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Field theoretic formulation of fluid mechanics according to the geometric algebra

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Abstract. A simple and coherent approach to fluid mechanics is presented using a proper formalism of geometric algebra. The analogy between the equations of electromagnetism and fluid mechanics provides reinterpretation of the equations for two constituent (vorticity and the Lamb vector) fields. Identifying certain quantities as the source fields, the guiding Navier–Stokes (NS) equations of fluid mechanics can be formulated as a set of four geometrically distinct field equations, resembling exactly the Maxwell equations for the constituent magnetic and electric fields. The same set of equations works for all the cases of compressible, incompressible, viscous and the inviscid fluid motions with appropriately modified source terms. The analogy is completed by defining the combined 'fluidomechanic' bivector field in space–time algebra and further extended to the fluidic analogue of the Poynting theorem, Poynting vector and Lorentz force.

Keywords. Navier–Stokes equations; vorticity; turbulence; Maxwell's equations; Poynting theorem; Lorentz force; Clifford's geometric algebra; space–time algebra.

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1. Introduction

Fluid mechanics is a branch of physics for studying moving and stationary fluids to improve our understanding of the behaviour of fluids under various forces and physical conditions and the forces they produce. The investigations help us to select the proper fluid for different practical applications. For applications involving transportation, power generation and conversion, the search for an acceptable method of harnessing nuclear energy to name a few, the importance of studying fluid mechanics cannot be overstated! It has almost ubiquitous applications across inter-disciplinary boundaries of physics, engineering, biology, atmospheric science, geology, oceanography and so on.

The relevant variables and equations describing the motion of a fluid are closely analogous to those of electromagnetism. This has been traced by several researchers in the past and Maxwell himself pointed out parallels between the electromagnetic vector potential and the fluid velocity (vector). Similarity with the mathematical structures of electromagnetism (EM) and fluid mechanics (FM) may be readily recognised from the definitions of the constituent fields of the two theories. Biot–Savart law, which is usually known to describe the magnetic field generated by a stationary electric current, is also used in aerodynamics to calculate the velocity induced by the vorticity field. However, in comparison to the magnetic case, the roles of vorticity and current are reversed. Quite recently, Osano and Adams compared the evolution equation of the electromagnetic vector potential with the Naiver–Stokes (NS) equation and the two evolution equations for the magnetic and vorticity fields with non-vanishing dissipation terms [1]. Also the NS equations, rewritten in terms of the vorticity and the Lamb vector fields exactly resemble the Maxwell equations for the constituent magnetic and electric fields [2–4].

On the other hand, the term eddy current in electricity comes from analogous currents seen in fluid dynamics, causing localised areas of turbulence known as eddies giving rise to persistent vortices. Somewhat similarly, eddy currents can take time to build up and can persist for short time interval in conductors due to their inductance. Also, recent works based on a 'fluidic' viewpoint, developed an NS-like equation in electrodynamics by using the appropriate electromotive force. The new approach suggests possible applications in

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producing electric fields of the required configuration in plasma medium [5]. However, as Feynman has observed [6] "electrodynamics is really much easier than hydrodynamics", and in fact, discussing EM first really helps to understand the complications of fluid mechanics better.

After the initial success in describing the electro magnetic theory, Gibbs–Helmholtz's vector algebra (VA) dominates for generations as a major mathematical framework of theoretical physics. However, various formulations using vector algebra suffer from several unwarranted features and inadequacies and incorporate in addition, matrix, tensor and spinorial algebras for the complete description of physical theories. The cross product of VA is definable only in 3D and produces a pseudovector which lacks an absolute direction, to be fixed according to the convention. Through its definition of cross product, VA introduces a handedness (chirality) even when there is no chirality in the entity being modelled.

It should be emphasised that, the cross product, triple product etc. of vectors and curl of a vector field and hence the vector algebra and vector calculus as such, can be defined only in 3D. Generalisation to any other dimension is not possible – in two dimension, no third dimension exists to accommodate the (cross) product vector, and in higher dimensions there are too many orthogonal directions [7]! By removing the inadequacies of vector algebra, the Grassmann exterior algebra or the 'algebra of extension', provides an efficient and useful generalisation to any finite dimension and the basic framework for Clifford geometric algebra. The exterior or wedge ($\wedge$) product of Grassmann algebra between any two vectors $\mathbf{u}$ and $\mathbf{v}$ belonging to the vector space of arbitrary dimensionality can be simply and precisely defined as $\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u}$. Using the associative wedge product, two, three or any number of linearly independent vectors in a given dimension can be wedged together to produce higher definite-grade multivector blades – bivector, trivector, etc. The wedge product also implies a 'closure property'. For example, a bivector $\mathbf{u} \wedge \mathbf{v}$ in two dimensions and a trivector $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$ in three dimensions, both having only one component each that flips sign under reflection, represent the highest grade element and pseudoscalar of the respective dimensions. On the other hand, a trivector $\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \mathbf{v}_3$ in 2D and a quadrivector $\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \mathbf{v}_3 \wedge \mathbf{v}_4$ in 3D collapse back down to scalar zero so as to prevent construction of any element of grade higher than the dimensionality of the space.

Physical quantities like angular velocity, angular momentum, torque in a force field, magnetic field $\mathbf{B}$ at a point due to a current-element (given according to Biot–Savart law) and vorticity $\mathbf{W}$ in fluid motion, usually represented by pseudovectors of VA, are properly represented by bivectors in Clifford geometric algebra (GA). Correspondingly, the flux and helicity (density) of the bivector (magnetic, vorticity) fields are aptly represented by pseudoscalars. Also, particles with spin 0 and odd parity are called pseudoscalar particles, e.g. 'pseudoscalar mesons'. Hestenes and others [8,9] have revealed that GA provides a comprehensive description for the most advanced concepts in theoretical physics, such as classical mechanics, electromagnetism, fluid mechanics, theory of relativity, quantum mechanics, computer science, etc. [9]. GA is now being increasingly recognised as the natural algebra for describing the physics of n-space and applied to a range of problems in varied research fields. It is also claimed that its superior geometric intuition is straightforward and simple enough to be taught in high schools replacing Gibbs–Helmholtz VA [10,11]!

Describing the magnetic field as a bivector (field), geometric algebra provides a final unification of the usual four Maxwell's equations of the VA in a single, compact equation for the combined electromagnetic field. Furthermore, a single space–time force equation in terms of the combined electromagnetic field encapsulates both the Lorentz force equation and the power equation. Moreover, the formulation facilitates a natural introduction of the putative concept of magnetic monopole and offers a profound dual symmetric description by rendering the equations for both the constituent fields, symmetric and inhomogeneous. All these are discussed in [12] by the present author.

In the following, a similar study on fluid mechanics is discussed to facilitate a comprehensive introduction to fluid mechanics in the appropriate framework of GA. Formal similarities of physical concepts, notions and mathematical structures between the theories of electromagnetism and fluid mechanics are highlighted with greater clarifications in GA. The divergence and the evolution equation of the Lamb vector $\mathbf{l}$ give the 'fluidomechanic' sources – the charge and the current densities, respectively. Under simple approximations, the guiding NS equations of motion can be expressed, in terms of two appropriate field variables, exactly in the same structure of four Maxwell equations and can be similarly unified using the combined 'fluidomechanic' field in geometric algebra. The same set of equations works for all the cases of compressible, incompressible, viscous and inviscid fluid motions. Only the source fields, the scalar charge density and the vector current density have additional terms in a compressible and/or viscous fluid flow. Complete expressions for the source fields are derived in the present work.

Some papers [13,14] claiming 'geometric algebraic approach' in fluid mechanics have recently been published. However, they have actually represented the final expressions of vector algebraic description, in the language of geometric algebra at the end. Panakkal *et al* have also mixed the languages of matrix, tensor and differential form in their description. A straightforward and unified field theoretic formulation of fluid mechanics using purely geometric algebraic description is initiated in this paper which is expected to provide new insights and open up new directions of investigation. It is also intended to give a broad-based exposure of this powerful apparatus of theoretical physics to advanced graduate students – looking forward for its inclusion in university curriculum. After brief introductions to the basic theory of fluid mechanics, geometric algebra and calculus in the following two sections, the field theoretic formulation of FM will be developed next in the proper language of GA.

