{8}B_3^3-\frac{1}{2}B_2B_3,$$
$$P_0=B_0+\frac{1}{16}B_2B_3^2-\frac{1}{4}B_1B_3-\frac{1}{128}B_3^4.$$

We denote $F(\psi_1)=\psi_1^4+P_2\psi_1^2+P_1\psi_1+P_0$. Then the complete discrimination system of the fourth order is given as follows:

$$\begin{aligned}
&D_1=1,\qquad D_2=-P_2,\\
&D_3=-2P_2^3+8P_0P_2-9P_1^2,\\
&D_4=-P_1^2P_2^3+4P_0P_2^4+36P_0P_2P_1^2-32P_0^2P_2^2\\
&\qquad+64P_0^3-\frac{27}{4}P_1^4,\\
&E_2=9P_1^2-32P_0P_2.
\end{aligned} \tag{18}$$

Case 2.1. If $D_4=0$, $D_3=0$, $D_2=0$, we have $F(\psi_1)=\psi_1^4$. Then, eq. (17) turns into

$$\pm(\eta-\eta_0)=\int\frac{\sqrt{(\psi_1-\frac{1}{4}B_3)(\psi_1-\frac{1}{4}B_3-\gamma)}}{\psi_1^2}\mathrm{d}\psi_1. \tag{19}$$

By letting

$$a=\frac{1}{4}B_3\left(\frac{1}{4}B_3+\gamma\right),\quad b=-\frac{1}{2}B_3-\gamma,$$

when $a<0$, we get

$$\pm(\eta-\eta_0)=-\frac{\sqrt{(u-k)(u-k-\gamma)}}{u-k+\frac{1}{4}B_3}+\frac{b}{2\sqrt{-a}}\arcsin\times\left(\frac{a(u-k+\frac{1}{4}B_3)+2a}{\sqrt{b^2-4a}(u-k+\frac{1}{4}B_3)}\right)$$

$$+2\ln(\sqrt{u-k}+\sqrt{u-k-\gamma}). \tag{20}$$

When $a=0$, we have

$$\pm(\eta-\eta_0)=-2\frac{\sqrt{(u-k)(u-k-\gamma)}}{u-k+\frac{1}{4}B_3} +2\ln(\sqrt{u-k}+\sqrt{u-k-\gamma}) \tag{21}$$

and when $a>0$, we get

$$\pm(\eta-\eta_0)=\frac{\sqrt{(u-k)(u-k-\gamma)}}{u-k+\frac{1}{4}B_3} +\frac{b}{2\sqrt{a}}\operatorname{arccosh}\left(\frac{2a+a(u-k+\frac{1}{4}B_3)}{\sqrt{b^2-4a}(u-k+\frac{1}{4}B_3)}\right) +2\ln(\sqrt{u-k}+\sqrt{u-k-\gamma}). \tag{22}$$

Case 2.2. If $D_4=0$, $D_3=0$, $D_2>0$, $E_2>0$, we have

$$F(\psi_1)=\left(\psi_1-\sqrt{-\frac{P_2}{2}}\right)^2\left(\psi_1+\sqrt{-\frac{P_2}{2}}\right)^2.$$

Suppose

$$\psi_1>\sqrt{-\frac{P_2}{2}}.$$

Then, eq. (17) can be rewritten as

$$\pm(\eta-\eta_0)=\int\frac{\sqrt{(\psi_1-\frac{1}{4}B_3)(\psi_1-\frac{1}{4}B_3-\gamma)}}{(\psi_1-\sqrt{-\frac{P_2}{2}})(\psi_1+\sqrt{-\frac{P_2}{2}})}\mathrm{d}\psi_1. \tag{23}$$

Then, we have

$$\begin{aligned}\pm(\eta-\eta_0)=&\,2\ln\left(\sqrt{u-k}+\sqrt{u-k-\gamma}\right)\\ &-b\ln\left(\sqrt{\frac{1}{u-c_1}+\frac{b+2\sqrt{-\frac{P_2}{2}}-\gamma}{2\left(a+b\sqrt{-\frac{P_2}{2}}-\frac{P_2}{2}\right)}}\right.\\ &\left.+\sqrt{\frac{1}{u-c_1}+\frac{b+2\sqrt{-\frac{P_2}{2}}+\gamma}{2\left(a+b\sqrt{-\frac{P_2}{2}}-\frac{P_2}{2}\right)}}\right)\\ &-b\ln\left(\sqrt{\frac{1}{u-c_2}+\frac{b-2\sqrt{-\frac{P_2}{2}}-\gamma}{2\left(a-b\sqrt{-\frac{P_2}{2}}-\frac{P_2}{2}\right)}}\right.\\ &\left.+\sqrt{\frac{1}{u-c_2}+\frac{b-2\sqrt{-\frac{P_2}{2}}+\gamma}{2\left(a-b\sqrt{-\frac{P_2}{2}}-\frac{P_2}{2}\right)}}\right),\end{aligned} \tag{24}$$

