



A theoretical description of the first-forbidden β -decay transitions within Pyatov's restoration method

SERDAR ÜNLÜ^{1,*}, HASAN BİRCAN², NECLA ÇAKMAK³ and CEVAD SELAM⁴

¹Department of Physics, Burdur Mehmet Akif Ersoy University, Burdur, Turkey

²Department of Physics, Kütahya Dumlupınar University, Kütahya, Turkey

³Department of Physics, Karabük University, Karabük, Turkey

⁴Muş Alparslan University, Muş, Türkiye

*Corresponding author. E-mail: serdarunlu@mehmetakif.edu.tr

MS received 15 December 2022; revised 7 March 2023; accepted 21 March 2023

Abstract. We describe the first-forbidden beta transitions by using Pyatov's restoration method within the framework of proton–neutron quasiparticle random phase approximation (pn-QRPA). A detailed formalism related to how to obtain the energies and wave functions of the first-forbidden excitations is clearly given in the present work. A comparison of the calculated results for various nuclei with the corresponding experimental data is given to demonstrate an application of the present approximation.

Keywords. Restoration; first-forbidden beta transition; proton–neutron quasiparticle random phase approximation.

PACS Nos 21.30.-x; 23.40.-s; 21.60.Ev

1. Introduction

Investigation of weak interaction processes remains a prominent issue in the domain of nuclear physics and also significantly contributes to the explanation of astrophysical processes such as nuclear synthesis [1] and supernova explosions [2,3]. The allowed spin–isospin transitions (Gamow–Teller (GT) transitions) are the most common weak interaction processes. In situations where the allowed GT transitions are not favoured, first-forbidden transitions become important, particularly for medium and heavy mass nuclei. A theoretical study of GT and first-forbidden transitions plays a significant role in determining quantitative constraints for the tensor force [4].

The theoretical description of single and double β -decay rates is still an open question for the nuclear structure theories [5]. As known, the mean-field approximation is not successful in reproducing the experimental data related to decay observables due to the exclusion of

two-body effective interactions. However, the proton–neutron quasiparticle random phase approximation (pn-QRPA) has been considered the most potent method for single and double β -decay calculations [5–46]. The behaviour of double β -decay amplitude was searched using the pn-QRPA method [5–37,40,41,46]. The violations in isospin symmetry and spin–isospin symmetry significantly affect double β -decay rates which are obtained within the pn-QRPA [18,28,34,37,40,46]. The first-forbidden contributions to $2\nu\beta^-\beta^-$ decay rates were computed [23,41]. The contributions from the allowed and first-forbidden transitions for single- β -decay rates were obtained [38,39,42–45]. Recently, the allowed and first-forbidden β -decay processes have been investigated using self-consistent approximations in the framework of the pn-QRPA method [47–51]. The first-forbidden β -decay properties have been calculated by using gross theory [47]. The β -decay half-lives and delayed neutron emission probabilities for double magic ^{78}Ni and ^{132}Sn nuclei have been obtained using Landau–Migdal interaction in the particle–hole channel within a self-consistent continuum QRPA model [48]. A large-scale evaluation of β -decay rates of r-process nuclei with the inclusion of first-forbidden transitions has been given within a fully self-consistent microscopic

C. Selam: Retired from Muş Alparslan University, Muş, Türkiye.

theoretical framework [49]. The β^- -decay strengths for axially-deformed nuclei have been computed using the Skyrme finite-amplitude method, and the experimental data related to the β -decay rates and spin-dipole resonance have been reproduced by fitting Skyrme parameters [50]. The β -decay rates for neutron-rich nuclei have been studied using Skyrme energy density-functional theory [51].

Pyatov's restoration method is an efficient way to define effective interaction potential, which was initially introduced to restore broken Galilean invariance of pairing interaction [52]. Then, it has been extended to the restoration of symmetry violations which stem from the mean-field approximation [34,37,40,46,53–68]. The symmetry restoring treatment of the pairing potential has been discussed in the case of a separable monopole pairing potential on a quasiparticle basis [53]. A comparison between the effective interactions used in the symmetry restoration methods is given for intrinsic excitations in superfluid nuclei [54]. The broken Galilean invariance in superfluid nuclei and its connection with quadrupole pairing interactions are studied for even-mass Sn isotopes [55]. The electric dipole (E1) and magnetic dipole (M1) transitions have been investigated within the translational and rotational invariant approximations, respectively [56–59]. The restoration of broken isospin invariance of nuclear Hamiltonian makes an important contribution to understand the isobar analogue resonance and superallowed Fermi transitions ($\Delta L = 0, \Delta S = 0$) [60–66]. The description of spin-isospin transitions within the restoration of SU(4) symmetry violations plays a significant role in understanding GT transitions ($\Delta L = 0, \Delta S = \pm 1$) [67,68] and double β -decay [34,37,40,46].

First-forbidden beta transition probabilities ($\Delta L = \pm 1, \Delta S = 0, \pm 1$) are sensitive to the violations in translational invariance since the corresponding transition operator contains the dipole component (r). Charge-exchange spin-dipole transitions ($\Delta L = \pm 1, \Delta S = \pm 1$) show the sensitivity to the violations in both SU(4) symmetry and translational invariance. Due to these symmetry violations, the charge-exchange spin-dipole operator does not commute with the total Hamiltonian operator. It is well known that Coulomb and spin-orbit terms in the Hamiltonian break the SU(4) symmetry. Also, the kinetic energy part of the total Hamiltonian does not commute with the charge-exchange spin-dipole operator. In other words, the remaining part of the total Hamiltonian besides these terms commutes with the transition operator. However, this commutativity is broken in the mean-field level of approximation like other symmetry properties. At this point, Pyatov's restoration procedure can be applied to restore a broken commutator correlation between the total Hamiltonian operator

and the first-forbidden beta transition operator. Thus, it becomes possible to give a theoretical description of the first-forbidden transitions, which is free of effective interaction strength parameters. A detailed explanation of the restoration is given in the next section.

2. Model and method

2.1 Restoration of broken commutator correlation

2.1.1 *Mean-field basis.* The mean-field potential is described in the following form:

$$U_{mf} = U_{cent} + U_{ls} + U_C. \quad (1)$$

The central part consists of the isoscalar (U_0) and the isovector (U_1) terms as follows:

$$U_{cent} = U_0 + U_1, \\ U_0 = -V_0 f(r), \quad U_1 = V_1(r) t_z.$$

r -dependent isovector potential can be defined as

$$V_1(r) = 2\eta V_0 f(r) \frac{N - Z}{A}.$$

The distribution function is given by

$$f(r) = \frac{1}{1 + e^{\frac{r-R_0}{a}}}.$$

The isospin components for neutron and proton are written as

$$t_z = 1/2 \text{ (neutron)}, \quad t_z = -1/2 \text{ (proton)}.$$

The spin-orbit term is defined as

$$U_{ls} = V_{ls}(r) (\mathbf{l} \cdot \mathbf{s}).$$

r -dependent spin-orbit potential can be defined as

$$V_{ls}(r) = -\xi \frac{1}{r} \frac{dU_{cent}}{dr}$$

and the Coulomb term is given as

$$U_C = V_C(r) \left(\frac{1}{2} - t_z \right).$$

r -dependency in Coulomb potential can be introduced as

$$V_C(r) = e^2 \frac{Z-1}{r} \frac{3r}{2R_C} - \frac{1}{2} \left(\frac{r}{R_C} \right)^3, \quad r \leq R_C,$$

$$V_C(r) = e^2 \frac{Z-1}{r}, \quad r > R_C.$$

The Woods–Saxon potential with Chepurnov parametrisation [69] is usually suitable for single-particle levels.