2. Fluid mechanics

A fluid cannot support shearing force for any length of time and it flows. The ease with which a fluid flows is the measure of its viscosity and even for viscous fluids, there is no shearing force when it is at rest. Fluids are regarded as continuous media. In a continuum approach, 'fluid particles' and 'points' in a fluid are to be interpreted as infinitely small volume elements, containing many molecules and still regarded as points [15]. The motion of a fluid is described in terms of certain relevant variables. To reduce complexity, we start with a simple system in terms of its flow velocity $\mathbf{v}$, density ρ, pressure p and the coefficient of viscosity η and discuss its motion according to GA.

In FM, the equation of motion of a viscous fluid, developed by Navier and Stokes, describes the evolution of the velocity field vector under given initial conditions [15,16]. Newton's second law of motion under fluid stresses due to the pressure gradient and viscosity in combination with the basic conservation and continuity equations provide a set of equations. The Eulerian description which is used in most problems of FM uses the coordinate system fixed in space, like field theories describing EM or gravity. The field functions, defined at a given point in space at a given time, refer to the fixed points in space and not to specific fluid particles, the latter move about in space in course of time.

With a constant control volume, there is no source or sink of the mass, i.e. $Q = 0$, the equation of continuity describing the change of the density field $\rho = \rho(x, t)$, given by

$$\begin{aligned} &\partial_t \rho + \nabla \cdot (\rho\, \mathbf{v}) = 0 \\ &\Rightarrow \partial_t \rho + \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v} = 0 \end{aligned} \tag{1}$$

is the same as the equation for the conservation of mass. Here, $\partial_t \rho + (\mathbf{v} \cdot \nabla)\rho \; (= d_t \rho)$ is termed as the material (or total time) derivative and the concept of the material derivative, the rate of change of an intensive property of a 'fluid particle', appears in a velocity field. For an incompressible fluid setting the material derivative of density $d_t \rho$ equal to zero, the equation of continuity renders the velocity field solenoidal ($\nabla \cdot \mathbf{v} = 0$), making useful simplification in a number of problems. In terms of the fluid body force (per unit volume) $\mathbf{f}_b$, the equation of motion may be expressed as

$$\begin{aligned} \rho d_t \mathbf{v} &= \rho\{\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}\} = \mathbf{f}_b \\ &= -\nabla p - \rho \nabla (gz) + \rho \mathbf{k} \\ &\Rightarrow \partial_t \mathbf{v} = -(\mathbf{v} \cdot \nabla)\mathbf{v} - \nabla(\rho^{-1} p + gz) + \mathbf{k}, \end{aligned} \tag{2}$$

where the instantaneous/average fluid density ρ is assumed to be constant and the force acting on the unit fluid volume is equal to $-\nabla p$ plus the external force of gravity $-\rho \nabla (gz)$ and the fluid stress $\rho \mathbf{k}$ due to viscosity. Also, even when the flow is steady, i.e. $\partial_t \mathbf{v} = 0$, the acceleration is non-zero as long as $(\mathbf{v} \cdot \nabla)\mathbf{v} \neq 0$, that is, if the velocity field changes in space along itself.

The viscous stress tends to diffuse the fluid velocity gradients and is proportional to the coefficient of kinematic viscosity η (coefficient of dynamic viscosity μ divided by density ρ) and depends on temperature. Yet, if the temperature differences are small within the fluid, then η can be taken outside the derivative [16], producing $\mathbf{k} = \eta\{\nabla^2 \mathbf{v} + 3^{-1} \nabla(\nabla \cdot \mathbf{v})\}$. The inertia of a continuous medium is, therefore, described by a non-linear term $(\mathbf{v} \cdot \nabla)\mathbf{v}$, whereas the linear Newtonian friction law is expected to hold for small rates of strain as higher powers of η are neglected. For common fluids such as air and water, the linear relationship is found to be surprisingly accurate for most applications. For inviscid fluid flow ($\eta = 0$), the last term on r.h.s. vanishes and the NS equation reduces to Euler equation, developed earlier in 1757 by Euler [17], to describe the flow of an incompressible frictionless (inviscid) fluid. Nevertheless, the equations can be applied to both incompressible and to the more general compressible flow. The compressible or incompressible fluid flow offers a sort of 'gauge freedom' in fluid mechanics, although this is not simply a freedom or choice, actually it has implications about the physical nature of the flow. The relation $\nabla \cdot \mathbf{v} = 0$ for the incompressible fluid flow is akin to the Coulomb gauge in electromagnetism, whereas a Lorentz gauge-like equation for the average potential fields, relative to a compressible fluid flow may also be envisaged [18,19]. Actually, the NS equation is a generalisation of the Euler equation. Navier in 1821 first introduced viscous friction for the more realistic and vastly more difficult problem of viscous fluid motion. Stokes then improved on this work to its final form, though complete solutions were obtained only for the case of simple 2D flows.

The study of fluid mechanics is formulated as a boundary value problem, solving the NS equations (1) and (2) with appropriate boundary conditions [16]. For a Newtonian fluid, the coefficient of viscosity η is assumed to be constant. Using vector calculus in 3D fluid continuum, the nonlinear convective term $(\mathbf{v} \cdot \nabla)\mathbf{v}$ gives rise to pseudovector vorticity field (curl of the velocity vector) in the presence of circulation or rotationality of the flow velocity $\mathbf{v}$. Exact solutions are possible for the degenerate cases, in which the nonlinear terms vanish, as in the steady laminar flow – the well-known Hagen–Poiseuille flow through a capillary tube of uniform cross-section [20]. Other examples include Couette flow [15,16] (between the rotating cylinders and also between the moving parallel planes) and the oscillatory Stokes boundary layer [21].

In most of the practical problems, the presence of the nonlinear convective term $(\mathbf{v} \cdot \nabla)\mathbf{v}$ cannot be ignored. Analytical solutions for the resulting set of nonlinear partial differential equations range from difficult to practically impossible. Only in a few special cases, the NS equation is solvable in closed form. Interesting examples of such solutions to the full nonlinear equations, include Jeffery–Hamel flow, Von Kármán swirling flow, stagnation point flow [22], Landau–Squire jet [15] and Taylor–Green vortex [23]. The existence of these exact solutions, however, does not guarantee stability – turbulence may develop at higher Reynolds numbers. As the velocity increases, the complex vortices and turbulence, or chaos that occur in 3D fluid flows make the calculation intractable to any but approximate numerical methods for simulating the nonlinear term.

However, in a parallel field theoretic approach using vorticity and the Lamb vector as two independent field variables, equations of the fluid motion (the NS equations) are reformulated in terms of the divergence and the evolution equation of the two fields. The nonlinearities are absorbed in the divergence and the evolution equation of the Lamb vector field and assume special physical significance as the input source fields. In the process, a more tractable set of linear field equations can be developed to describe the dynamics of fluid flow. The source terms depend on the geometry and the total energetics of the flow and are also apt for modelling. With the input source terms, we are thus led to a ‘closed’ set of linear equations and the dynamics described in this manner is called the ‘metafluid dynamics’ by Marmanis [3]. While Marmanis considered an incompressible fluid, the Maxwell-type equations for a compressible fluid is constructed by Abreu *et al* [19] by taking into account a dissipation term from the beginning.

In this paper, using proper GA formalism, a set of four similar equations are developed for the most general case of compressible viscous fluid. Both Eulerian inviscid flow and incompressible flow are described with the same set by modifying the source fields with $\eta = 0$ and solenoidal velocity field, i.e. $\nabla \cdot \mathbf{v} = 0$. Moreover, the appropriate space–time force equation in this comprehensive formulation provides, both the power equation and the Lorentz-type force law as in the case of EM.

