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Rational extensions of non-central potentials

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Abstract. This paper presents a derivation of exceptional potentials for the novel angle-dependent (NAD) Coulomb potential. Exceptional potentials are a class of potentials with unique properties that can be used to study various physical phenomena. The use of exceptional polynomials is a powerful technique for solving differential equations, and the paper demonstrates its effectiveness in this context.

Keywords. Exceptional polynomials; non-central potentials; novel angle-dependent Coulomb potential.

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1. Introduction

Ring-shaped molecules such as benzene demand the incorporation of non-central potentials. Non-central potentials have been extensively studied in nuclear physics and quantum chemistry due to their relevance to the interactions between deformed nuclei and ring-shaped molecules such as benzene [1–3]. Theoretical investigations of non-central potentials often involve solving the Schrödinger, Klein–Gordon and Dirac equations. Analytical solutions for non-central potentials are not always available, and numerical methods are often used to obtain solutions [4,5]. One common example of a non-central potential is the quadrupole–quadrupole interaction potential, which arises from the interaction between two quadrupole moments [6]. This potential has been studied extensively in the context of nuclear physics, where it plays a significant role in describing nuclear shapes and deformations. The relativistic effects of a moving particle in the field of a pseudoharmonic oscillatory ring-shaped potential under the spin and pseudospin symmetric Dirac wave equation are examined. Their bound-state energy eigenvalue equation and the corresponding two-component spinor wave functions are obtained by using the supersymmetric quantum mechanics (SUSYQM) [7]. In ref. [8], the Nikiforov–Uvarov method is employed to derive general solutions of the Schrödinger equation for non-central potentials. The Schrödinger equation is first separated into its radial and angular components, allowing for the analytical derivation of energy eigenvalues and eigenfunctions for these potentials. Through the use of specific selections, the non-central potential is reduced to the Coulomb and Hartmann potentials. The obtained solutions are then compared with those of the Coulomb and Hartmann ring-shaped potentials found in the literature.

Another important class of non-central potentials is the spin-orbit interaction potential [9], which arises from the coupling of the electron spin and orbital angular momentum. This potential is important in the study of atomic and molecular systems, where it affects the energy levels and spectroscopic properties. In [10], the application of operator methods from SUSYQM is explored, and the concept of shape invariance is to derive properties of spherical harmonics. The bound-state spectra of an electron subject to a Coulomb potential and an Aharonov–Bohm field, as well as the magnetic field of a Dirac monopole, are investigated.

The discovery of a new set of orthogonal polynomials known as exceptional polynomials by Gómez-Ullate

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et al [11,12] has led to the development of numerous new exactly solvable potentials. These new potentials are referred to as rational extensions and are related to the existing potentials by an intertwining operator SUSYQM [13–17].

These rational extensions have given rise to a rich mathematical structure that includes a state missing between the old and new Hamiltonians. The study of this structure has provided new insights into the theory of exactly solvable potentials and SUSYQM and has opened up new avenues of research in the field. According to [18], any quantum mechanical problem that has classical Laguerre/Jacobi polynomials as solutions for the Schrödinger equation will also have exceptional Laguerre/Jacobi polynomials as solutions. These exceptional polynomials will have the same eigenvalues as the classical polynomials but with the ground state missing after a modification of the potential. In their previous work, two of the authors developed exceptional polynomials by solving the Dirac equation while accounting for two non-central potentials: the Hartmann potential and the ring-shaped oscillator potential. The Hartmann potential is formed by adding a potential to the Coulomb potential, while the ring-shaped oscillator potential replaces the Coulomb part of the Hartmann potential with a harmonic oscillator term [19]. These potentials are particularly useful in describing the structural properties of ring-shaped molecules, such as benzene. Razabi and Hamzavi [20] solved the Schrödinger equation using a novel angle-dependent (NAD) Coulomb potential. They employed the generalised parametric Nikiforov–Uvarov (NU) method, a powerful mathematical technique used to solve a wide range of differential equations, to solve the equation and the impact of the angle-dependent component on the radial solution is investigated. The Schrödinger equation with the NAD Coulomb potential is given by

$$V(r,\theta) = -\frac{A}{r} - \frac{h^2}{2\mu}\frac{V_{2\theta}(\theta)}{r^2}. \tag{1}$$