Let us consider a system of nucleons in a spherically symmetric average field with pairing forces. In

this case, the single quasiparticle Hamiltonian in the second-quantisation representation is given by

$$\hat{H}_{sqp} = \sum_{jm} \varepsilon_j(\tau) \hat{\alpha}_{jm}^\dagger(\tau) \hat{\alpha}_{jm}(\tau), \quad \tau = n, p, \quad (2)$$

where $\hat{\alpha}_{jm}^\dagger$ and $\hat{\alpha}_{jm}$ are one-quasiparticle creation and annihilation operators, respectively. The proton and neutron pairing gaps are defined as $\Delta_p = C_p/\sqrt{A}$ and $\Delta_n = C_n/\sqrt{A}$ [70], respectively. The pairing strength parameters C_p and C_n are chosen in such a way that the experimental pairing gaps [71] are reproduced.

2.1.2 Charge-exchange spin-dipole transition operator. The charge-exchange spin-dipole operator for β^+ transitions is defined as follows:

$$\begin{aligned} \hat{T}_{\lambda\mu}^+ = & \sum_{j_n m_n j_p m_p} \langle j_n m_n | [r(\hat{Y}_1 \otimes \hat{\sigma}_1)]_{\lambda\mu} | j_p m_p \rangle \\ & \times \hat{\alpha}_{j_n m_n}^\dagger \hat{\alpha}_{j_p m_p}, \end{aligned} \quad (3)$$

where $\hat{Y}_1 \otimes \hat{\sigma}_1$ is a tensor product of the spin and dipole operators and $\hat{\alpha}_{j_n m_n}^\dagger$ ($\hat{\alpha}_{j_p m_p}$) is a one-particle creation (annihilation) operator, λ and μ are the corresponding nuclear spin for the transition and its projection, respectively. By means of Bogolyubov quasiparticle transformations [72], it is possible to define the transition operator for even and odd- A nuclei separately.

(a) *Even- A nuclei*

The corresponding transition operator for β^+ -decay can be written as a combination of quasiboson creation and annihilation operators

$$\begin{aligned} \hat{T}_{\lambda\mu}^+ = & \sum_{np} \bar{b}_{np}(\lambda) \hat{C}_{np}^\dagger(\lambda\mu) \\ & + (-1)^{\lambda+\mu} b_{np}(\lambda) \hat{C}_{np}(\lambda, -\mu). \end{aligned} \quad (4)$$

The transition operator for β^- transitions is defined as a Hermitian conjugate operator:

$$\hat{T}_{\lambda\mu}^- = [\hat{T}_{\lambda\mu}^+]^\dagger.$$

One-quasiboson creation and annihilation operators are given by

$$\begin{aligned} \hat{C}_{np}^\dagger(\lambda\mu) = & \sqrt{\frac{2\lambda+1}{2j_n+1}} \sum_{m_n m_p} \langle j_p m_p \lambda \mu | j_n m_n \rangle (-1)^{j_p-m_p} \\ & \times \hat{\alpha}_{j_n m_n}^\dagger \hat{\alpha}_{j_p, -m_p}^\dagger \end{aligned}$$

and

$$\hat{C}_{np}(\lambda\mu) = [\hat{C}_{np}^\dagger(\lambda\mu)]^\dagger.$$

The reduced matrix elements ($\bar{b}_{np}(\lambda)$, $b_{np}(\lambda)$) are given as defined by Varshalovich *et al* [73]

$$\bar{b}_{np}(\lambda) = \frac{\langle j_n || [r(\hat{Y}_1 \otimes \hat{\sigma}_1)]_{\lambda\mu} || j_p \rangle}{\sqrt{2\lambda+1}} u_n v_p,$$

$$b_{np}(\lambda) = \frac{\bar{b}_{np}(\lambda)}{u_n v_p} u_p v_n,$$

where v and u are single-particle and hole amplitudes, respectively.

(b) *Odd- A nuclei*

For odd- A nuclei, the scattering terms coming from Bogolyubov quasiparticle transformations are also considered as follows:

$$\begin{aligned} \hat{T}_{\lambda\mu}^+ = & \sum_{np} \bar{b}_{np}(\lambda) \\ & \times \hat{C}_{np}^\dagger(\lambda\mu) + (-1)^{\lambda+\mu} b_{np}(\lambda) \hat{C}_{np}(\lambda, -\mu) \\ & + \bar{d}_{np}(\lambda) \hat{D}_{np}^\dagger(\lambda\mu) \\ & + (-1)^{\lambda+\mu+1} d_{np}(\lambda) \hat{D}_{np}(\lambda, -\mu) \end{aligned} \quad (5)$$

and

$$\hat{D}_{np}^\dagger(\lambda\mu) = \sqrt{\frac{2\lambda+1}{2j_n+1}} \sum_{m_n m_p} \langle j_p m_p \lambda \mu | j_n m_n \rangle \hat{\alpha}_{j_n m_n}^\dagger \hat{\alpha}_{j_p m_p}.$$

Also,

$$\begin{aligned} \hat{D}_{np}(\lambda\mu) = & [\hat{D}_{np}^\dagger(\lambda\mu)]^\dagger \\ \bar{d}_{np}(\lambda) = & \frac{\bar{b}_{np}(\lambda)}{u_n v_p} u_p u_n, \quad d_{np}(\lambda) = \frac{\bar{b}_{np}(\lambda)}{u_n v_p} v_p v_n \end{aligned}$$

2.1.3 Effective interaction. A charge-exchange spin-dipole operator can be defined as a combination of β^- and β^+ decay operators in the following form:

$$\hat{F}_{\lambda\mu}^\rho = \frac{1}{2} (\hat{T}_{\lambda\mu}^+ + \rho (-1)^{\lambda+\mu} \hat{T}_{\lambda, -\mu}^-), \quad \rho = \pm. \quad (6)$$

Many-body Hamiltonian does not commute with the charge-exchange spin-dipole operator because of Coulomb, spin-orbit and kinetic energy terms:

$$[\hat{H}, \hat{F}_{\lambda\mu}^\rho] = [\hat{U}_C + \hat{U}_{ls} + \hat{P}^2/2m, \hat{F}_{\lambda\mu}^\rho]. \quad (7)$$

In other words, the remaining part of the Hamiltonian must commute with the transition operator.

$$[\hat{H} - \hat{U}_C - \hat{U}_{ls} - \hat{P}^2/2m, \hat{F}_{\lambda\mu}^\rho] = 0. \quad (8)$$

This commutativity is broken in the mean-field level of approximation as follows:

$$[\hat{H}_{sqp} - \hat{U}_C - \hat{U}_{ls} - \hat{P}^2/2m, \hat{F}_{\lambda\mu}^\rho] \neq 0. \quad (9)$$

According to the mean-field potential defined in eq. (1), eq. (9) can be rewritten in the following form:

$$[\widehat{U}_{\text{cent}}, \widehat{F}_{\lambda\mu}^\rho] \neq 0. \quad (10)$$

Isoscalar potential already commutes with the transition operator, and so the violation in commutator correlation stems from the isovector term in the mean-field potential.

$$[\widehat{U}_1, \widehat{F}_{\lambda\mu}^\rho] \neq 0. \quad (11)$$

Hence, the nucleon–nucleon effective interaction potential should be considered in such a way that the broken commutator correlation is restored.

$$[\widehat{U}_1 + \widehat{h}^\lambda, \widehat{F}_{\lambda\mu}^\rho] = 0. \quad (12)$$

The effective interaction potential consists of the particle–hole (ph) and the particle–particle (pp) terms, and is defined within Pyatov’s restoration method:

$$\begin{aligned} \widehat{h}^\lambda = & \sum_{\rho} \frac{1}{4\gamma_{\text{ph}}^\rho(\lambda)} \sum_{\mu} [\widehat{U}_1, \widehat{F}_{\lambda\mu}^\rho]_{\text{ph}}^\dagger \cdot [\widehat{U}_1, \widehat{F}_{\lambda\mu}^\rho]_{\text{ph}} \\ & + \sum_{\rho} \frac{1}{4\gamma_{\text{pp}}^\rho(\lambda)} \sum_{\mu} [\widehat{U}_1, \widehat{F}_{\lambda\mu}^\rho]_{\text{pp}}^\dagger \cdot [\widehat{U}_1, \widehat{F}_{\lambda\mu}^\rho]_{\text{pp}}. \end{aligned} \quad (13)$$