3. Geometric algebra and calculus

The GA developed by Clifford provides a natural unification of the algebras of Grassmann and Hamilton into a single structure. The higher grade elements of exterior algebra extends the usual 3D vector space, spanned by 4 unit bases $(1,\ \{\hat{\alpha}_i\})$, to a multivector space spanned by $8\,(= 2^3)$ unit bases:

$$1,\ \{\hat{\alpha}_i\},\ \{\hat{\alpha}_i \wedge \hat{\alpha}_j\} \text{ and } I_3\,(= \hat{\alpha}_1 \wedge \hat{\alpha}_2 \wedge \hat{\alpha}_3),$$

with $i, j = 1, 2, 3$ and $i \neq j$. 1 and I_3 $(I_3^2 = -1)$ are unit scalar and pseudoscalar respectively and form a pair of unit dual bases. Similarly, the unit vector and bivector bases $\hat{\alpha}_i$ and $\hat{\alpha}_j \wedge \hat{\alpha}_k$ $(= I_3\,\hat{\alpha}_i = \hat{\alpha}_i\, I_3)$ form another pair of unit dual bases. The product $I_3\,\hat{\alpha}_i$ implies contraction (or usual dot product) of I_3 with $\hat{\alpha}_i$. Also in 3D, the pseudovector $\mathbf{u} \times \mathbf{v}$ may be regarded as the dual of the bivector $\mathbf{u} \wedge \mathbf{v}$ since $\mathbf{u} \wedge \mathbf{v} = I_3(\mathbf{u} \times \mathbf{v}) = (\mathbf{u} \times \mathbf{v})I_3$.

Dot product of two vectors produces a scalar, and hence the name scalar product in vector algebra. After introducing the exterior product, Grassmann [24] also introduced the dot product between two vectors, defined similarly, but always to be carried out first in a sequence, i.e. $\mathbf{u} \cdot \mathbf{v} \wedge \mathbf{w} \equiv (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$. In tensor algebra, this product is termed as inner product or contraction which reduces the total rank by 2 and the product is not always a scalar. For the higher rank tensors, it is extended with the provision of multiple inner products or contractions, producing different lower rank tensors in addition to scalars. Similarly, the exterior algebra also uses multiple inner products or contractions of two multivector blades. For example, contraction between a vector and a bivector results in a vector whereas, with two bivectors one gets a bivector from a single contraction and a scalar from a double contraction.

The inner and exterior products of two vectors complement each other: while the inner product lowers the grade, the other raises it, one is commutative and the other is anticommutative. However, they are not invertible in general. By combining both exterior (wedge) and inner (dot) products of Grassmann algebra, Clifford's ingenuous contribution was to define a new associative product, the geometric product [25]. For any two arbitrary vectors **u** and **v**, the product is defined as

$$\mathbf{uv} = \mathbf{u}.\mathbf{v} + \mathbf{u} \wedge \mathbf{v} = \mathcal{C} \text{ (say)} \Rightarrow \mathbf{u}.\mathbf{v} = (\mathbf{uv} + \mathbf{vu})/2 \quad \text{and} \quad \mathbf{u} \wedge \mathbf{v} = (\mathbf{uv} - \mathbf{vu})/2,$$

and it endows the basic vector space with an algebraic structure that embraces vector, complex, quaternion and the spin algebra in a single formalism and sets apart Clifford algebra from others. The product $\mathcal{C}$ is the sum of a scalar s $(= \mathbf{u} \cdot \mathbf{v})$ and a bivector $\mathbf{A}$ $(= \mathbf{u} \wedge \mathbf{v})$ and represents a mixed multivector. Here, the multivectors of definite grade are represented with bold capitals and calligraphics represent multivectors of mixed grade. The even subalgebra generated by the geometric product **uv** contains only scalars and bivectors and in two dimensions the even subalgebra is isomorphic to the complex numbers, while in three dimensions it represents the four-component quaternion defined by Hamilton. In fact, the geometric product appears as a more fundamental product of the algebra, since both the inner and the exterior products can be derived from it. The associativity and almost invertibility of the geometric product makes this algebra a formidable tool of mathematical physics.

Elements of GA represent both physical quantities and operations. Projections, rejections, reflections along vectors and rotations in planes – all these operations are handled much more efficiently in GA than in traditional vector and matrix algebras [7]. The unit quaternion, called rotor in GA, encode rotation and using the elliptic functional form introduced by Hestenes [26], the rotor is found to be more efficient than the conventional rotation matrix. In a sequence of rotations, interpolation with quaternionic representation is far more convenient than that with the rotation matrix.

3.1 *Geometric calculus*

Retaining the gradient and divergence operations and discarding curl $(= \nabla \times)$ of the 3D vector calculus, the 'exterior calculus' introduces the exterior derivative $\nabla \wedge$ of a vector field **f**, defined as

$$\begin{aligned}\nabla \wedge \mathbf{f} &= \hat{\alpha}_i \partial_{x_i} \wedge \hat{\alpha}_j f_j = \hat{\alpha}_i \wedge \hat{\alpha}_j \, \partial_{x_i} f_j \\ &\equiv I_3(\nabla \times \mathbf{f}) = (\nabla \times \mathbf{f}) I_3 \\ &\Rightarrow \nabla \times \mathbf{f} = -I_3(\nabla \wedge \mathbf{f}) = -(\nabla \wedge \mathbf{f}) I_3 \text{ etc.}\end{aligned}$$

In geometric calculus, like the geometric product, a generalised gradient operation on multivectors is similarly defined with $\nabla \equiv \nabla. + \nabla\wedge$ and using this operator a complete unification of the field equations for both EM [12] and FM (as we will see here) can be achieved. Hestenes also generalised the calculus of differential forms according to GA and given an invariant formulation of the Hamiltonian mechanics in terms of geometric calculus [27].

3.2 *The space–time algebra and calculus*

The GA, like exterior algebra can be seamlessly extended from 2, 3 dimensions to two arbitrary higher dimensions. Replacing the Euclidean metric by the Minkowski metric with appropriate basis vectors, the 3D GA can be extended to the algebra of 4D Minkowski space–time or simply space–time algebra [8,28]. The four basis vectors $\hat{\alpha}_\mu$, $\mu = 0, 1, 2, 3$, satisfying $\hat{\alpha}_\mu \cdot \hat{\alpha}_\nu = \eta_{\mu\nu}$, generates the sequence $(- + + +)$ of algebraic signs on the main diagonal of the flat space–time metric in which $\hat{\alpha}_0^2 = -1 = -\hat{\alpha}_k^2$ (the opposite signature of $(+ - - -)$ can also be used). Here, the Greek indices run from 0 to 3 and Latin indices run from 1 to 3. $\hat{\alpha}_k$s are evidently similar to the usual orthogonal basis vectors of 3D space, where $\hat{\alpha}_0$ is the time-like basis vector of the 4D space–time. The corresponding space–time algebra is spanned by $2^4 = 16$ multivector unit bases: 1, $\{\hat{\alpha}_\mu\}$, $\{\hat{\alpha}_j \hat{\alpha}_0, \hat{\alpha}_j \hat{\alpha}_k\}$, $\{I_4 \hat{\alpha}_\mu\}$ and I_4 $(= \hat{\alpha}_0 \hat{\alpha}_1 \hat{\alpha}_2 \hat{\alpha}_3)$, with $\mu = 0, 1, 2, 3\,;\ j, k = 1, 2, 3$ and $j \neq k$. The geometric product, for the entire set of basis vectors, is similarly defined as

$$\begin{aligned}\hat{\alpha}_\mu \hat{\alpha}_\nu &= \hat{\alpha}_\mu \wedge \hat{\alpha}_\nu, \ \mu \neq \nu \\ &= \hat{\alpha}_\mu \cdot \hat{\alpha}_\nu, \quad \mu = \nu,\end{aligned}$$

1 and I_4 $(I_4^2 = -1)$ are unit scalar and pseudoscalar, respectively and forms a pair of unit dual bases. The dual sets of $\hat{\alpha}_j \hat{\alpha}_0$ (time-like) and $\hat{\alpha}_j \hat{\alpha}_k$ (space-like) bases together represent six orthogonal space–time bivector bases. Also, $I_4 \hat{\alpha}_\mu$, the dual of $\hat{\alpha}_\mu$, represents four trivector bases of which $I_4 \hat{\alpha}_0 = \hat{\alpha}_1 \hat{\alpha}_2 \hat{\alpha}_3$, may be identified as the unit pseudoscalar I_3 of the associated 3D space.

In 4D space–time, the highest grade-4 blades are pseudoscalars and the grade-3 trivector blades, dual of the vectors, are called antivectors. The geometric product is a sum of the multiple inner products or contractions and the exterior product and is termed a multivector. All these multivectors, linear combinations of blades of different grades, are elements of a geometric algebra.

The even subalgebra of higher-dimensional space, the spinors, generalises the rotation–dilation produced by quaternions in 3D space. It may be noted that, rotors can handle much more complex rotations and in the non-Euclidean space. For example, rotors in 4D space–time continuum produce Lorentz boost in addition to the usual rotations on three orthogonal spatial planes.