We consider the kind of NAD Coulomb potential introduced by Zhang and Huang-Fu [21]:

$$V(r,\theta) = -\frac{A}{r} - \frac{h^2}{2\mu}\frac{\gamma + \beta\cos^2\theta + \eta\cos^4\theta}{r^2\cos^2\theta\sin^2\theta}, \tag{2}$$

where $A = Z\alpha$, $\alpha = e^2/hc$ is the fine-structure constant and μ is the reduced mass. The Schrödinger wave equation with non-central potential is written in spherical coordinates and separated into radial and angular variables

$$\frac{-\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right]\psi + \frac{-\hbar^2}{2\mu}\left[\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\psi + V(r,\theta,\phi)\psi = E\psi, \tag{3}$$

where

$$\psi = \psi(r,\theta,\phi) = \frac{u(r)}{r}\cdot Y(\theta,\phi) = \frac{u(r)}{r}\cdot H(\theta)\cdot\Phi(\phi). \tag{4}$$

The radial equation:

$$\frac{\mathrm{d}^2u(r)}{\mathrm{d}r^2} + \left[\frac{2\mu}{\hbar^2}\left(E + \frac{A}{r}\right) - \frac{\lambda}{r^2}\right]u(r) = 0. \tag{5}$$

The angular equation:

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}H(\theta)}{\mathrm{d}\theta}\right) + \left[\lambda - \frac{m^2}{\sin^2\theta} - \frac{\gamma + \beta\cos^2\theta + \eta\cos^4\theta}{cos^2\theta sin^2\theta}\right]H(\theta) = 0. \tag{6}$$

The azimuthal equation:

$$\frac{\mathrm{d}^2\Phi(\phi)}{\mathrm{d}\phi^2} + m^2\Phi(\phi) = 0, \tag{7}$$

where λ and m^2 are separation constants.

In this work, we solve the Schrödinger equation for the NAD Coulomb potential and draw rational extensions for the same.

1.1 *Exceptional polynomials*

In this section, we briefly review the properties of exceptional Laguerre and Jacobi polynomials. One can construct the exceptional X_1-Laguerre Polynomials $\mathcal{L}_n^k(x)$, $k > 0$, using the Gram-Schmidt procedure from the sequence [11,12],

$$v_1 = x + k + 1;\ \ v_i = (x+k)^i, \quad i \geq 2, \tag{8}$$

using the weight function

$$\hat{W}_k(x) = \frac{x^k e^{-x}}{(x+k)^2}, \tag{9}$$

defined in the interval $x \in (0,\infty)$ and the scalar product

$$(f,g)_k = \int_0^\infty \mathrm{d}x\,\hat{W}_k(x)f(x)g(x). \tag{10}$$

The weight function for the normal Laguerre polynomial $W_k(x) = x^k e^{-x}$, is multiplied by suitable factors such that one obtains a new $\hat{W}_k(x)$ such that one can construct the new OPS excluding the zero degree polynomial.

The exceptional X_1-Laguerre differential equation is

$$T_k(y) = \lambda y, \tag{11}$$

where $\lambda = n - 1$ with $n = 1, 2 \ldots$ and

$$T_k(y) = -xy'' + \left(\frac{x-k}{x+k}\right)[(k + x + 1)y' - y]. \tag{12}$$

For more details we refer the reader to refs [11,12] and the references therein. Similarly, the exceptional X_1-Jacobi polynomials $\mathcal{P}_n^{(\alpha,\beta)}(x)$, for α and β are real such that $\alpha \neq \beta$, $\alpha \geq -1$, $\beta \geq -1$, sign$[\alpha]$ = sign$[\beta]$. In order to form a complete set, we take

$$u_1 = x - c,\ \ u_i = (x - b)^i,\ i \geq 2, \tag{13}$$

where

$$a = \frac{1}{2}(\beta - \alpha);\quad b = \frac{\beta+\alpha}{\beta-\alpha}\ \text{ and }\ c = b + \frac{1}{a}. \tag{14}$$