As seen in eq. (13), the commutator correlation in eq. (12) contains two unknown strength parameters. Hence, the particle–hole ($\gamma_{\text{ph}}^\rho(\lambda)$) and the particle–particle ($\gamma_{\text{pp}}^\rho(\lambda)$) strength parameters can be determined analytically using two different commutator correlations, which are defined by adding a constant taking value as $0 < c < 1$.

$$[c\widehat{U}_1 + \widehat{h}_{\text{ph}}^\lambda, \widehat{F}_{\lambda\mu}^\rho] = 0 \quad (14)$$

$$[(1-c)\widehat{U}_1 + \widehat{h}_{\text{pp}}^\lambda, \widehat{F}_{\lambda\mu}^\rho] = 0 \quad (15)$$

$$\widehat{h}_{\text{ph}}^\lambda = \sum_{\rho} \frac{1}{4\gamma_{\text{ph}}^\rho(\lambda)} \sum_{\mu} [\widehat{U}_1, \widehat{F}_{\lambda\mu}^\rho]_{\text{ph}}^\dagger \cdot [\widehat{U}_1, \widehat{F}_{\lambda\mu}^\rho]_{\text{ph}}$$

$$\widehat{h}_{\text{pp}}^\lambda = \sum_{\rho} \frac{1}{4\gamma_{\text{pp}}^\rho(\lambda)} \sum_{\mu} [\widehat{U}_1, \widehat{F}_{\lambda\mu}^\rho]_{\text{pp}}^\dagger \cdot [\widehat{U}_1, \widehat{F}_{\lambda\mu}^\rho]_{\text{pp}}$$

The following expressions for particle–hole and particle–particle strength parameters are obtained using the commutator correlations in eqs (14) and (15), respectively.

$$\gamma_{\text{ph}}^\rho(\lambda) = \frac{\rho(-1)^{\lambda+\mu}}{2c} \langle G.S. | [[\widehat{U}_1, \widehat{F}_{\lambda,-\mu}^\rho]_{\text{ph}}, \widehat{F}_{\lambda\mu}^\rho] | G.S. \rangle \quad (16)$$

$$\gamma_{\text{pp}}^\rho(\lambda) = \frac{\rho(-1)^{\lambda+\mu}}{2(1-c)} \langle G.S. | [[\widehat{U}_1, \widehat{F}_{\lambda,-\mu}^\rho]_{\text{pp}}, \widehat{F}_{\lambda\mu}^\rho] | G.S. \rangle. \quad (17)$$

(a) *The strength parameters for even-A nuclei*

The commutator correlation between the isovector potential and transition operator is obtained as follows:

$$\begin{aligned} [\widehat{U}_1, \widehat{F}_{\lambda\mu}^\rho]_{\text{ph}} = & \sum_{np} f_{np}^\rho(\lambda) (\widehat{C}_{np}^\dagger(\lambda\mu) \\ & + \rho(-1)^{\lambda+\mu} \widehat{C}_{np}(\lambda, -\mu)) \\ f_{np}^\rho(\lambda) = & \frac{V_1(r)}{2} (\bar{b}_{np}(\lambda) - \rho b_{np}(\lambda)) \end{aligned} \quad (18)$$

and

$$\begin{aligned} [\widehat{U}_1, \widehat{F}_{\lambda\mu}^\rho]_{\text{pp}} = & \sum_{np} d_{np}^\rho(\lambda) (\widehat{C}_{np}^\dagger(\lambda\mu) \\ & + \rho(-1)^{\lambda+\mu} \widehat{C}_{np}(\lambda, -\mu)) \\ d_{np}^\rho(\lambda) = & \frac{V_1(r)}{2} (\bar{d}_{np}(\lambda) - \rho d_{np}(\lambda)). \end{aligned} \quad (19)$$

The double commutators for even-A nuclei ($|G.S.\rangle = |0\rangle$)

$$\gamma_{\text{ph}}^\rho(\lambda) = \frac{\rho(-1)^{\lambda+\mu}}{2c} \langle 0 | [[\widehat{U}_1, \widehat{F}_{\lambda,-\mu}^\rho]_{\text{ph}}, \widehat{F}_{\lambda\mu}^\rho] | 0 \rangle \quad (20)$$

$$\gamma_{\text{pp}}^\rho(\lambda) = \frac{\rho(-1)^{\lambda+\mu}}{2(1-c)} \langle 0 | [[\widehat{U}_1, \widehat{F}_{\lambda,-\mu}^\rho]_{\text{pp}}, \widehat{F}_{\lambda\mu}^\rho] | 0 \rangle \quad (21)$$

are solved and the final expressions for strength parameters are obtained as follows:

$$\gamma_{\text{ph}}^\rho(\lambda) = -\frac{1}{c} \sum_{np} f_{np}^\rho(\lambda) b_{np}^\rho(\lambda) \quad (22)$$

$$\gamma_{\text{pp}}^\rho(\lambda) = -\frac{1}{1-c} \sum_{np} d_{np}^\rho(\lambda) b_{np}^\rho(\lambda) \quad (23)$$

and $b_{np}^\rho(\lambda)$ is defined as

$$b_{np}^\rho(\lambda) = \frac{1}{2} (\bar{b}_{np}(\lambda) + \rho b_{np}(\lambda)).$$

(b) *The strength parameters for odd-A nuclei*

For odd-A nuclei, the commutator correlation between the isovector potential and transition operator is given as

$$\begin{aligned} [\widehat{U}_1, \widehat{F}_{\lambda\mu}^\rho]_{\text{ph}} = & \sum_{np} f_{np}^\rho(\lambda) (\widehat{C}_{np}^\dagger(\lambda\mu) + \rho(-1)^{\lambda+\mu} \widehat{C}_{np}(\lambda, -\mu)) \\ & + d_{np}^\rho(\lambda) (\widehat{D}_{np}^\dagger(\lambda\mu) + \rho(-1)^{\lambda+\mu+1} \widehat{D}_{np}(\lambda, -\mu)) \end{aligned} \quad (24)$$

and

$$\begin{aligned}
 & [\widehat{U}_1, \widehat{F}_{\lambda\mu}^\rho]_{pp} \\
 &= \sum_{np} d_{np}^\rho(\lambda) (\widehat{C}_{np}^\dagger(\lambda\mu) + \rho(-1)^{\lambda+\mu} \widehat{C}_{np}(\lambda, -\mu)) \\
 &+ f_{np}^\rho(\lambda) (\widehat{D}_{np}^\dagger(\lambda\mu) \\
 &+ \rho(-1)^{\lambda+\mu+1} \widehat{D}_{np}(\lambda, -\mu)). \quad (25)
 \end{aligned}$$

The double commutators for odd- A nuclei are given as

$$\begin{aligned}
 \gamma_{ph}^\rho(\lambda) &= \frac{\rho(-1)^{\lambda+\mu}}{2c} \\
 &\times \langle j_k m_k | [[\widehat{U}_1, \widehat{F}_{\lambda, -\mu}^\rho]_{ph}, \widehat{F}_{\lambda\mu}^\rho] | j_k m_k \rangle, \quad (26)
 \end{aligned}$$

$$\begin{aligned}
 \gamma_{pp}^\rho(\lambda) &= \frac{\rho(-1)^{\lambda+\mu}}{2(1-c)} \\
 &\times \langle j_k m_k | [[\widehat{U}_1, \widehat{F}_{\lambda, -\mu}^\rho]_{pp}, \widehat{F}_{\lambda\mu}^\rho] | j_k m_k \rangle, \quad (27)
 \end{aligned}$$

where the valence nucleon is represented by $|G.S.\rangle = |j_k m_k\rangle = \widehat{\alpha}_{j_k m_k}^\dagger |0\rangle$.