Quaternions and spinors have equivalent algebraic properties as well as the same geometric significance [26]. In fact, after a long time, Pauli in the formulation of quantum mechanical spinor algebra and Dirac in his theory of the relativistic electron, though not appreciating fully, have rediscovered Clifford algebra. It was Hestenes [29] who finally demonstrated that both the Pauli and Dirac algebras are indeed expressible in the language of GA and has carried out an extensive reformulation of the theory of spinors – without invoking anything quantum-mechanical!

The gradient operator in space–time calculus is correspondingly expressed as $\hat{\alpha}^{\mu}\partial_{x_\mu} = -\hat{\alpha}_0 c^{-1}\partial_t + \nabla$ which is equivalent to the Dirac operator $\Box$, using four γ matrices instead of the space–time basis vectors $\hat{\alpha}_\mu$. For a check, we note: $(-\hat{\alpha}_0 c^{-1}\partial_t + \nabla)\cdot(-\hat{\alpha}_0 c^{-1}\partial_t + \nabla) = -c^{-2}\partial_t^2 + \nabla^2$ is the d'Alembertian $\Box^2$ – the Laplacian of Minkowski space.

4. Fluid mechanics using geometric algebra

Using a common identity of geometric calculus (GC) for the nonlinear term: $(\mathbf{v}\cdot\nabla)\mathbf{v} = \mathbf{v}\cdot(\nabla\wedge\mathbf{v}) + \frac{1}{2}\nabla v^2$; with $v \equiv |\mathbf{v}|$, eq. (2) becomes

$$\partial_t \mathbf{v} = -\mathbf{v}\cdot(\nabla\wedge\mathbf{v}) - \frac{\nabla v^2}{2} - \nabla(\rho^{-1}p + gz) + \mathbf{k}$$
$$\Rightarrow \partial_t \mathbf{v} = -\mathbf{v}\cdot\mathbf{W} - \nabla\Phi + \eta\{\nabla^2\mathbf{v} + 3^{-1}\nabla(\nabla\cdot\mathbf{v})\}, \quad (3)$$

where

$$\mathbf{W} = \nabla\wedge\mathbf{v}. \quad (4)$$

The bivector $\mathbf{W}$ ($\equiv I_3\,\mathbf{w}$, the vector $\mathbf{w}$ being the dual of $\mathbf{W}$) provides the proper representation of the vorticity field (just like the magnetic field in electromagnetic theory) and the term Φ $(= \frac{v^2}{2} + \frac{p}{\rho} + gz)$ is the Bernoulli energy function or simply the Bernoulli head. In order to include the effect of viscous dissipative terms from the beginning, we now redefine the Lamb vector field as: $\mathbf{l} = \mathbf{v}\cdot\mathbf{W} - \mathbf{k}$ and from eq. (3) we get

$$\mathbf{l} = -\partial_t\mathbf{v} - \nabla\Phi \quad (5)$$

– an expression for the Lamb vector $\mathbf{l}$ in terms of Φ and $\mathbf{v}$, exactly similar to that for the electric field ($\mathbf{e} = -\nabla\phi - \partial_t\mathbf{a}$), described by the conservative scalar potential ϕ due to the electric charge distribution and the non-conservative term involving the time derivative of the electromagnetic vector potential $\mathbf{a}$. Putting $\eta = 0 \Rightarrow \mathbf{l} \equiv \mathbf{l}_{\text{invisc}} = \mathbf{v}\cdot\mathbf{W}$ (the vector part $\mathbf{v}\cdot\mathbf{W}$ of the geometric product $\mathbf{vW}$ $(= \mathbf{v}\cdot\mathbf{W} + \mathbf{v}\wedge\mathbf{W})$ constitutes the Lamb vector. The pseudoscalar part $\mathbf{v}\wedge\mathbf{W}$ is the helicity density of the vorticity field [30,31], in analogy with the helicity density $(\mathbf{a}\wedge\mathbf{B})$ of the magnetic field in electromagnetism. It appears in the expression for the fluidic current density (in eq. (10)) and measures the linkage or knottedness of vortex lines in the flow and also may be considered as the handedness (or chirality) of the flow as it changes sign from a right-handed to a left-handed frame of reference.), and both eqs (3) and (5) describe the Euler equation for the inviscid flow.

Though the examples of true inviscid fluids or superfluids are limited, inviscid flow has many applications in fluid dynamics. The Euler equation is as well applicable in many fluid dynamical problems involving low viscosity and large Reynolds number. However, the assumed negligible viscosity is no longer valid near a solid boundary.

In a steady flow ($\partial_t\mathbf{v} = 0$) of an ideal fluid along a streamline, the Euler equation gives Bernoulli equation. Since $\mathbf{v}$ is orthogonal to the vector $\mathbf{v}\cdot\mathbf{W}$, taking scalar product with $\mathbf{v}$ one gets from eq. (5): $\mathbf{v}\cdot\nabla\Phi = 0$; which implies that Φ is constant along a streamline. This is Bernoulli's theorem of fluid dynamics. In addition, if the flow is irrotational, i.e. vorticity $\mathbf{W} = 0$, we get $\nabla\Phi = 0$ or Φ = constant everywhere. The Bernoulli's theorem is actually a statement of conservation of energy and can also be deduced using simple arguments [6]. Another equation named after Torricelli, expressing the magnitude of the final velocity ($v = \sqrt{2gz}$) of a fluid flowing out of an orifice at a depth z from the top of a full reservoir, where g is the acceleration due to gravity, can be shown to be a particular case of Bernoulli's theorem.

The Lamb vector for the inviscid case $\mathbf{v}\cdot\mathbf{W}$, also known as vortex force or the NS swirl field, is identically zero in irrotational flows ($\mathbf{W} = 0$). Also, if the flow velocity $\mathbf{v}$ is normal to the vorticity $\mathbf{W}$ plane, $\mathbf{l}$ is again zero, resulting in what is known as the Beltrami flow. Both the vorticity and the Lamb vector are derived from the velocity field and as such do not give any new information that is not available from the velocity field. According to Stokes' theorem, vorticity is related to the flow's circulation (line integral of the velocity along a closed path) per unit area of an infinitesimal loop. Vorticity gives a microscopic measure of the rotation at any point in the fluid. Vortex is a fluid structure within which any fluid particle experiences a 'rotation' and is associated with the vorticity bivector field. Vorticity has a magnitude which is twice the angular velocity of the rotating fluid structure and the plane of the bivector represents the plane of rotation [6]. Whereas vorticity is essentially a fluid related term, the angular velocity term is generally used for all sorts of rotational motion. Vortex can be either laminar or turbulent. Vorticity emphasises the rotational content of the fluid motion and the evolution and the divergence of the other dynamical variable, the Lamb vector field represents the source fields in the field theoretic study of the problem.

In this formulation, the Bernoulli energy function Φ plays the part of the scalar potential instead of enthalpy, which is used by several researchers to describe the scalar potential [4,32]. Here, Φ appears to be a better choice to represent the scalar potential. In this approach, the velocity field $\mathbf{v}$ plays as usual the role of the vector potential along with Φ as the scalar potential and the field variables $\mathbf{W}$ and $\mathbf{l}$ are the two other essential elements of the theory.

A flow situation in which the kinetic energy is significantly absorbed due to the action of fluid viscosity gives rise to a laminar flow regime at low velocity and density, smaller characteristic linear dimensions and at higher viscosity. However, the most important and interesting problem in fluid dynamics is the phenomenon of turbulence, distinguished by high Reynolds number, high diffusivity and dissipation and 3D vortical fluctuations. Turbulence is impossible in irrotational flows and vorticity is necessary (but not sufficient) for turbulent flow to start and is characterised by many spatio-temporal scales, produced and sustained by continuous transfer of energy and momentum from the larger to smaller scales. In turbulent flow regime, the system's inertial forces dominate over the viscous forces, the flow energy is extracted and transferred to the swirling motion and the reverse current created – the 'eddies'. The energy transferred is then get cascaded to smaller eddies and so on, until all the kinetic energy gets transferred to thermal energy by viscosity. It is possible to present turbulent flow as an interplay between vorticity and the Lamb fields [3]. However, due to the complex nature of the nonlinear NS equation, in most cases, the equations are solved for describing average quantities through simulation and approximations [33].