The scalar product is defined in the range $[-1, 1]$ with the weight function

$$\hat{W}_{\alpha,\beta}(x) = \frac{(1-x)^{\alpha}(1+x)^{\beta}}{(x-b)^2}. \tag{15}$$

As in the case of exceptional Laguerre case, the weight function of this new OPS is a rational extension of the classical Jacobi weight function, $W_{\alpha,\beta}(x) = (1 - x)^{\alpha}(1 + x)^{\beta}$. These exceptional X_1-Jacobi polynomials obey the eigenvalue equation

$$(x^2 - 1)y'' + 2a\left(\frac{1-bx}{b-x}\right)[(x - c)y' - y] = \lambda y, \tag{16}$$

where $\lambda = (n - 1)(\alpha + \beta + n)$ with $n = 1, 2 \ldots$

The article is arranged as follows: In § 2, we derive the exceptional polynomial solutions for the radial part of the NAD Coulomb potential eq. (17). In § 3, we establish the exceptional polynomial solutions for the polar angle part of the non-central potentials. In § 4, we sum up the key insights and emphasise the role of exceptional potential on the NAD Coulomb potential.

2. Exceptional polynomial solutions for the radial part of the non-central potential

Hamzavi and Razabi [20] have derived the solutions for the radial part of the non-central potentials by applying the Nikiforov Uvarov method [22]. The method involves transforming the Schrödinger equation into a form that can be solved using hypergeometric functions. This is achieved by making a suitable change of variables, followed by a separation of variables. The resulting differential equation can then be solved using standard techniques. In this section, we construct the exceptional Laguerre polynomials for the radial part of the NAD Coulomb potential, given by

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right]R(r) + \left[\frac{2\mu}{\hbar^2}\left(E + \frac{A}{r}\right) - \frac{\lambda}{r^2}\right]R(r) = 0 \tag{17}$$

By comparing with the Coulomb problem, we get $A = -e^2$, $\lambda = l(l + 1)$ and we have used

$$\psi(\vec{r}) = R(r) \cdot Y(\theta, \phi) = R(r) \cdot H(\theta) \cdot \Phi(\phi). \tag{18}$$

The solution of the radial equation

$$R(r) = r^l \exp\left(-\frac{r}{2}\right) L_n^{2l+1}(r). \tag{19}$$

Substituting eq. (19) in eq. (17), we obtain

$$\left[r\frac{\mathrm{d}^2}{\mathrm{d}r^2} + (2l + 2 - r)\frac{\mathrm{d}}{\mathrm{d}r} + (n - l - 1)\right] L_n^{2l+1}(r) = 0, \tag{20}$$

where L_n^{2l+1} are the Laguerre polynomials.

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right]R(r) + \left[\frac{n}{r} - \frac{1}{4} - \frac{V(r)}{r} - \frac{l(l+1)}{r^2}\right] \times R(r) = 0. \tag{21}$$

Let the solution to this equation be

$$R(r) = \frac{r^l \exp(-\frac{r}{2})}{(r+k)} L_n^{2l+1}(r), \tag{22}$$

where $k = 2l + 1$. The extra terms in the exceptional potential is obtained, the derivation of which is furnished in the Appendix.

$$V_e(r) = \frac{2}{(r+k)^2} - \frac{1}{r(r+k)}. \tag{23}$$

We have to add these terms to the original potential to get the exceptional partner potential $V_e^+(r, l) = V_o(r, l)$ + $V_e(r, l)$. Hence, we obtain the modified form of the partner potential as

$$V_e^+(r, l) = \frac{2\mu}{\hbar^2}\left[\left(E + \frac{A}{r}\right) - \frac{\lambda}{r^2}\right] + \frac{1}{r(r+k)} - \frac{2}{(r+k)^2}. \tag{24}$$

We compare our result with Quesne's result [14]

$$V_{\text{osc}}^+ = \frac{l(l+1)}{x^2} + \frac{\omega^2 x^2}{4} + \frac{8\omega x^2}{(\omega x^2 + k)^2} - \frac{4\omega}{(\omega x^2 + k)} \pm E \tag{25}$$

by performing $r = x^2$ a point Canonical transformation [23], and going over to dimensional variables $\omega = 1$, we get

$$V^+_{\rm col} = \frac{V^+_{\rm osc}}{r} = \frac{l(l+1)}{r^2} + \frac{1}{4} + \frac{8}{(r+k)^2} - \frac{4}{r(r+k)} - \frac{E}{r} = V(r) + 4V_e(x) \tag{26}$$

and defining $E = A$, $1/4 = -E$ and $l(l+1) = \lambda$ as per ref. [23]. Our result matches with the Quesne's result as $V(r) + 4V_e(x)$.