The final form of the strength parameter also contains a core–nucleon interaction term as follows:

$$\begin{aligned}
 \gamma_{ph}^\rho(\lambda) &= -\frac{1}{c} \sum_{np} f_{np}^\rho(\lambda) b_{np}^\rho(\lambda) \\
 &+ \sum_p d_{kp}^\rho(\lambda) g_{kp}^\rho(\lambda) \quad (\text{odd-neutron}) \quad (28)
 \end{aligned}$$

$$\begin{aligned}
 \gamma_{pp}^\rho(\lambda) &= -\frac{1}{1-c} \sum_{np} d_{np}^\rho(\lambda) b_{np}^\rho(\lambda) \\
 &+ \sum_p f_{kp}^\rho(\lambda) g_{kp}^\rho(\lambda) \quad (\text{odd-neutron}) \quad (29)
 \end{aligned}$$

$$\begin{aligned}
 \gamma_{ph}^\rho(\lambda) &= -\frac{1}{c} \sum_{np} f_{np}^\rho(\lambda) b_{np}^\rho(\lambda) \\
 &+ \sum_n d_{nk}^\rho(\lambda) g_{nk}^\rho(\lambda) \quad (\text{odd-proton}) \quad (30)
 \end{aligned}$$

$$\begin{aligned}
 \gamma_{pp}^\rho(\lambda) &= -\frac{1}{1-c} \sum_{np} d_{np}^\rho(\lambda) b_{np}^\rho(\lambda) \\
 &+ \sum_n f_{nk}^\rho(\lambda) g_{nk}^\rho(\lambda) \quad (\text{odd-proton}) \quad (31)
 \end{aligned}$$

and $g_{np}^\rho(\lambda)$ is defined as

$$g_{np}^\rho(\lambda) = \frac{1}{2} (\bar{d}_{np}(\lambda) + \rho d_{np}(\lambda)).$$

2.2 First-forbidden excitations and transition rates

The collective Hamiltonian for the first-forbidden transitions can be defined as follows:

$$H_\lambda = H_{sqp} + h_\lambda. \quad (32)$$

The following equation is solved to determine the corresponding energies and wave functions for the first-forbidden excitations:

$$[\widehat{H}_\lambda, \widehat{\Psi}(\lambda)] |G.S.\rangle = \omega^\lambda \widehat{\Psi}(\lambda) |G.S.\rangle. \quad (33)$$

2.2.1 First-forbidden excitations in even- A nuclei.

The first-forbidden excitations for even- A nuclei are represented by a phonon-creation operator in the following form:

$$\begin{aligned}
 \widehat{\Psi}_i(\lambda\mu) |g.s.\rangle &= \widehat{Q}_i^\dagger(\lambda\mu) |0\rangle \\
 &= \left[\sum_{np} X_{np}^i(\lambda) \widehat{C}_{np}^\dagger(\lambda\mu) \right. \\
 &\left. + (-1)^{\lambda+\mu} Y_{np}^i(\lambda) \widehat{C}_{np}(\lambda, -\mu) \right] |0\rangle. \quad (34)
 \end{aligned}$$

The commutator correlation between the phonon creation and annihilation operators is given by

$$[\widehat{Q}_i(\lambda\mu), \widehat{Q}_j^\dagger(\lambda\mu)] = \delta_{ij}. \quad (35)$$

The total Hamiltonian can be diagonalised by solving the equation of motion:

$$[\widehat{H}_\lambda, \widehat{Q}_i^\dagger(\lambda\mu)] |0\rangle = \omega_i^\lambda \widehat{Q}_i^\dagger(\lambda\mu) |0\rangle. \quad (36)$$

The orthonormalisation condition giving the forward (X_{np}^i) and backward (Y_{np}^j) amplitudes is obtained using eq. (35):

$$\sum_{np} (X_{np}^i(\lambda))^* X_{np}^j(\lambda) - (Y_{np}^i(\lambda))^* Y_{np}^j(\lambda) = \delta_{ij}. \quad (37)$$

2.2.2 First-forbidden excitations in odd- A nuclei.

The suitable version of the pn-QRPA for odd-mass nuclei is called the proton–neutron quasiparticle phonon nuclear model (pn-QPNM). According to pn-QPNM, the first-forbidden excitations are represented by the following operator:

$$\begin{aligned}
 \widehat{\Psi}_k^n(\lambda) |G.S.\rangle &= (\widehat{\Omega}_k^n(\lambda))^\dagger |0\rangle = N_k^n(\lambda) \widehat{\alpha}_{j_k m_k}^\dagger |0\rangle \\
 &+ \left[\sum_{iv} R_{ni}^{kv}(\lambda) \langle j_\nu m_\nu \lambda \mu | j_k m_k \rangle \widehat{Q}_i^\dagger(\lambda\mu) \widehat{\alpha}_{j_\nu m_\nu}^\dagger \right] |0\rangle, \quad (38)
 \end{aligned}$$

where $k(\nu)$ indices correspond to valence nucleons for the initial (final) nucleus as neutron (proton) or proton (neutron). The corresponding energies for the first-forbidden excitations in the neighbour nuclei are determined by solving the equation of motion as

$$[\widehat{H}_\lambda, (\widehat{\Omega}_k^n(\lambda))^\dagger] |0\rangle = \omega_k^n(\lambda) (\widehat{\Omega}_k^n(\lambda))^\dagger |0\rangle. \quad (39)$$