Using the formal similarity between the field variables of fluid mechanics (eqs (4) and (5)) and the corresponding (field) variables of electromagnetic theory, together with suitably defined source terms, a similar set of linear field equations (for these two field variables) can be developed to describe the dynamics of the fluid motions. In the following, the 'fluidomechanic' charge and the current densities will be defined by the divergence and the evolution equations of the Lamb vector, respectively. Now, from the very definition of the vorticity field $\mathbf{W}$ (eq. (4)), we directly get

$$\nabla \wedge \mathbf{W} = 0, \tag{6}$$

just like the Maxwell equation for the magnetic field. In the most general case of viscous flow, the vorticity equation (also known as Helmholtz equation), describing the evolution of the vorticity field can be derived from eq. (3) and is given by

$$\partial_t \mathbf{W} = \nabla \wedge \partial_t \mathbf{v} = -\nabla \wedge \mathbf{l} \equiv -\nabla \wedge (\mathbf{v} \cdot \mathbf{W}) + \nabla \wedge \mathbf{k}. \tag{7}$$

The equation also represents the law of angular momentum conservation applied to the fluid flow [6]. In both the evolution equations (3) and (7), the coefficient of viscosity η enters as a multiplicative factor to the Laplacian of the respective field. The first term of the right hand side of eq. (7) may be expanded as: $-\nabla \wedge (\mathbf{v} \cdot \mathbf{W}) = -\mathbf{W}(\nabla \cdot \mathbf{v}) - (\mathbf{v} \cdot \nabla)\mathbf{W} + (\mathbf{W} \wedge \nabla) \cdot \mathbf{v}$, since $\nabla \wedge \mathbf{W} = 0$.

The first term $-\mathbf{W}(\nabla \cdot \mathbf{v})$ describes the stretching of vorticity due to flow compressibility, the second term $-(\mathbf{v} \cdot \nabla)\mathbf{W}$ is the advection term whereas the third term $(\mathbf{W} \wedge \nabla) \cdot \mathbf{v}$ describes the stretching or tilting of vorticity due to the flow velocity gradients. The random field of turbulence, however, exhibits certain organised structures of vorticity. Vorticity increases when vortex lines are stretched, enhancing dissipation [30]. The Burgers–Rott vortex, apart from serving as an example which provides an exact solution to the NS equations (for viscous flow), furnishes an illustration of the vortex stretching mechanism.

Next, we consider the divergence of the Lamb vector. By taking the divergence of both sides in eq. (5) we get

$$\begin{aligned}\nabla \cdot \mathbf{l} &= -\partial_t \nabla \cdot \mathbf{v} - \nabla^2 \Phi = n(\mathbf{r}, t) \equiv n \\ &= -\nabla^2 \Phi \quad \text{for incompressible flow.}\end{aligned} \tag{8}$$

It is important to note that, the divergence of the Lamb vector is the same for both inviscid and viscous incompressible flow. The Laplacian of Φ, a function of position and time, is identically zero for irrotational flows. More important is the specific way in which it connects with the vorticity. The function $n(\mathbf{r}, t)$ represents the fluidomechanic (also called 'turbulent') charge density and for incompressibile flow, $n = -\nabla^2 \Phi$. In turbulent flow, this function will be significantly greater than in a laminar flow. There is a tendency for Φ to accumulate in regions where the divergence of the Lamb vector is greater than 0.

Using simple identity of geometric calculus, the divergence of the Lamb vector can also be expressed as

$$\begin{aligned}\nabla \cdot \mathbf{l} &= \nabla \cdot (\mathbf{v} \cdot \mathbf{W} - \mathbf{k}) \\ &= \mathbf{W} : \mathbf{W} - \mathbf{v} \cdot (\nabla \cdot \mathbf{W}) - \nabla \cdot \mathbf{k} \\ &\equiv \mathbf{W} : \mathbf{W} - \mathbf{v} \cdot \nabla^2 \mathbf{v} + (\mathbf{v} \cdot \nabla - \frac{4}{3}\eta \nabla^2) \nabla \cdot \mathbf{v}.\end{aligned} \tag{9}$$

For the incompressible fluid flow, $\nabla \cdot \mathbf{v} = 0 \Rightarrow \nabla \cdot \mathbf{l} = \mathbf{W} : \mathbf{W} - \mathbf{v} \cdot \nabla^2 \mathbf{v} = -\nabla^2 \Phi$. Comparing eq. (8) with eq. (9), we can write $\nabla \cdot \mathbf{l} = -\nabla^2 \Phi + (\mathbf{v} \cdot \nabla - \frac{4}{3}\eta \nabla^2) \nabla \cdot \mathbf{v} = n$. The first term on the right-hand side of eq. (9) – the square magnitude of the vorticity $\mathbf{W}^2 = \mathbf{W} : \mathbf{W}$, a scalar field representing the strength of the vorticity field defines another important quantity *enstrophy* density. Applying the vector integral theorem on the NS equation (3), it follows that for an incompressible fluid, $2^{-1}\partial_t v^2 = \eta \mathbf{W} : \mathbf{W}$, i.e., the time rate of change of the flow energy (per unit mass) is proportional to the viscosity coefficient η times the enstrophy density [30]. The negative value implies the decline of the flow energy or dissipation effect. Large value of enstrophy due to the stretching of vortex filaments indicates turbulent flow [34].

The importance and the physical properties of the Lamb vector and its divergence have been studied and explored by several researchers [35,36]. While the overall Lamb vector divergence can be positive, zero or negative, positive contributions can only arise owing to the second term $-\mathbf{v} \cdot (\nabla \cdot \mathbf{W})$, the so-called 'flexion product'. Lamb vector divergence is identically zero for the irrotational ($\mathbf{W} = 0$) and Beltrami ($\mathbf{l} = 0$) flows and more generally, whenever the Lamb vector is solenoidal. In such cases, the two parts may be locally balanced, i.e., $\mathbf{W} : \mathbf{W} = \mathbf{v} \cdot (\nabla^2 \mathbf{v})$ and need not be zero separately. Turbulence occurs frequently in regions where the sign of the Lamb vector divergence switches between negative and positive.

Finally, we derive the evolution of the Lamb vector, which is not adequately described and studied in the literature. In the first place, from eq. (5) we get the expression: $\partial_t \mathbf{l} = -\partial_t^2 \mathbf{v} - \nabla \partial_t \Phi$. More explicit and useful expression for the evolution equation of the Lamb vector is obtained from the very definition: $\mathbf{l} = \mathbf{v} \cdot \mathbf{W} - \mathbf{k}$ as $\partial_t \mathbf{l} = \partial_t (\mathbf{v} \cdot \mathbf{W}) - \partial_t \mathbf{k} = \partial_t \mathbf{v} \cdot \mathbf{W} + \mathbf{v} \cdot \partial_t \mathbf{W} - \partial_t \mathbf{k}$, and use of eqs (5) and (7) yields

$$\begin{aligned}
\partial_t \mathbf{l} &= (-\mathbf{l} - \nabla\Phi) \cdot \mathbf{W} + \mathbf{v} \cdot (-\nabla \wedge \mathbf{l}) - \partial_t \mathbf{k} \\
&= -\mathbf{l} \cdot \mathbf{W} - (\nabla\Phi) \cdot \mathbf{W} - \mathbf{v} \cdot (\nabla \wedge \mathbf{l}) - \partial_t \mathbf{k} \\
&= -v^2\, \nabla \cdot \mathbf{W} - \mathbf{v}\, \nabla \cdot \mathbf{l} + \{\nabla \cdot (\mathbf{v} \wedge \mathbf{W})\} \cdot \mathbf{v} \\
&\quad + (\mathbf{v} \wedge \mathbf{W}) : \nabla \wedge \mathbf{v} + \mathbf{W} \cdot \nabla(\Phi + v^2) \\
&\quad - 2\, (\mathbf{l} \cdot \nabla)\mathbf{v} + \mathbf{l}(\nabla \cdot \mathbf{v}) - \nabla(\mathbf{v} \cdot \mathbf{k}) \\
&\quad + \nabla \cdot (\mathbf{v} \wedge \mathbf{k}) - \partial_t \mathbf{k} \quad \Rightarrow \partial_t \mathbf{l} + v^2 \nabla \cdot \mathbf{W} = -\mathbf{j},
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{j} &= \mathbf{v}\, n - \{\nabla \cdot (\mathbf{v} \wedge \mathbf{W})\} \cdot \mathbf{v} \\
&\quad - (\mathbf{v} \wedge \mathbf{W}) : \mathbf{W} - \mathbf{W} \cdot \nabla(\Phi + v^2) \\
&\quad + 2\, (\mathbf{l} \cdot \nabla)\mathbf{v} - \mathbf{l}(\nabla \cdot \mathbf{v}) + \nabla(\mathbf{v} \cdot \mathbf{k}) \\
&\quad - \nabla \cdot (\mathbf{v} \wedge \mathbf{k}) + \partial_t \mathbf{k},
\end{aligned}$$