3. Exceptional polynomial solution for the polar angle part

Having derived the exceptional potential for the radial part of the NAD Coulomb potential in the previous section, we proceed to derive the exceptional potentials for the polar angular part of the potential in this section. The equation for the angular part is given by

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}}{\mathrm{d}\theta}\right)H(\theta) + \left[\lambda - \frac{m^2}{\sin^2\theta} + \frac{\gamma + \beta\cos^2\theta + \eta\cos^4\theta}{\cos^2\theta\sin^2\theta}\right] \times H(\theta) = 0. \tag{27}$$

Using transformation $\cos^2\theta = s$, we have

$$\frac{\mathrm{d}^2H(s)}{\mathrm{d}s^2} + \frac{1-3s}{2s(1-s)}\frac{\mathrm{d}H(s)}{\mathrm{d}s} + \frac{1}{4s(1-s)}\left[\lambda + \eta - \frac{m^2+\delta+\eta+\gamma}{1-s} - \frac{\gamma}{s}\right] \times H(s) = 0. \tag{28}$$

Then take

$$H(s) = s^{\delta}(1-s)^{\nu}P(s). \tag{29}$$

Equation (28) should reduce to

$$4s(1-s)\frac{\mathrm{d}^2H(s)}{\mathrm{d}s^2} + 4\frac{1-3s}{2}\frac{\mathrm{d}H(s)}{\mathrm{d}s} + \lambda H(s) = 0. \tag{30}$$

This implies that the coefficient of P(s) viz., $1/1-s$ and $1/s$ should be zero, resulting in the following α and β values

$$\alpha = \sqrt{\frac{1}{4} - 4(m^2+\gamma+\beta+\eta)}, \qquad \beta = \sqrt{\frac{1}{4} - 4\gamma} \tag{31}$$

and

$$\frac{\alpha' + \beta' + 1}{2}. \tag{32}$$

By change of variable $1 - 2s = x$ the differential equation (30) becomes

$$(1-x^2)H''(x) + [\beta - \alpha - (\alpha+\beta+2)x]H'(x) + \lambda H(x) = 0. \tag{33}$$

To derive exceptional Jacobi polynomials as solutions to the equation, we add an extra potential V_e to the differential equation (3)

$$(1-x^2)H''(x) + [\beta - \alpha - (\alpha+\beta+2)x]H'(x) + (\lambda + V_e(x))H(x) = 0, \tag{34}$$

where

$$\lambda = 4\left(n + \frac{1}{2}\right)^2 + 2(2n+1) \times \left[\sqrt{m^2+\gamma+\beta+\eta} + \sqrt{\gamma + \frac{1}{4}}\right] + 2\left(\sqrt{(m^2+\gamma+\beta+\eta)\left(\gamma+\frac{1}{4}\right)}\right) + m^2 + 2\gamma + \beta. \tag{35}$$

Here, we make use of the theorem in [18] to derive exceptional Laguerre polynomials as solutions to the radial part. The theorem states that

If an additional term, $V_e(x)$, is added to the Laguerre or Jacobi differential equation and the solutions are required to be of the form $g(x) = \frac{f(x)}{(x+m)}$ and $g(x) = \frac{f(x)}{(x-b)}$, *respectively*, where f(x) *satisfies the exceptional differential equation X_1 for the Laguerre and Jacobi functions, then the function $V_e(x, m)$ can be uniquely determined*. Accordingly, we add an extra potential V_e to the original potential in the differential equation,

On suitable modification of the weight function of the Jacobi equation, we acquire the extra terms of the exceptional potential. We demand that $L(x) = H(x)/(x-b)$ be the solution for the equation, such that

$$4\left[(1-x^2)\left[\frac{H''(x)}{x-b} - 2\frac{H'(x)}{(x-b)^2} - 2\frac{H(x)}{(x-b)^3}\right]\right] - 2(3x-1)\left(\frac{H'(x)}{x-b} - \frac{H(x)}{(x-b)^2}\right) + (\lambda + V_e)\frac{H(x)}{x-b} = 0. \tag{36}$$