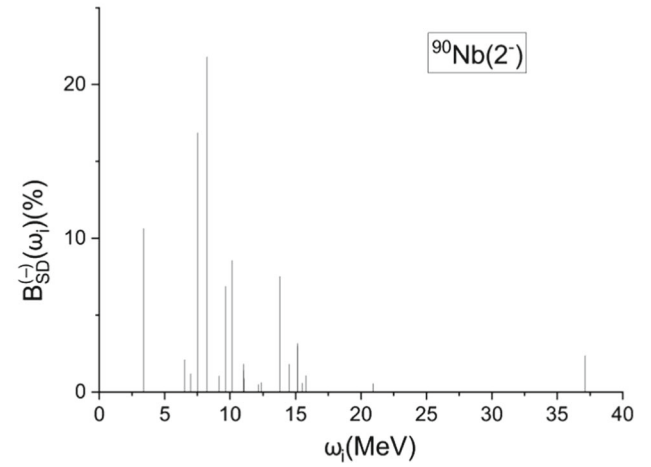
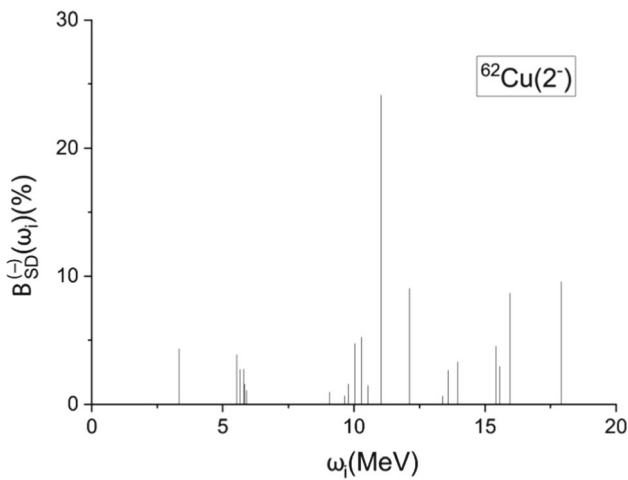
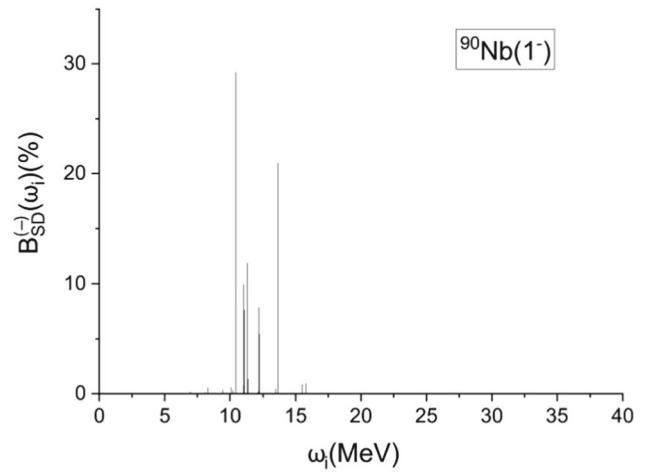
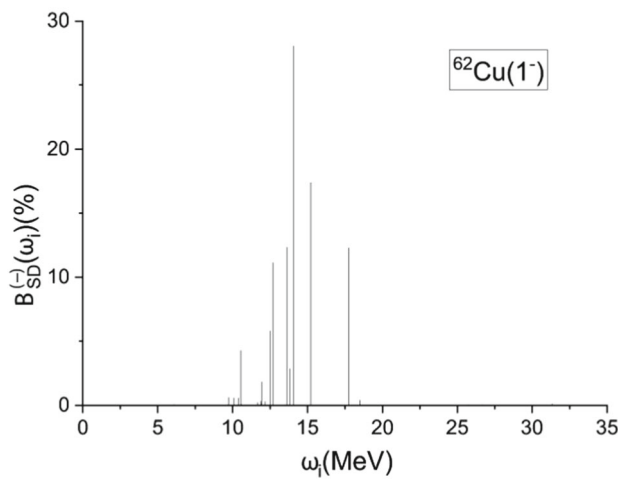
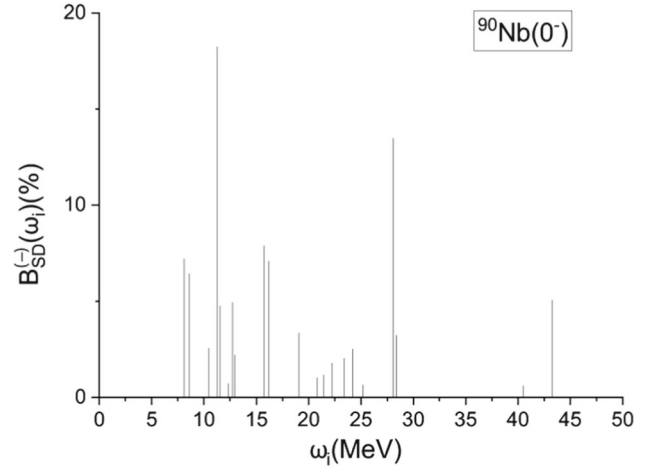
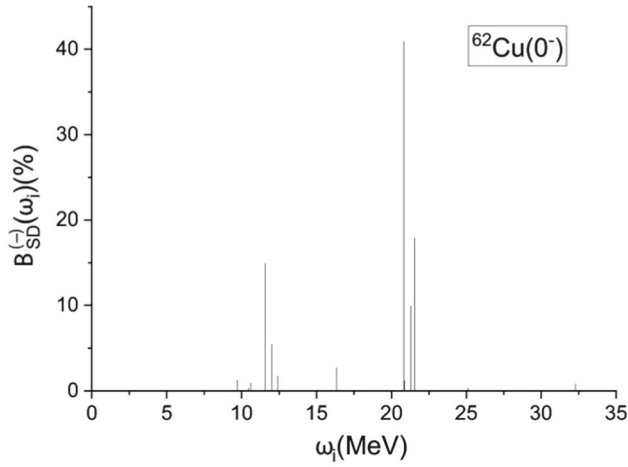


Figure 1. Percentage contributions of charge-exchange spin-dipole excitations in ^{62}Cu .

Figure 2. Percentage contributions of charge-exchange spin-dipole excitations in ^{90}Nb .

The wave-function amplitudes are found from the following normalisation condition:

$$|N_k^n(\lambda)|^2 + \sum_{iv} |R_{ni}^{kv}(\lambda)|^2 = 1. \quad (40)$$

2.2.3 Transition rates. Generally, the β -decay probabilities are given by the following formula:

$$B(J_i \rightarrow J_f) = \frac{1}{2J_i + 1} |\langle J_f || M_{\beta^\pm} || J_i \rangle|^2. \quad (41)$$

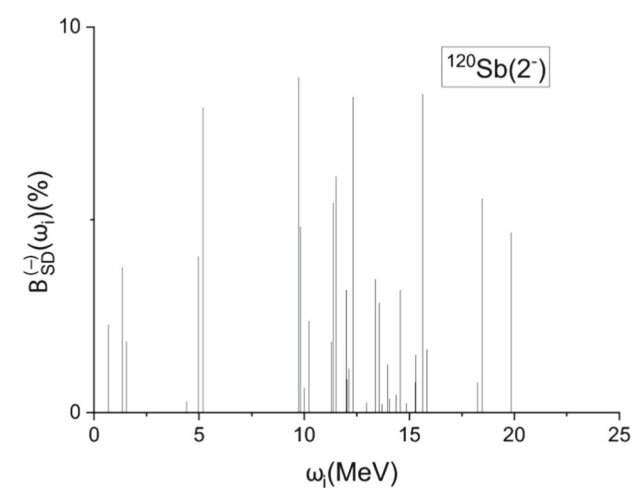
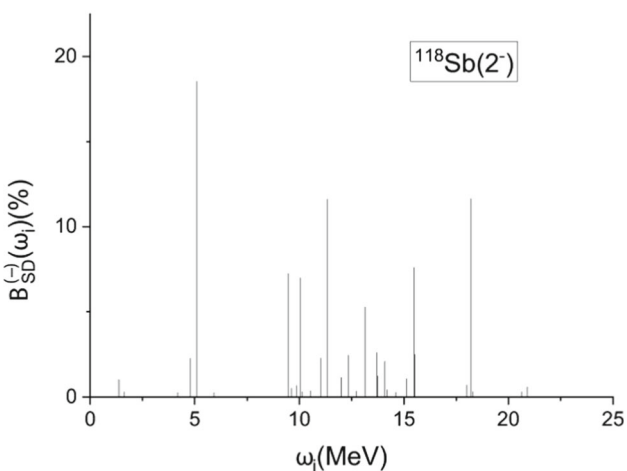
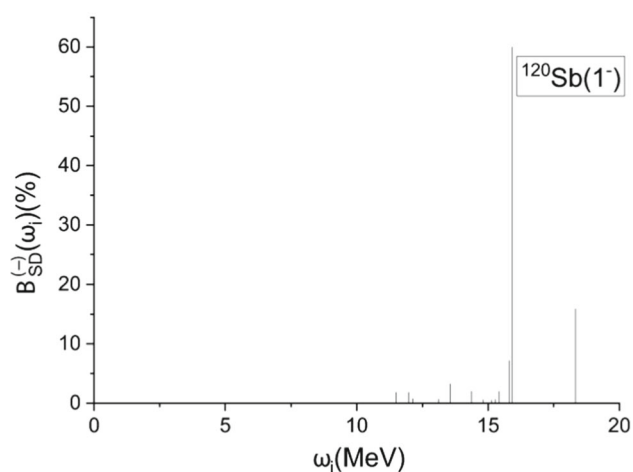
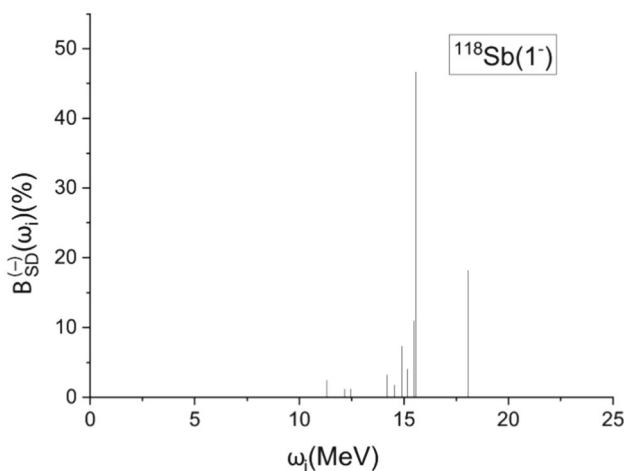
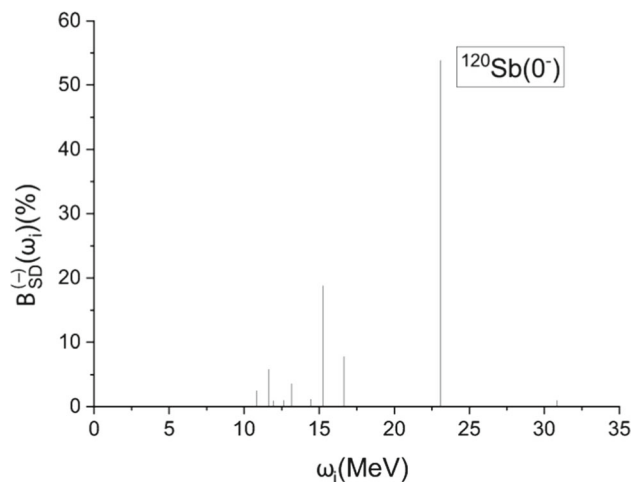
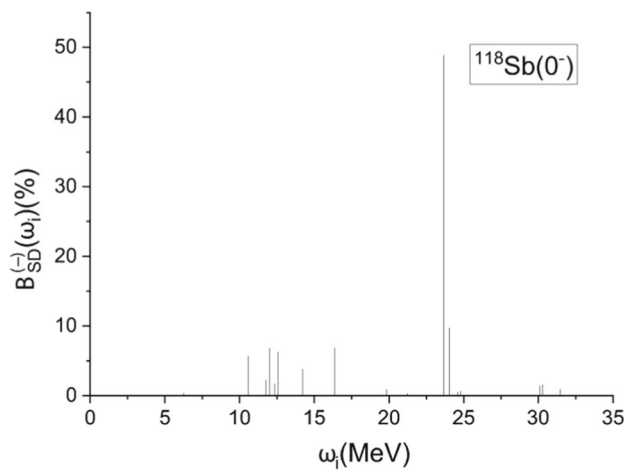