where

$$a=\frac{1}{4}B_3\left(\frac{1}{4}B_3+\gamma\right),$$
$$b=-\frac{1}{2}B_3-\gamma,$$
$$c_1=k+\sqrt{-\frac{P_2}{2}}-\frac{1}{4}B_3,$$
$$c_2=k-\sqrt{-\frac{P_2}{2}}-\frac{1}{4}B_3.$$

Case 2.3. If $D_4=0$, $D_3=0$, $D_2<0$, $E_2<0$, we have

$$F\left(\psi_1\right)=(\psi_1^2+\frac{P_2}{2}\Big)^2.$$

Similarly, we can get

$$\begin{aligned}\pm\frac{1}{2\gamma^2}(\eta-\eta_0)=&\,\frac{1}{\sqrt{a_1f_1}}\left(\frac{a_1}{h_1}-\frac{g_1}{C_0}\right)\arctan\left(t\sqrt{\frac{a_1}{f_1}}\right)\\ &+\frac{1}{\sqrt{a_1g_1}}\left(\frac{f_1}{C_0}-\frac{a_1}{h_1}\right)\\ &\times\arctan\left(t\sqrt{\frac{a_1}{g_1}}\right)\\ &+\frac{h_1}{C_0a_1}\ln|t+1|+\frac{2h_1}{C_0a_1}\ln|t-1|\\ &+\frac{3(f_1-g_1)}{2C_0a_1}\ln\left|t^2+\frac{f_1}{a_1}\right|,\end{aligned} \tag{25}$$

where

$$a_1=\frac{1}{16}B_3^2+\frac{P_2}{2},$$
$$b_1=-\frac{1}{2}B_3\left(\frac{1}{4}B_3+\gamma\right)-P_2,$$
$$d_1=\left(\frac{1}{4}B_3+\gamma\right)^2+\frac{P_2}{2},$$
$$h_1=\sqrt{b_1^2-4a_1d_1},$$
$$f_1=\frac{b_1}{2}-\frac{1}{2}h_1,$$
$$g_1=\frac{b_1}{2}+\frac{1}{2}h_1,$$
$$C_0=\frac{b_1h_1}{a_1^2}+\frac{h_1}{a_1}+\frac{b_1^2h_1-h_1^3}{4a_1^3}.$$

Case 2.4. If $D_4=0$, $D_3=0$, $D_2>0$, $E_2=0$, then we have $F(\psi_1)=(\psi_1-\alpha)^3(\psi_1+3\alpha)$, with the solution given by

$$\pm(\eta-\eta_0)=\int\frac{\sqrt{(\psi_1-\frac{1}{4}B_3)(\psi_1-\frac{1}{4}B_3-\gamma)}}{(\psi_1-\alpha)\sqrt{(\psi_1-\alpha)(\psi_1+3\alpha)}}\mathrm{d}\psi_1, \tag{26}$$

where

$$\alpha^2 = -\frac{P_2}{6}.$$

In the same way, let

$$\sqrt{\left(\psi_1 - \frac{1}{4}B_3\right)\left(\psi_1 - \frac{1}{4}B_3 - \gamma\right)} = t\left(\psi_1 - \frac{1}{4}B_3\right),$$

we obtain

$$\pm\frac{\left(\frac{1}{4}B_3 - \alpha\right)\sqrt{\left(\frac{1}{4}B_3 - \alpha\right)\left(\frac{1}{4}B_3 + 3\alpha\right)}}{2\gamma^2}(\eta - \eta_0) = \int\left\{\frac{1}{t^2 + R_1} - \frac{1}{(t^2-1)(t^2+R_1)}\right\} \times \frac{1}{\sqrt{(t^2+R_1)(t^2+R_2)}}\mathrm{d}t, \tag{27}$$

where

$$R_1 = \frac{\alpha - \frac{1}{4}B_3 - \gamma}{\frac{1}{4}B_3 - \alpha},$$

$$R_2 = \frac{3\alpha - \frac{1}{4}B_3 - \gamma}{\frac{1}{4}B_3 + 3\alpha}$$

and

$$\frac{1}{4}B_3 > \alpha.$$

Case 2.5. If $D_4 = 0$, $D_3 > 0$, $D_2 > 0$, we have

$$F(\psi_1) = (\psi_1 - \alpha)(\psi_1 - \beta)\left(\psi_1 + \frac{\alpha + \beta}{2}\right)^2.$$