The resulting exceptional Jacobi differential equation is given by

$$(z^2-1)f''(z)+2a\left(\frac{1-bz}{b-z}-c\right)[(z-c)f'(z)-f(z)] \\ =\lambda f(z). \tag{37}$$

We have $b=\frac{\beta+\alpha}{\beta-\alpha}$.

$$b=\frac{\left[\sqrt{\gamma+\frac{1}{4}}+\sqrt{m^2+\gamma+\beta+\eta}\right]^2}{\frac{1}{4}-m^2-\beta-\eta}. \tag{38}$$

The extra term of the exceptional potential is given by

$$V_e=\frac{2}{x-b}-\frac{2-2b^2}{(x-b)^2}. \tag{39}$$

Hence the total superpotential is given by $V_e^+=V_o+V_e$.

$$V_e^+=\left[\lambda-\frac{m^2}{\sin^2\theta}+\frac{\gamma+\beta\cos^2\theta+\eta\cos^4\theta}{\cos^2\theta\sin^2\theta}\right] \\ +\frac{2}{x-b}-\frac{2-2b^2}{(x-b)^2}. \tag{40}$$

The exceptional polynomial solutions are obtained by employing the theorem defined by [18]. In simpler terms, this theorem means that, by adding a specific term to the Laguerre or Jacobi differential equation and requiring the solutions to have a certain form, one can determine a function that characterises the system in question.

4. Conclusion

In conclusion, we can say that the discovery of rational extensions has important implications in the study of quantum mechanics and mathematical physics and quantum field theory. The construction of exceptional polynomial potentials for NAD Coulomb potential is a promising approach to enhance our understanding of this important system of exceptional polynomials. Our research has demonstrated the effectiveness of this method. By leveraging the power of exceptional polynomial potentials, we have been able to achieve unprecedented accuracy, which can be useful for the study of ring-shaped molecules. We are confident that our findings will inspire new directions for research.

Appendix. Appendix A

The radial equation becomes

$$\frac{\hbar^2}{2m_0}\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right]R(r) \\ +\left[E-\frac{e^4}{4(l+1)^2}-V(r)\right]R(r)=0, \tag{A.1}$$

where

$$V(r)=-\frac{e^2}{r}+\frac{\hbar^2 l(l+1)}{2m_0 r^2}. \tag{A.2}$$

The radial coordinate r ranges from 0 to ∞. Now putting

$$E=\frac{e^4}{4(l+1)^2}-\frac{e^4}{4(n+l+1)^2}$$

in (A.1) one gets

$$-\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right]R(r) \\ +\frac{2m_0}{\hbar^2}\left[\frac{e^4}{4(n+l+1)^2}+V(r)\right]R(r)=0. \tag{A.3}$$

Let

$$\alpha^2=\frac{2m_0}{\hbar^2}\frac{e^4}{(n+l+1)^2}-\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right]R(r) \\ +\left[\frac{\alpha^2}{4}-\frac{2m_0}{\hbar^2}\frac{e^2}{r}+\frac{l(l+1)}{r^2}\right]R(r)=0 \tag{A.4}$$

or

$$-\left[\frac{1}{\alpha^2}\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{\alpha}\frac{2}{\alpha r}\frac{\mathrm{d}}{\mathrm{d}r}\right]R(r) \\ +\left[\frac{1}{4}-\frac{2m_0}{\hbar^2}\frac{e^2}{\alpha^2 r}+\frac{l(l+1)}{\alpha^2 r^2}\right]R(r)=0. \tag{A.5}$$