Figure 3. Percentage contributions of charge-exchange spin-dipole excitations in ^{118}Sb .

Figure 4. Percentage contributions of charge-exchange spin-dipole excitations in ^{120}Sb .

The β -decay matrix elements are usually defined as follows:

$$M_{\beta^\pm}(J_i \rightarrow J_f, \lambda, \mu) = \langle J_f | \widehat{T}_{\lambda\mu}^\pm | J_i \rangle. \quad (42)$$

First-forbidden β -decay matrix elements are computed by using ξ -approximation [70]. According to this approximation, the decay matrix elements for $\lambda^\pi = 0^-$

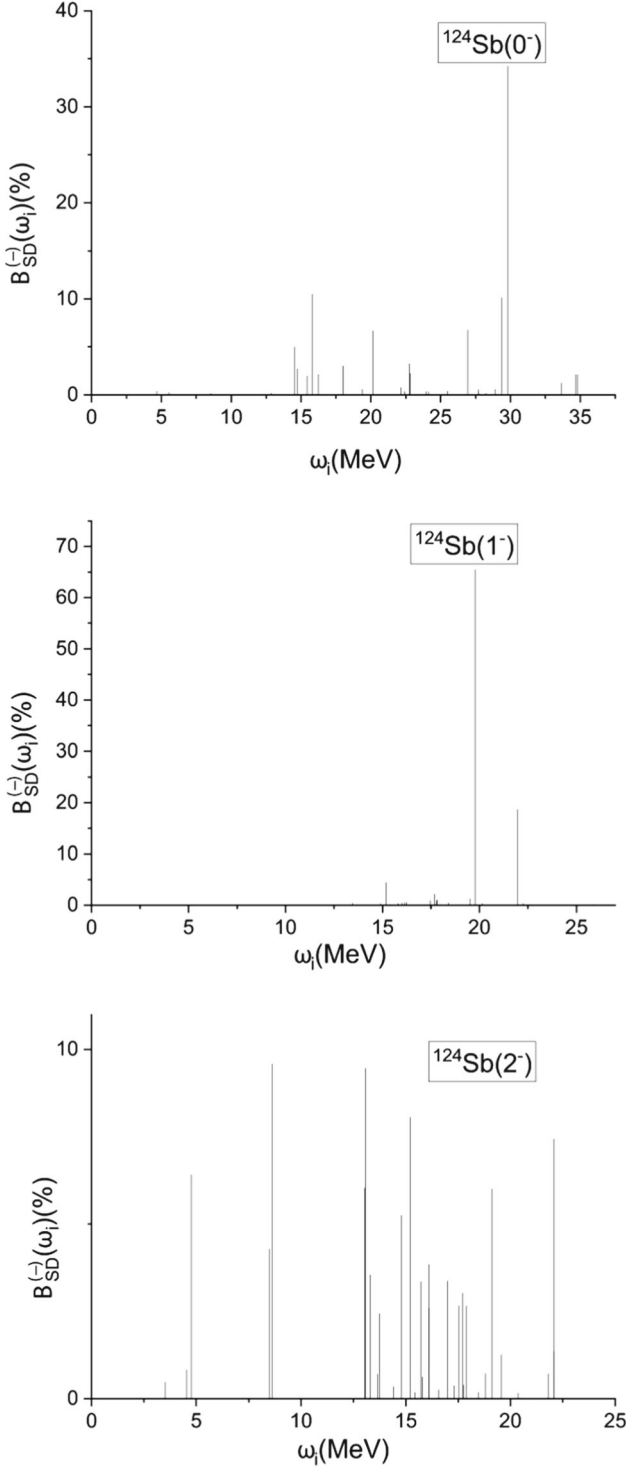


Figure 5. Percentage contributions of charge-exchange spin-dipole excitations in ^{124}Sb .

and $\lambda^\pi = 1^-$ transitions consist of relativistic and non-relativistic terms:

$$M_{\beta^\pm}(0^-) = \mp M^\pm(\rho_A, \lambda = 0) - i \frac{m_e c}{\hbar} \xi M^\pm(j_A, \kappa = 1, \lambda = 0) \quad (43)$$

and

$$M_{\beta^\pm}(1^-) = M^\pm(j_V, \kappa = 0, \lambda = 1, \mu) \mp i \frac{m_e c}{\sqrt{3}\hbar} \xi M^\pm(\rho_V, \lambda = 1, \mu) + i \sqrt{\frac{2}{3}} \frac{m_e c}{\hbar} \xi M^\pm(j_A, \kappa = 1, \lambda = 1, \mu). \quad (44)$$

The total decay rate can be given as follows:

$$f_0 t (B(J_i \rightarrow J_f, \lambda = 0) + B(J_i \rightarrow J_f, \lambda = 1)) = \frac{D g_V^2}{4\pi}. \quad (45)$$

The transition probabilities $B(\lambda^\pi = 0^-, 1^-; \beta^\pm)$ are specified by

$$B(J_i \rightarrow J_f, \lambda = 0) = \frac{1}{2J_i + 1} |\langle J_f || M_{\beta^\pm}(0^-) || J_i \rangle|^2 \quad (46)$$

and

$$B(J_i \rightarrow J_f, \lambda = 1) = \frac{1}{2J_i + 1} |\langle J_f || M_{\beta^\pm}(1^-) || J_i \rangle|^2. \quad (47)$$

Finally, the non-relativistic matrix element for $\lambda^\pi = 2^-$ is specified by

$$M_{\beta^\pm}(2^-) = M^\pm(\lambda = 2, \kappa = 1, j_A, \mu). \quad (48)$$

Transitions with $\lambda^\pi = 2^-$ are referred to as unique first-forbidden transitions and their decay rates can be expressed as

$$f_1 t B(J_i \rightarrow J_f, \lambda = 2) = \frac{3 D g_V^2}{4 \cdot 4\pi}, \quad (49)$$

where

$$B(J_i \rightarrow J_f, \lambda = 2) = \frac{1}{2J_i + 1} |\langle J_f || M_{\beta^\pm}(2^-) || J_i \rangle|^2, \quad (50)$$

$D = 6250$ s and the effective ratio of the axial and vector coupling constants is taken as $(g_A/g_V) = -1.24$.