with

$$\begin{aligned}
n &= \nabla \cdot \mathbf{l}, \\
\mathbf{k} &= -\eta\{\nabla^2 \mathbf{v} + (3^{-1}\nabla)\nabla \cdot \mathbf{v}\}
\end{aligned}$$

and

$$\begin{aligned}
&\partial_t \mathbf{k} = -\eta\{\nabla^2 + (3^{-1}\nabla)\nabla \cdot\} \\
&\quad (\mathbf{l} + \nabla\Phi) \text{ etc.} \\
&\quad (\text{see Appendix}). \qquad (10)
\end{aligned}$$

Identifying n and $\mathbf{j}$ as the source quantities, possessing spatial and temporal structures, it is possible to present turbulence as an interplay between vorticity and the Lamb vector. The viscous dissipation terms are absorbed in the source terms through eqs (8) and (10).

4.1 *Field theoretic formulation according to the geometric algebra*

The guiding NS equation of fluid mechanics, expressed in terms of appropriately defined vorticity $\mathbf{W}$ and the Lamb vector $\mathbf{l}$ according to geometric calculus as a set of four geometrically distinct (scalar, vector, bivector and pseudoscalar) field equations (6)–(10), are now rearranged as

$$\begin{aligned}
&\nabla \cdot \mathbf{l} = n; \;\; v^2\, \nabla \cdot \mathbf{W} + \partial_t \mathbf{l} = -\mathbf{j}; \\
&\partial_t \mathbf{W} + \nabla \wedge \mathbf{l} = 0 \quad \text{and} \quad \nabla \wedge \mathbf{W} = 0, \qquad (11)
\end{aligned}$$

where eqs (8) and (10) furnish complete expressions for the charge and current densities (n and $\mathbf{j}$) respectively. The same set of four equations can be used for Eulerian inviscid flow by modifying the source fields with $\eta = 0$, whereas for incompressible flow the velocity field is solenoidal, i.e. $\nabla \cdot \mathbf{v} = 0$. The modified expressions of n and $\mathbf{j}$ for incompressible and/or inviscid flow according to eq. (10) are equivalent with the corresponding expressions quoted in [3,13,19], using vector calculus. Both the sources (the charge and the current densities) are inputs and to be determined by observing the geometry and the total energetics of the flow.

Earlier, Troshkin [2] examined the significance of the analogy between the incompressible Euler equation and Maxwell equations. He studied the turbulent fluctuations in an ideal turbulent medium with the system of equations for average velocities and Reynolds stresses using perturbation technique. Marmanis [3] subsequently presented an analogy between incompressible NS equation and Maxwell equation and applied the formulation to study the turbulent fluid flow for high Reynolds numbers, with averaged field quantities. He has not considered dissipation terms arguing that, for a theory of high Reynolds number, the viscous corrections are important only in the very small scales and are eventually filtered out by the averaging procedure. However, the derivation of the expressions in this important study is marred with repeated and simultaneous use of cross and wedge products. In a more general and independent formulation, Kambe [4] derived a set of analogous equations for inviscid compressible fluid flow with source terms. Inclusion of viscous effects in the formulation is also indicated. In another interesting paper, extending the analogy between fluid mechanics and electromagnetism, Thompson and Moeller [18] formulated a set of Maxwell equations for the charged fluid 'plasma'.

Abreu *et al* [19], on the other hand, obtained the Maxwell-type equations for a compressible fluid with dissipation term, considering the viscosity from the beginning. However, the formulation omitted certain source terms. The authors have also constructed the Lagrangian for this fluid. To analyse the applications of this formalism in quark-gluon plasma (QGP), they also developed the non-Abelian generalisation of some results. Besides, a correlation function and the dispersion relation have been analysed as functions of the Reynolds number.

The fluid mechanical analogues of the Lorentz force law and the Poynting theorem of EM can also be developed in this formulation. Using the identity, $\nabla\cdot(\mathbf{W}\cdot\mathbf{l}) = (\nabla\cdot\mathbf{W})\cdot\mathbf{l} - (\nabla\wedge\mathbf{l}) : \mathbf{W}$ and the vorticity equation $\partial_t\mathbf{W} = -\nabla\wedge\mathbf{l}$, we can develop analogous 'Poynting theorem' in FM by writing the rate of energy supplied to the system:

$$\begin{aligned}\partial_t u = \mathbf{j}\cdot\mathbf{l} &= -\{\partial_t\mathbf{l} - v^2\nabla\cdot\mathbf{W}\}\cdot\mathbf{l}\\ &= -\frac{\partial_t\mathbf{l}^2}{2} - v^2\{(\nabla\wedge\mathbf{l}) : \mathbf{W} + \nabla\cdot(\mathbf{W}\cdot\mathbf{l})\}\\ &= -\partial_t\frac{\mathbf{l}^2 - v^2\mathbf{W}^2}{2} - v^2\nabla\cdot(\mathbf{W}\cdot\mathbf{l})\\ &= -\partial_t u_f - \nabla\cdot\mathbf{s}_f \quad\Rightarrow \partial_t(u+u_f) + \nabla\cdot\mathbf{s}_f\\ &= 0,\end{aligned} \tag{12}$$

where $u_f = 2^{-1}(\mathbf{l}^2 - v^2\mathbf{W}^2)$ is the energy density of the fluid and $\mathbf{s}_f = v^2\,\mathbf{W}\cdot\mathbf{l}$ is the corresponding 'Poynting vector' ($\nabla\cdot\mathbf{s}_f$ representing the flux of the energy flowing out). Also, in analogy with the classical electromagnetic field, the Lagrangian density may be written as $\mathcal{L} = \frac{\rho}{2}v^2 - n\,\Phi + \mathbf{v}\cdot\mathbf{j}$, where $n\,\Phi - \mathbf{v}\cdot\mathbf{j}$ represents the potential part. Then, applying the Euler–Lagrangian equation and using the appropriate field equations, analogous expression for the Lorentz force density can be derived as

$$\mathbf{f} = n\,\mathbf{l} - \mathbf{j}\cdot\mathbf{W}. \tag{13}$$

Earlier, Scofield and Huq [37], using the tensor calculus, also derived the fluid mechanical analogue of the Lorentz force and the Poynting theorem (of electromagnetic theory) and discussed the implications. From the analysis, they observed that the fluidic Lorentz force and the Poynting theorem describe new channels of stress-energy propagation and dissipation. In the following, we will see that a single space–time force equation, just as in the case of EM, provides a complete account for both the Poynting theorem and the Lorentz force law.

4.2 *Unification of the four geometrically distinct field equations*

Now, using the formalism of geometric calculus by introducing a paravector differential operator $(c^{-1}\partial t + \nabla)$ and the fluidomechanic parabivector field $\mathcal{F} = \mathbf{l} + c\,\mathbf{W}$, we can combine all the four equations (11) into a single multivector equation in terms of the paravector source term $\mathcal{J} = c\,n - \mathbf{j}$ as

$$\begin{aligned}(c^{-1}\partial t + \nabla)\mathcal{F} &= c^{-1}\partial t\mathcal{F} + \nabla\cdot\mathcal{F} + \nabla\wedge\mathcal{F}\\ &= n - c^{-1}\mathbf{j} \equiv v^{-1}\mathcal{J},\end{aligned} \tag{14}$$

where the dimensional parameter c of the differential operator is equated with the magnitude of the instantaneous fluid velocity v at the end. Operating both sides of eq. (14) with $(-c^{-1}\partial t + \nabla)$ and equating the scalar parts, we get the equation of continuity expressing fluidomechanic charge conservation:

$$\partial_t n + \nabla\cdot\mathbf{j} = 0 \tag{15}$$

– a new equation which relates n and $\mathbf{j}$, where the turbulent current is the flux of the turbulent charge. With suitable applications, further significance of this equation may be gained.