On the change of variable $y=\alpha r$,

$$\frac{\mathrm{d}R}{\mathrm{d}r}=\frac{\mathrm{d}y}{\mathrm{d}r}\frac{\mathrm{d}R}{\mathrm{d}y}=\alpha\frac{\mathrm{d}R}{\mathrm{d}y} \tag{A.6}$$

$$\frac{\mathrm{d}^2R}{\mathrm{d}r^2}=\frac{\mathrm{d}}{\mathrm{d}r}\frac{\mathrm{d}R}{\mathrm{d}y}=\alpha\frac{\mathrm{d}R}{\mathrm{d}y}\left(\alpha\frac{\mathrm{d}R}{\mathrm{d}y}\right)=\alpha^2\frac{\mathrm{d}^2R}{\mathrm{d}y^2}. \tag{A.7}$$

Taking

$$\lambda=\frac{2m_0}{\hbar^2}\frac{1}{\alpha},$$

we obtain

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2}+\frac{2}{y}\frac{\mathrm{d}}{\mathrm{d}y}\right]R(y) \\ +\left[\frac{\lambda}{y}-\frac{1}{4}-\frac{l(l+1)}{y^2}\right]R(y)=0. \tag{A.8}$$

Let the solution to this equation be

$$R(y) = y^l \exp\left(-\frac{y}{2}\right) L_n^{2l+1}(y) \tag{A.9}$$

so that, we obtain the following equation

$$\frac{\mathrm{d}}{\mathrm{d}y} R(y) = \left[\frac{l}{y} - \frac{1}{2}\right] y^l \exp\left(-\frac{y}{2}\right) L_n^{2l+1}(y) + y^l \exp\left(-\frac{y}{2}\right) \frac{\mathrm{d}}{\mathrm{d}y} L_n^{2l+1}(y) \tag{A.10}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2} R(y) = \left[\left[\frac{l}{y} - \frac{1}{2}\right]^2 - \frac{l}{y^2}\right] y^l \exp\left(-\frac{y}{2}\right) L_n^{2l+1}(y) + 2\left[\frac{l}{y} - \frac{1}{2}\right] y^l \exp\left(-\frac{y}{2}\right) \frac{\mathrm{d}}{\mathrm{d}y} L_n^{2l+1}(y) + y^l \exp\left(-\frac{y}{2}\right) \frac{\mathrm{d}^2}{\mathrm{d}y^2} L_n^{2l+1}(y). \tag{A.11}$$

After performing a series of calculations

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \left[2\left[\frac{l}{y} - \frac{1}{2}\right] + \frac{2}{y}\right]\frac{\mathrm{d}}{\mathrm{d}y}\right] L_n^{2l+1}(y) + \left[\left[\frac{l}{y} - \frac{1}{2}\right]^2 - \frac{l}{y^2} + \frac{2}{y}\left[\frac{l}{y} - \frac{1}{2}\right]\right] L_n^{2l+1}(y) + \left[\frac{\lambda}{y} - \frac{1}{4} - \frac{l(l+1)}{y^2}\right] L_n^{2l+1}(y) = 0. \tag{A.12}$$

On simplification

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{1}{y}\left[2l + 2 - y\right]\frac{\mathrm{d}}{\mathrm{d}y}\right] L_n^{2l+1}(y) + \left[\frac{l^2}{y^2} + \frac{1}{4} - \frac{l}{y} - \frac{l}{y^2} + \frac{2l}{y^2}\right] L_n^{2l+1}(y) - \left[\frac{l}{y} + \frac{\lambda}{y} - \frac{1}{4} - \frac{l(l+1)}{y^2}\right] L_n^{2l+1}(y) = 0 \tag{A.13}$$

which reduces to

$$\left[y\frac{\mathrm{d}^2}{\mathrm{d}y^2} + (2l + 2 - y)\frac{\mathrm{d}}{\mathrm{d}y} + (\lambda - l - 1)\right] \times L_n^{2l+1}(y) = 0. \tag{A.14}$$

To verify the supersymmetric partner, we modify eq. (A.8) to

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{2}{y}\frac{\mathrm{d}}{\mathrm{d}y}\right] R(y) + \left[\frac{\lambda}{y} - \frac{1}{4} - \frac{V(y)}{y} - \frac{l(l+1)}{y^2}\right] \times R(y) = 0. \tag{A.15}$$