The relativistic and the non-relativistic β -decay multiple operators [70] for various excitations are given as follows:

For $J^\pi = 0^-$ excitation,

$$M^\pm(\rho_A, \lambda = 0) = \frac{g_A}{c\sqrt{4\pi}} \sum_k \hat{t}_\pm(k) (\sigma_k \cdot \vartheta_k) M^\pm(j_A, \kappa = 1, \lambda = 0) = g_A \sum_k \hat{t}_\pm(k) r_k (Y_1(\hat{r}_k) \sigma_k)_0.$$

Table 1. The centroid energies for the charge-exchange spin-dipole transitions.

Transition	$\bar{\omega} (0^-)$	$\bar{\omega} (1^-)$	$\bar{\omega} (2^-)$	$\omega(\Delta L = 1, \Delta S = 1)$ [74]
$^{62}\text{Ni} \rightarrow ^{62}\text{Cu}$	22.23	17.65	15.25	≈ 20
$^{90}\text{Zr} \rightarrow ^{90}\text{Nb}$	23.42	17.34	15.82	≈ 20
$^{118}\text{Sn} \rightarrow ^{118}\text{Sb}$	23.32	18.81	14.08	≈ 20
$^{120}\text{Sn} \rightarrow ^{120}\text{Sb}$	22.55	19.13	14.58	≈ 20
$^{124}\text{Sn} \rightarrow ^{124}\text{Sb}$	24.84	19.90	14.50	≈ 20

The energies are given in unit of MeV.

Table 2. The β^- decay $\log ft$ values for some odd- A nuclei.

BE	$j_n \rightarrow j_p$	PW	Exp.
^{95}Sr	$(1/2)^+ \rightarrow (1/2)^-$	6.311	6.161 [75]
^{137}Xe	$(7/2)^- \rightarrow (7/2)^+$	6.831	6.567 [76]
^{139}Ba	$(7/2)^- \rightarrow (7/2)^+$	6.811	6.845 [77]
^{205}Hg	$(1/2)^- \rightarrow (1/2)^+$	5.455	5.257 [78]
^{209}Pb	$(9/2)^+ \rightarrow (9/2)^-$	5.869	5.534 [79]
^{211}Pb	$(9/2)^+ \rightarrow (9/2)^-$	6.08	5.99 [80]

BE stands for beta emitter and PW stands for present work.

For $J^\pi = 1^-$ excitation,

$$M^\pm(j_V, \kappa = 0, \lambda = 1, \mu) = \frac{g_V}{c\sqrt{4\pi}} \sum_k \hat{t}_\pm(k) (\vartheta_k)_{1\mu}$$

$$M^\pm(\rho_V, \lambda = 1, \mu) = g_V \sum_k \hat{t}_\pm(k) r_k Y_{1\mu}(\hat{r}_k)$$

$$M^\pm(j_A, \kappa = 1, \lambda = 1, \mu) = g_A \sum_k \hat{t}_\pm(k) r_k (Y_1(\hat{r}_k) \sigma_k)_{1\mu}$$

For $J^\pi = 2^-$ excitation,

$$M^\pm(j_A, \kappa = 1, \lambda = 2, \mu) = g_A \sum_k \hat{t}_\pm(k) r_k (Y_2(\hat{r}_k) \sigma_k)_{2\mu}$$

2.3 Application of various isotopes

Charge-exchange spin-dipole β^- strength distributions for ^{62}Ni , ^{90}Zr and $^{118,120,124}\text{Sn}$ isotopes are illustrated in figures 1–5. Let us note that the horizontal axis shows the calculated energies with respect to the ground state of the neighbour odd–odd nuclei. As seen, the first-forbidden 2^- excitations in odd–odd nuclei show more fragmentation than the 0^- and 1^- excitations. The strength distributions for $\lambda^\pi = 1^-$ excitations are usually obtained in a narrow energy region. The energy spectra for $\lambda^\pi = 1^-$ excited states do not exhibit a peak above 20 MeV except for the ^{124}Sb isotope. It can be concluded that the β^- transition strength for $\lambda^\pi = 1^-$ excitations shifts to higher energies for heavier Sb isotopes. The decay strength for $\lambda^\pi = 0^-$ excitations is

concentrated in a higher energy region than 1^- and 2^- excitations. The energy spectra for $\lambda^\pi = 0^-$ excitations do not show a peak below 10 MeV except for ^{90}Nb . The first-forbidden 0^- excited states in ^{90}Nb exhibit more fragmentation than other 0^- spectra. For these isotopes, the experimental data, which were obtained from ($^3\text{He}, t$) reactions at $E(^3\text{He}) = 450$ MeV, show a charge-exchange spin-dipole resonance around 20 MeV [74]. To make a comparison with the experimental data, the centroid energies for three components (0^- , 1^- and 2^- states) of the spin-dipole transitions are calculated according to the following formula:

$$\bar{\omega} = \frac{\sum_i \omega_i |M_{\beta^-}^i|^2}{\sum_i |M_{\beta^-}^i|^2}$$

The calculated centroid energies are tabulated and presented in table 1. The last column shows the experimental data for spin-dipole resonance. It is possible to say that the calculated centroid energies usually show a good agreement with the experimental position of spin-dipole resonance.

Another application of the present method is an investigation of the first-forbidden transitions for odd- A nuclei. In this respect, a comparison of the calculated β -decay $\log ft$ values for various odd- A nuclei with the corresponding experimental data is given in tables 2 and 3. These tables demonstrate that the calculated $\log ft$ values are usually close to the corresponding experimental values.

Table 3. The β^+ -decay $\log ft$ values for some odd- A nuclei.

BE	$j_p \rightarrow j_n$	PW	Exp
^{191}Au	$(3/2)^+ \rightarrow (3/2)^-$	6.2	6.9 [81]
^{197}Tl	$(1/2)^+ \rightarrow (1/2)^-$	5.96	6.42 [82]
^{199}Tl	$(1/2)^+ \rightarrow (1/2)^-$	5.91	6.54 [83]
^{201}Tl	$(1/2)^+ \rightarrow (1/2)^-$	5.90	6.09 [84]

BE stands for beta emitter and PW stands for present work.

3. Conclusion

Charge-exchange spin-dipole transitions are described by using Pyatov's restoration method within the framework of pn-QRPA method. The mathematical formalism is provided to be free of effective interaction strength parameters by following the restoration procedure. The present formalism is applied for the β^- decay strength distributions of ^{62}Ni , ^{90}Zr , ^{118}Sn , ^{120}Sn , ^{124}Sn isotopes and β -decay $\log ft$ values of some odd- A nuclei. It can be said that the present approximation is successful in reproducing the experimental data related to charge-exchange spin-dipole transitions.

In the near future, the calculations will be extended to other decay emitters. We predict that applying Pyatov's restoration procedure for restoring a broken commutator correlation will ensure a good motivation to describe the electric or magnetic multiple transitions, which have no relation with any symmetry property.

Acknowledgements

The present work is supported by the Scientific and Technological Research Council of Turkey under the grant number 121F206.