5. Reformulation of fluid mechanics in space–time algebra

Finally, extending the algebra of 3D space to the algebra of 4D Minkowski space–time [8,28] with the replacement of the Euclidean metric by the Minkowski metric, a more compact and elegant description of FM can be achieved. Specially, this reformulation in terms of space–time algebra leads to possible relativistic investigation of the fluid motion.

Replacing the paravector differential operator by the space–time gradient (Dirac operator) $\square = \hat{\alpha}^{\mu}\partial_{x_\mu} = -\hat{\alpha}_0 c^{-1}\partial_t + \nabla$ and the parabivector field by the appropriate space–time bivector $\mathbf{F} = \mathbf{l}\wedge\hat{\alpha}_0 - c\,\mathbf{W}$, eq. (14) takes the form

$$\square\mathbf{F} = \square\cdot\mathbf{F} + \square\wedge\mathbf{F} = c^{-1}\bar{\mathbf{j}} \equiv v^{-1}\bar{\mathbf{j}}, \tag{16}$$

where the source field is accordingly represented by the space–time current density vector $\bar{\mathbf{j}} = \hat{\alpha}_0 c\,n + \mathbf{j}$ and c of the Dirac operator is substituted at the end with v, the magnitude of the instantaneous fluid velocity. The left-hand side of eq. (16) contains both vector and trivector parts, whereas the right-hand side contains only a space–time vector. From the two constituent vector ($\square\cdot\mathbf{F} = v^{-1}\bar{\mathbf{j}}$) and trivector ($\square\wedge\mathbf{F} = 0$) equations, equating the time-like and space-like bases of the two sides of the equations separately, one gets back the set of four equations (11). It is important to note that the unification of the separate equations for divergence and curl in a single equation is non-trivial—both the unified equations (14) and (16) can be inverted directly to determine the combined field. Also, by defining the space–time vector potential $\bar{\mathbf{a}} = \hat{\alpha}_0\Phi - c\,\mathbf{v}$, one can consistently express the space–time bivector field as $\mathbf{F} = \square \wedge \bar{\mathbf{a}}$ and since $\square \cdot (\square \cdot \mathbf{F})$ is zero, we get $\square \cdot \bar{\mathbf{j}} = 0$ – eq. (15) expressing charge conservation. Demir and Tanişli [32] reformulated the Maxwell-type equations for compressible inviscid fluid on the basis of space–time algebra in a compact and elegant form. Moreover, the fluid wave equation in terms of potentials are derived 'in a form similar to electromagnetic and gravitational counterparts'. In this formulation, the scalar potential is represented by the enthalpy and they have not considered viscous dissipation terms.

Both the power equation and the Lorentz force law can also be obtained, as in the case of EM, from the appropriate space–time force equation as

$$\begin{aligned}\bar{\mathbf{f}} &\equiv \partial_t \bar{\mathbf{p}} = c^{-1}\,\bar{\mathbf{j}} \cdot \mathbf{F}\\ &\Rightarrow \hat{\alpha}_0\, c^{-1}\, \partial_t u + \partial_t \mathbf{p}\\ &= (\hat{\alpha}_0\, n + c^{-1}\,\mathbf{j}) \cdot (\mathbf{l} \wedge \hat{\alpha}_0 - c\,\mathbf{W})\\ &= n\,\mathbf{l} + c^{-1}\,\hat{\alpha}_0\,\mathbf{j} \cdot \mathbf{l} - \mathbf{j} \cdot \mathbf{W}.\end{aligned}$$

Separation of the temporal and spatial parts of the space–time force yields both power and Lorentz force equations. Written explicitly:

$$\begin{aligned}\partial_t u = \mathbf{j} \cdot \mathbf{l} &= -\partial_t \frac{\mathbf{l}^2 - v^2\mathbf{W}^2}{2}\\ &\quad - v^2 \nabla \cdot (\mathbf{W} \cdot \mathbf{l})\\ &= -\partial_t u_f - \nabla \cdot \mathbf{s}_f\end{aligned}$$

and

$$\mathbf{f} = \partial_t \mathbf{p} = n\,\mathbf{l} - \mathbf{j} \cdot \mathbf{W},$$

as in eqs (12) and (13).

In this formulation, as in the case of EM [12], from the scalar part $\langle \mathbf{F}^2 \rangle_0 = \mathbf{l}^2 + v^2\mathbf{W}^2$ of the square of the fluidomechanic bivector field $\mathbf{F}$ which is independent of the reference frame, the Lagrangian density $\mathcal{L}$ can be obtained, although this is not true for the field energy density u_f – which is an observer-dependent quantity. One can also derive the field equation (eq. (16)) from the Euler–Lagrange equation, as an alternative. In the presence of the source terms, using the interaction energy $\bar{\mathbf{j}} \cdot \bar{\mathbf{a}}$ $(= \langle \bar{\mathbf{j}}\,\bar{\mathbf{a}} \rangle_0)$ together with the invariant scalar $\langle \mathbf{F}^2 \rangle_0$, the total Lagrangian density in the appropriate form is

$$\mathcal{L} = \left(\left\langle \frac{\mathbf{F}^2}{2} + v^{-1}\bar{\mathbf{j}}\,\bar{\mathbf{a}} \right\rangle_0 \right) \equiv \left\langle \frac{(\square \wedge \bar{\mathbf{a}})^2}{2} + v^{-1}\,\bar{\mathbf{j}}\,\bar{\mathbf{a}} \right\rangle_0. \quad (17)$$

In functional form we may write: $\mathcal{L} \equiv \mathcal{L}(\bar{\mathbf{a}}, \square \wedge \bar{\mathbf{a}})$ and the space–time Euler–Lagrange equation may be accordingly expressed as

$$\frac{\partial\,\mathcal{L}}{\partial\,\bar{\mathbf{a}}} - \square \frac{\partial\,\mathcal{L}}{\partial(\square \wedge \bar{\mathbf{a}})} = 0.$$

Substitution of $\mathcal{L}$ from eq. (17) in the above equation finally reproduces eq. (16): $\square \cdot (\square \wedge \bar{\mathbf{a}}) = \square \cdot \mathbf{F} = v^{-1}\,\bar{\mathbf{j}}$. The remaining equation also follows as $\square \wedge \mathbf{F} = \square \wedge \square \wedge \bar{\mathbf{a}} = 0$. The potential formulation thus facilitates the formulation of a Lagrangian field theory of FM starting with a scalar-valued Lagrangian density. The approach also leads to the conservation laws of energy and momentum.

6. Concluding remarks

The advantages of using the geometric algebraic formulation of fluid mechanics are discussed and clarified. Formal similarities between physical concepts, notions and in mathematical structures, specially between the theories of electromagnetism and fluid mechanics are highlighted. In the first place, reinterpreting the guiding NS equation(s) with the constituent vorticity and the Lamb fields and identifying certain quantities as the source fields, it is possible to present a set of four geometrically distinct field equations – scalar, vector, bivector and pseudoscalar equations, respectively, in the usual 3D space. The same set of equations works for all the cases of compressible, incompressible, viscous and inviscid fluid flows. Only the source fields, the charge density and the vector current density have additional terms in a compressible and/or viscous fluid flow. With the two source fields n and $\mathbf{j}$, we get the equation of continuity – a new equation expressing fluidic charge conservation. Determining the distribution of 'fluidomechanic sources' experimentally or numerically, the set of linear equations can be solved for specific problems.

The analogy with the Maxwell equation of electromagnetism is completed with the unification of all the four equations in terms of an appropriate combined fluidomechanic bivector field $\mathbf{F}$ in space–time algebra. Secondly, a full-fledged field theoretic description is obtained with the derivation of the fluidic analogue of the Poynting theorem, Poynting vector and Lorentz force from the space–time force equation. As an alternative, it is shown that one can also derive the field equation (16) from the appropriate Euler–Lagrange equation using the Lagrangian density $\mathcal{L}$ obtained from the combined field $\mathbf{F}$.