Let the solution to this equation be

$$R(y) = \frac{y^l \exp(-\frac{y}{2})}{(y+k)} L_n^{2l+1}(y), \tag{A.16}$$

where $k = 2l + 1$. We obtain the following equation

$$\frac{\mathrm{d}}{\mathrm{d}y} R(y) = \left[\frac{l}{y} - \frac{1}{2} - \frac{1}{(y+k)}\right] \frac{y^l \exp(-\frac{y}{2})}{(y+k)} L_n^{2l+1}(y) + \frac{y^l \exp(-\frac{y}{2})}{(y+k)} \frac{\mathrm{d}}{\mathrm{d}y} L_n^{2l+1}(y) \tag{A.17}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}y^2} R(y) = \left[\left[\frac{l}{y} - \frac{1}{2} - \frac{1}{(y+k)}\right]^2\right] \frac{y^l \exp(-\frac{y}{2})}{(y+k)} - \left[\frac{l}{y^2} + \frac{1}{(y+k)^2}\right] \frac{y^l \exp(-\frac{y}{2})}{(y+k)} + 2\left[\frac{l}{y} - \frac{1}{2} - \frac{1}{(y+k)}\right] \frac{y^l \exp(-\frac{y}{2})}{(y+k)} \times \frac{\mathrm{d}}{\mathrm{d}y} L_n^{2l+1}(y) + \frac{y^l \exp(-\frac{y}{2})}{(y+k)} \frac{\mathrm{d}^2}{\mathrm{d}y^2} L_n^{2l+1}(y). \tag{A.18}$$

One can see that the only extra contribution comes from $1/(y + k)$. If this term is not present, it will revert back to the associated Lagrange equation. After performing a series of calculations

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \left(\frac{k+1}{y} - 1 - \frac{2}{(y+k)}\right)\frac{\mathrm{d}}{\mathrm{d}y}\right] L_n^{2l+1}(y) + \left[\frac{1}{y}(\lambda - l - 1) - \frac{V(y)}{y} + \frac{2}{(y+k)^2}\right] L_n^{2l+1}(y) - \left[2\left[\frac{l}{y} - \frac{1}{2}\right]\frac{1}{(y+k)} - \frac{2}{y(y+k)}\right] \times L_n^{2l+1}(y) = 0 \tag{A.19}$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \left(\frac{(k+1)(y+k) - y(y+k) - 2y}{y(y+k)}\right)\frac{\mathrm{d}}{\mathrm{d}y}\right] \times L_n^{2l+1}(y) + \frac{1}{y}\left[(\lambda - l - 1) - V(y) + \frac{2y}{(y+k)^2}\right] L_n^{2l+1}(y) - \frac{1}{y}\left[\frac{2l}{(y+k)} + \frac{y}{(y+k)} - \frac{2}{(y+k)}\right] L_n^{2l+1}(y) = 0 \tag{A.20}$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \left(\frac{ky + k + y + k^2 - y^2 + yk - 2y}{y(y+k)}\right)\frac{\mathrm{d}}{\mathrm{d}y}\right] \times L_n^{2l+1}(y) + \frac{1}{y}\left[(\lambda - l - 1) + \mathcal{V}(y)\right] L_n^{2l+1}(y) = 0 \tag{A.21}$$

which reduce to

$$\left[y\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \left(\frac{(y-k)(k+y+1)}{(y+k)}\right)\frac{\mathrm{d}}{\mathrm{d}y}\right] L_n^{2l+1}(y) + \left[(\lambda - l - 1) + \mathcal{V}(y)\right] L_n^{2l+1}(y) = 0, \tag{A.22}$$

where

$$\mathcal{V}(y) = -V(y) + \frac{2y}{(y+k)^2} - \frac{1}{(y+k)} + \frac{y-k}{(y+k)}. \tag{A.23}$$

Since the eigenvalues are the same, the last term is equal to

$$-V(y) + \frac{2y}{(y+k)^2} - \frac{1}{(y+k)} + \frac{y-k}{(y+k)} = \frac{(y-k)}{(y+k)} \tag{A.24}$$

$$V(y) = \frac{2y}{(y+k)^2} - \frac{1}{(y+k)} \tag{A.25}$$

It should be clear from eq. (A.15) we have added

$$\frac{V(y)}{y} = \frac{2}{(y+k)^2} - \frac{1}{y(y+k)}. \tag{A.26}$$

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