References

- [1] J J Cowan and F K Thielemann, *Phys. Today* **57**, 47 (2004)
- [2] I N Borzov, *Nucl. Phys. A* **777**, 645 (2006)
- [3] J-Un Nabi, *Adv. Space Res.* **46**, 1191 (2010)
- [4] C L Bai *et al*, *Phys. Rev. C* **83**, 054316 (2011)
- [5] J Suhonen and O Civitarese, *Phys. Rep.* **300**, 123 (1998)
- [6] J Suhonen, *Nucl. Phys. A* **864**, 63 (2011)
- [7] A Bobyk, W A Kaminski and P Zareba, *Nucl. Phys. A* **669**, 221 (2000)
- [8] A A Raduta, C M Raduta and A Escuderos, [arXiv:nucl-th/0412104v1](https://arxiv.org/abs/nucl-th/0412104v1) (2004)
- [9] P Vogel and M R Zirnbauer, *Phys. Rev. Lett.* **57**, 3148 (1986)
- [10] O Civitarese, A Faessler and T Tomoda, *Phys. Lett. B* **194**, 11 (1987)
- [11] T Tomoda and A Faessler, *Phys. Lett. B* **199**, 475 (1987)
- [12] J Engel, P Vogel and M R Zirnbauer, *Phys. Rev. C* **37**, 731 (1988)
- [13] K Muto and H V Klapdor, *Phys. Lett. B* **201**, 420 (1988)
- [14] J Suhonen, T Taigel and A Faessler, *Nucl. Phys. A* **486**, 91 (1988)
- [15] K Muto, E Bender and H V Klapdor, *Z. Phys. A* **334**, 187 (1989)
- [16] G Pantis *et al*, *J. Phys. G* **18**, 605 (1992)
- [17] J Engel *et al*, *Phys. Lett. B* **208**, 187 (1988)
- [18] J Hirsch and F Krmpotic, *Phys. Rev. C* **41**, 792 (1990)
- [19] A Staudt, T T S Kuo and H V Klapdor-Kleingrothaus, *Phys. Rev. C* **46**, 871 (1992)
- [20] F Krmpotic, *Phys. Rev. C* **48**, 1452 (1993)
- [21] S S Hsiao, Y Tzeng and T T S Kuo, *Phys. Rev. C* **49**, 2233 (1994)
- [22] M K Cheoun *et al*, *Prog. Part. Nucl. Phys.* **32**, 315 (1994)
- [23] O Civitarese and J Suhonen, *Nucl. Phys. A* **607**, 152 (1996)
- [24] J Suhonen *et al*, *Phys. Rev. C* **55**, 714 (1997)
- [25] O Civitarese and J Suhonen, *Nucl. Phys. A* **653**, 321 (1999)
- [26] F Simkovic, L Pacearescu and A Faessler, *Nucl. Phys. A* **733**, 321 (2004)
- [27] A A Raduta *et al*, *Phys. Rev. C* **69**, 064321 (2004)
- [28] V A Rodin, M H Urin and A Faessler, *Nucl. Phys. A* **747**, 295 (2005)
- [29] R Alvarez-Rodriguez *et al*, *Prog. Part. Nucl. Phys.* **57**, 251 (2006)
- [30] M S Yousef *et al*, *Nucl. Phys. B: Proc. Suppl.* **188**, 56 (2009)
- [31] M S Yousef *et al*, *Phys. Rev. C* **79**, 014314 (2009)
- [32] O Moreno *et al*, *J. Phys. G: Nucl. Part. Phys.* **36**, 015106 (2009)
- [33] C D Konti, F Krmpotic and B V Carlson, [arXiv:1202.3511v1](https://arxiv.org/abs/1202.3511v1) [nucl-th] (2012)
- [34] L Arisoy and S Unlu, *Nucl. Phys. A* **883**, 35 (2012)
- [35] S Unlu, *Phys. Scr.* **87**, 045202 (2013)
- [36] J Suhonen and O Civitarese, *Nucl. Phys. A* **924**, 1 (2014)
- [37] S Unlu and N Cakmak, *Nucl. Phys. A* **939**, 13 (2015)
- [38] J-Un Nabi, N Cakmak and Z Iftikhar, *Eur. Phys. J. A* **52**, 1, 2016.
- [39] J-Un Nabi *et al*, *Nucl. Phys. A* **957**, 1 (2017)
- [40] S Unlu, N Cakmak and C Selam, *Nucl. Phys. A* **957**, 491 (2017)
- [41] S Unlu, N Cakmak and C Selam, *Nucl. Phys. A* **970**, 379 (2018)
- [42] S Cakmak and N Cakmak, *Nucl. Phys. A* **1015**, 122287 (2021)
- [43] Q Zheng *et al*, *Nucl. Phys. Rev.* **38**, 361 (2021)
- [44] H A Aygör *et al*, *Int. J. Mod. Phys. E* **30**, 2150011 (2021)
- [45] J-Un Nabi *et al*, *Nucl. Phys. A* **1015**, 122278 (2021)
- [46] S Unlu and N Cakmak, *Act. Phys. Pol. B* **53**, 3 (2022)
- [47] P Möller, B Pfeiffer and K L Kratz, *Phys. Rev. C* **67**, 055802 (2003)
- [48] I N Borzov, *Phys. At. Nucl.* **79**, 910 (2016)

- [49] T Marketin, L Huther and G Martinez-Pinedo, *Phys. Rev. C* **93**, 025805 (2016)
- [50] M T Mustonen and J Engel, *Phys. Rev. C* **93**, 014304 (2016)
- [51] E M Ney, J Engel, T Li and N Schunck, *Phys. Rev. C* **102**, 034326 (2020)
- [52] N I Pyatov and D I Salamov, *Nucleonica* **22**, 127 (1977)
- [53] O Civitarese and M C Licciardo, *Phys. Rev. C* **38**, 967 (1988)
- [54] O Civitarese and M C Licciardo, *Phys. Rev. C* **41**, 1778 (1990)
- [55] O Civitarese, A Faessler and M C Licciardo, *Nucl. Phys. A* **542**, 221 (1992)
- [56] E Guliyev *et al*, *Phys. Lett. B* **532**, 173 (2002)
- [57] E Guliyev, A A Kuliev and F Ertugral, *Eur. Phys. J. A* **39**, 323 (2009)
- [58] F Ertugral *et al*, *Cent. Eur. J. Phys.* **7**, 731 (2009)
- [59] H Quliyev *et al*, *Appl. Sci. Rep.* **14**, 199 (2016)
- [60] T Babacan *et al*, *J. Phys. G: Nucl. Part. Phys.* **30**, 759 (2004)
- [61] A Kucukbursa *et al*, *Pramana – J. Phys.* **63**, 947 (2004)
- [62] D I Salamov *et al*, *Pramana – J. Phys.* **66**, 1105 (2006)
- [63] A E Calik, M Gerceklioglu and D I Salamov, *Z. Naturforsch. A* **64**, 865 (2009)
- [64] A E Calik, M Gerceklioglu and D I Salamov, *Pramana – J. Phys.* **79**, 417 (2012)
- [65] A E Calik, M Gerceklioglu and C Selam, *Phys. At. Nucl.* **76**, 549 (2013)
- [66] H A Aygör *et al*, *Cent. Eur. J. Phys.* **12**, 490 (2014)
- [67] T Babacan, D I Salamov and A Kucukbursa, *Phys. Rev. C* **71**, 037303 (2005)
- [68] D I Salamov, S Unlu and N Cakmak, *Pramana – J. Phys.* **69**, 369 (2007)
- [69] V G Soloviev, *Theory of complex nuclei* (Pergamon Press, New York, 1976)
- [70] A A Bohr and B R Mottelson, *Nuclear structure* (Benjamin, New York, Amsterdam, 1969)
- [71] P Möller and J R Nix, *Nucl. Phys. A* **536**, 20 (1992)
- [72] W Greiner and J A Maruhn, *Nuclear models* (Springer-Verlag, Berlin, Heidelberg, 1996)
- [73] D A Varshalovich, A N Moskalev and V K Khersonskii, *Quantum theory of angular momentum* (World Scientific Publishing, Singapore, 1988)
- [74] H Akimune *et al*, *Nucl. Phys. A* **569**, 245c (1994)
- [75] S K Basu, G Mukherjee and A A Sonzogni, *Nucl. Data Sheets* **111**, 2555 (2010)
- [76] E Browne and J K Tuli, *Nucl. Data Sheets* **108**, 2173 (2007)
- [77] P K Joshiör *et al*, *Nucl. Data Sheets* **138**, 1 (2016)
- [78] F G Kondev, *Nucl. Data Sheets* **166**, 1 (2020)
- [79] J Chen and F G Kondev, *Nucl. Data Sheets* **126**, 373 (2015)
- [80] E A Mccutchan *et al*, *Nucl. Data Sheets* **114**, 661 (2013)
- [81] V R Vanin *et al*, *Nucl. Data Sheets* **108**, 2393 (2007)
- [82] H Xiaolong and Z Chunmei, *Nucl. Data Sheets* **104**, 283 (2005)
- [83] B Singh, *Nucl. Data Sheets* **108**, 79 (2007)
- [84] F G Kondev, *Nucl. Data Sheets* **187**, 355 (2023)