Equation (9) can be rewritten as

$$\pm(\eta - \eta_0) = \int \frac{\sqrt{(\psi_1 - \frac{1}{4}B_3)(\psi_1 - \frac{1}{4}B_3 - \gamma)}}{(\psi_1 + \frac{\alpha+\beta}{2})\sqrt{(\psi_1 - \alpha)(\psi_1 - \beta)}}\mathrm{d}\psi_1, \tag{28}$$

where $\alpha > \beta$, and $\beta \neq -3\alpha$, $\beta \neq -\frac{\alpha}{3}$.

Similar to the above case, let

$$\sqrt{\left(\psi_1 - \frac{1}{4}B_3\right)\left(\psi_1 - \frac{1}{4}B_3 - \gamma\right)} = t\left(\psi_1 - \frac{1}{4}B_3\right),$$

then we infer that

$$\pm\frac{\left(\frac{1}{4}B_3 + \frac{\alpha+\beta}{2}\right)\sqrt{\left(\frac{1}{4}B_3 - \alpha\right)\left(\frac{1}{4}B_3 + 3\alpha\right)}}{2\gamma^2}(\eta - \eta_0) = \int\left\{\frac{1}{t^2 + R_1} - \frac{1}{(t^2-1)(t^2+R_1)}\right\} \times \frac{1}{\sqrt{(t^2+R_2)(t^2+R_3)}}\mathrm{d}t. \tag{29}$$

Here,

$$R_1 = -\frac{\alpha + \beta + \frac{1}{2}B_3 + 2\gamma}{\frac{1}{2}B_3 + \alpha + \beta},$$

$$R_2 = \frac{\alpha - \frac{1}{4}B_3 - \gamma}{\frac{1}{4}B_3 - \alpha},$$

$$R_3 = \frac{\beta - \frac{1}{4}B_3 - \gamma}{\frac{1}{4}B_3 - \beta}.$$

Case 2.6. If $D_4 = 0$, $D_3 < 0$, we have $F(\psi_1) = (\psi_1 - \alpha)^2[(\psi_1 + \alpha)^2 + \beta^2]$. We get

$$\pm(\eta - \eta_0) = \int \frac{\sqrt{(\psi_1 - \frac{1}{4}B_3)(\psi_1 - \frac{1}{4}B_3 - \gamma)}}{(\psi_1 - \alpha)\sqrt{(\psi_1 + \alpha)^2 + \beta^2}}\mathrm{d}\psi_1, \tag{30}$$

where $\beta \neq 0$. From eq. (30), we have

$$\pm\frac{\frac{1}{4}B_3 - \alpha}{2\gamma^2}(\eta - \eta_0) = \int\left\{\frac{1}{t^2 + R_1} - \frac{1}{(t^2-1)(t^2+R_1)}\right\} \times \frac{1}{\sqrt{R_2 t^4 + R_3 t^3 + R_4}}\mathrm{d}t, \tag{31}$$

where

$$R_1 = \frac{\alpha - \frac{1}{4}B_3 - \gamma}{\frac{1}{4}B_3 - \alpha},$$

$$R_2 = \frac{1}{4}B_3 + \alpha - \beta^2,$$

$$R_3 = -2\left(\frac{1}{4}B_3 + \alpha\right)\left(\alpha + \frac{1}{4}B_3 + \gamma\right) - 2\beta^2$$

and

$$R_4 = \left(\alpha + \frac{1}{4}B_3 + \gamma\right) + \beta^2.$$

Case 2.7. If $(D_4 > 0, D_3 > 0, D_2 > 0)$, or $(D_4 < 0, D_2 > 0 \parallel D_4 < 0, D_2 < 0, D_3 < 0 \parallel D_4 < 0, D_2 = 0, D_3 \leq 0)$, or $(D_4 > 0, D_2 \leq 0 \parallel D_4 > 0, D_3 \leq 0, D_2 > 0)$, we have

$$\pm(\eta - \eta_0) = \int \sqrt{\frac{(\psi_1 - \frac{1}{4}B_3)(\psi_1 - \frac{1}{4}B_3 - \gamma)}{(\psi_1 - \alpha)(\psi_1 - \beta)(\psi_1 - \delta)(\psi_1 - \varphi)}} \mathrm{d}\psi_1, \tag{32}$$

where $\varphi = \alpha + \beta + \delta$.