On the other hand, concepts and formalisms of FM are also incorporated in electromagnetic theory. For example, the term eddy current in electricity comes from analogous currents seen in fluid dynamics, causing vigorous localised circulations known as eddies. Eddy currents are induced by changing magnetic fields and circulate in conductors in closed loops in the plane of the magnetic field like swirling eddies in a stream. Critical appreciations of the two formulations offer important insights to share between the two. New perspective from the turbulent hydrodynamics also helped in attaining illuminating observations on the electromagnetic field.

Starting from the microscopic Maxwell equations and with the introduction of polarisation and magnetisation fields, Liu [38] developed a set of irreversible, nonlinear 'hydrodynamic Maxwell' equations in continuous media. The author identified two additional thermodynamic forces which 'give rise to dissipative terms and represent mechanism for the EM fields to restore equilibrium in non-conducting media' and expected many more consequences for 'ferrofluids', superconductors and nematic liquid crystals with appropriate modification of the Maxwell's equations. Holland [39] proposed an alternative Eulerian model and argued that EM phenomena, conventionally described in a field-theoretic language, also admit a complementary description in terms of a many-particle system possessing an interaction potential. The method facilitates the construction of spin-0 state for the EM field and offers an analogue of the quantum potential formulation. In addition, using the classical trajectories, an expression for the propagator of the EM field is derived in the Eulerian picture.

As in FM, new extra terms are introduced using the convective derivative in moving EM systems. In developing electro- and magnetohydrodynamics, stellar evolution etc., NS-like equation must be solved to determine the time and spatial response of charges. Parallels are made between the inertia property of matter, electromagnetism and the hydrodynamic drag in potential flow. The methodological treatment provided by the 'fluidic electrodynamics' approach is rewarding on various accounts. A new approach to electrodynamics, based on a fluidic viewpoint, developed an NS-like equation by using the appropriate electromotive force. Recent works introducing new concepts suggest among others, possible applications in producing electric fields of the required configuration in the plasma medium [5]. The authors also mentioned several important approaches in solving complex problems of high-energy physics and cosmology by identifying conceptual similarities between disparate physical phenomena.

The newly developing field theoretic approach of the FM allows a fast transposition from electromagnetism to fluid dynamics and vice versa. It is natural to expect that the comprehensive theoretical framework presented in this paper will provide new insights and open up new directions of investigation with many more consequences.

Appendix

Using eqs (5) and (7), the evolution equation of the Lamb vector field $\mathbf{l}$ can be written as

$$\begin{aligned}\partial_t \mathbf{l} &= \partial_t(\mathbf{v}\cdot\mathbf{W}-\mathbf{k}) = \partial_t\mathbf{v}\cdot\mathbf{W}+\mathbf{v}\cdot\partial_t\mathbf{W}-\partial_t\mathbf{k}\\ &= (-\mathbf{l}-\nabla\Phi)\cdot\mathbf{W}+\mathbf{v}\cdot(-\nabla\wedge\mathbf{l})\\ &\quad -\eta\{\nabla^2+(3^{-1}\nabla)\nabla\cdot\}\partial_t\mathbf{v}\\ &= -\mathbf{l}\cdot\mathbf{W}-(\nabla\Phi)\cdot\mathbf{W}-\mathbf{v}\cdot(\nabla\wedge\mathbf{l})\\ &\quad -\eta\{\nabla^2+(3^{-1}\nabla)\nabla\cdot\}(-\mathbf{l}-\nabla\Phi).\end{aligned}\tag{A.1}$$

With two vector fields $\mathbf{l}$ and $\mathbf{v}$, a known identity of geometric calculus reads as

$$\begin{aligned}\nabla(\mathbf{v}\cdot\mathbf{l}) &= (\mathbf{v}\cdot\nabla)\mathbf{l}+(\mathbf{l}\cdot\nabla)\mathbf{v}+(\nabla\wedge\mathbf{v})\cdot\mathbf{l}+(\nabla\wedge\mathbf{l})\cdot\mathbf{v}\\ &\Rightarrow (\nabla\wedge\mathbf{l})\cdot\mathbf{v}\equiv -\mathbf{v}\cdot(\nabla\wedge\mathbf{l}) = -\nabla(\mathbf{v}\cdot\mathbf{k})\\ &\quad -(\mathbf{v}\cdot\nabla)\mathbf{l}-(\mathbf{l}\cdot\nabla)\mathbf{v}+\mathbf{l}\cdot\mathbf{W}\end{aligned}\tag{A.2}$$

since $\mathbf{v}\cdot(\mathbf{v}\cdot\mathbf{W})$ vanishes identically. Now, substitution of (A.2) in (A.1) gives

$$\begin{aligned}\partial_t\mathbf{l} &= -\nabla(\mathbf{v}\cdot\mathbf{k})-(\mathbf{v}\cdot\nabla)\mathbf{l}-(\mathbf{l}\cdot\nabla)\mathbf{v}+\mathbf{W}\cdot\nabla\Phi\\ &\quad +\eta\{\nabla^2+(3^{-1}\nabla)\nabla\cdot\}(\mathbf{l}+\nabla\Phi).\end{aligned}$$

Moreover, from the following identity of GC:

$$\begin{aligned}\nabla\cdot(\mathbf{v}\wedge\mathbf{l}) &= \mathbf{l}(\nabla\cdot\mathbf{v})+(\mathbf{v}\cdot\nabla)\mathbf{l}-\mathbf{v}(\nabla\cdot\mathbf{l})-(\mathbf{l}\cdot\nabla)\mathbf{v}\\ &\Rightarrow (\mathbf{v}\cdot\nabla)\mathbf{l} = \nabla\cdot(\mathbf{v}\wedge\mathbf{l})-\mathbf{l}(\nabla\cdot\mathbf{v})\\ &\quad +\mathbf{v}(\nabla\cdot\mathbf{l})+(\mathbf{l}\cdot\nabla)\mathbf{v}\end{aligned}$$

and we can write:

$$\begin{aligned}\partial_t\mathbf{l} &= -\nabla(\mathbf{v}\cdot\mathbf{k})-\nabla\cdot(\mathbf{v}\wedge\mathbf{l})+\mathbf{l}(\nabla\cdot\mathbf{v})-\mathbf{v}(\nabla\cdot\mathbf{l})\\ &\quad -2(\mathbf{l}\cdot\nabla)\mathbf{v}+\mathbf{W}\cdot\nabla\Phi\\ &\quad +\eta\{\nabla^2+(3^{-1}\nabla)\nabla\cdot\}(\mathbf{l}+\nabla\Phi).\end{aligned}\tag{A.3}$$

Expressing the second term on the right-hand side of the above equation as

$$\begin{aligned}&-\nabla\cdot(\mathbf{v}\wedge(\mathbf{v}\cdot\mathbf{W}))+\nabla\cdot(\mathbf{v}\wedge\mathbf{k})\\ &\quad = -\nabla\cdot\{v^2\mathbf{W}-(\mathbf{v}\wedge\mathbf{W})\cdot\mathbf{v}\}+\nabla\cdot(\mathbf{v}\wedge\mathbf{k})\\ &\quad = -\nabla v^2\cdot\mathbf{W}-v^2\nabla\cdot\mathbf{W}+\{\nabla\cdot(\mathbf{v}\wedge\mathbf{W})\}\cdot\mathbf{v}\\ &\qquad +(\mathbf{v}\wedge\mathbf{W}):\nabla\wedge\mathbf{v}+\nabla\cdot(\mathbf{v}\wedge\mathbf{k}),\end{aligned}$$

eq. (A.3) can be written as

$$\begin{aligned}\partial_t\mathbf{l} &= -\nabla(\mathbf{v}\cdot\mathbf{k})-v^2\,\nabla\cdot\mathbf{W}+\{\nabla\cdot(\mathbf{v}\wedge\mathbf{W})\}\cdot\mathbf{v}\\ &\quad +(\mathbf{v}\wedge\mathbf{W}):\mathbf{W}+\nabla\cdot(\mathbf{v}\wedge\mathbf{k})+\mathbf{l}(\nabla\cdot\mathbf{v})-\mathbf{v}(\nabla\cdot\mathbf{l})\\ &\quad -2\,(\mathbf{l}\cdot\nabla)\mathbf{v}+\mathbf{W}\cdot\nabla(\Phi+v^2)+\eta\nabla^2(\mathbf{l}+\nabla\Phi)\\ &\quad +3^{-1}\eta\nabla\{\nabla\cdot(\mathbf{l}+\nabla\Phi)\}.\end{aligned}$$

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