From Cases 2.4–2.7, these solutions can be represented by elliptic integral or elliptic functions. From all the cases we have discussed, the forms of travelling wave solutions of system (1) include solitary wave solutions, singular periodic solutions and double periodic solutions.

4. Physical representation

In this section, we show images of two types of solutions we obtained by adjusting the corresponding parameters. Other cases can be obtained in the same way.

Example 1. Take $k = 1, \gamma = 5, \alpha = 6, \beta = 2, \eta_0 = 0$, then solution (12) becomes

$$x - t = \ln\left|\frac{\sqrt{\frac{u-6}{u-3}} + 1}{\sqrt{\frac{u-6}{u-3}} - 1}\right| + \frac{1}{2}\ln\left|\frac{\sqrt{\frac{u-6}{u-3}} - \frac{1}{2}}{\sqrt{\frac{u-6}{u-3}} + \frac{1}{2}}\right|. \tag{33}$$

Therefore, the graph of solution (12) can be seen in figure 1.

Example 2. Take $k = 1, \gamma = 3, \alpha = 2, \eta_0 = 0$, then solution (14) becomes

$$\pm(x - t) = \sqrt{\frac{u-4}{u-3}} + \ln\left|\frac{\sqrt{\frac{u-4}{u-3}} \mp 1}{\sqrt{\frac{u-4}{u-3}} \pm 1}\right|. \tag{34}$$

Therefore, the graph of solution (14) can be seen in figure 2.

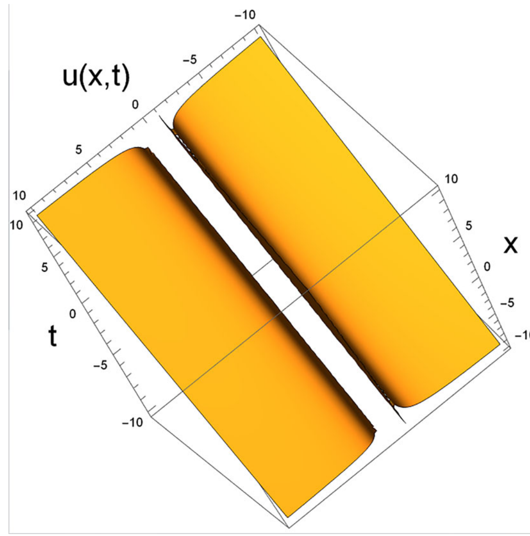

Figure 1. Expression of eq. (12).

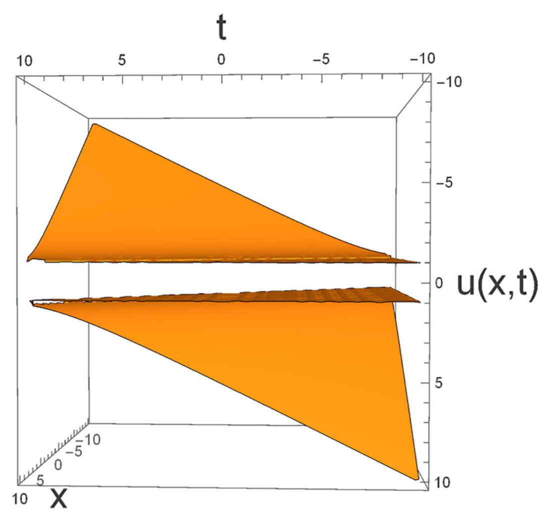

Figure 2. Expression of eq. (14).

5. Conclusion

This study has shown all travelling wave solutions of the two-component DGH system. By the direct integral method and CDSPM, we attained solitary wave solutions, singular periodic solutions and double periodic solutions. In addition, double periodic solutions were initially presented. These travelling wave solutions will help us to better understand the propagation forms of shallow water waves.

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Data availability The data used to support the findings of this study are available from the corresponding author upon request.

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