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Lie symmetry analysis, optimal system, exact solutions and dynamics of solitons of a (3 + 1)-dimensional generalised BKP–Boussinesq equation

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Abstract. The Lie symmetry technique is utilised to obtain three stages of similarity reductions, exact invariant solutions and dynamical wave structures of multiple solitons of a $(3+1)$-dimensional generalised BKP–Boussinesq (gBKP-B) equation. We obtain infinitesimal vectors of the gBKP-B equation and each of these infinitesimals depends on five independent arbitrary functions and two parameters that provide us with a set of Lie algebras. Thenceforth, the commutative and adjoint tables between the examined vector fields and one-dimensional optimal system of symmetry subalgebras are constructed to the original equation. Based on each of the symmetry subalgebras, the Lie symmetry technique reduces the gBKP-B equation into various nonlinear ordinary differential equations through similarity reductions. Therefore, we attain closed-form invariant solutions of the governing equation by utilising the invariance criteria of the Lie group of transformation method. The established solutions are relatively new and more generalised in terms of functional parameter solutions compared to the previous results in the literature. All these exact explicit solutions are obtained in the form of different complex wave structures like multiwave solitons, curved-shaped periodic solitons, strip solitons, wave–wave interactions, elastic interactions between oscillating multisolitons and nonlinear waves, lump waves and kinky waves. The physical interpretation of computational wave solutions is exhibited both analytically and graphically through their three-dimensional postures by selecting relevant values of arbitrary functional parameters and constant parameters.

Keywords. Lie symmetry method; generalised BKP–Bossinesq equation; invariant solutions; optimal system; solitary wave solutions; lump waves.

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1. Introduction

Nonlinear partial differential equations (PDEs) play an essential role in the analysis of complex nonlinear phenomena in nonlinear sciences. One needs to obtain explicit closed-form solutions to these nonlinear equations for a clear understanding of these complex nonlinear phenomena characterised by nonlinear PDEs. Nonlinear evolution equations are particular forms of nonlinear PDEs, which describe many nonlinear phenomena in the disciplines of nonlinear sciences and engineering physics such as, optical physics, plasma physics, water waves, chemical physics, fluid dynamics, oceanography, hydrodynamics and so on. For a deep understanding of such complex nonlinear phenomena in nature, seeking exact closed-form solutions of nonlinear PDEs play a crucial role in the study of nonlinear sciences. It is well known that numerous analytical mathematical methods are developed by researchers and mathematicians, to seek closed-form solutions of nonlinear PDEs and each technique is precise for obtaining various forms of exact explicit solutions. Here, our prime objective is to study localised solitary wave solutions that can be described as a travelling wave solution that maintains its shape while propagating at a constant velocity. These solitons/solitary wave solutions are obtained by cancelling dispersive and nonlinear effects in the medium. A variety of efficient mathematical methods such as the auxiliary equation method [1], Bäcklund and Darboux transform [2,3], Lie symmetry method [4–9], the exp-function methods [10,11], the direct algebraic method and modified extended direct algebraic method [12], the inverse scattering transform [13], the F-expansion method [14], Lax pair [15], Hirota

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technique [16], Kudryashov method [17], extended simplest equation method [18], bifurcation method [19], the (G'/G)-expansion method [20], generalised exponential rational function method [21,22] and so on have been proposed.

In nonlinear sciences, the Kadomtsev and Petviashvili (KP) equation, which describes the nonlinear waves, is introduced by Kadomtsev and Petviashvili [23], has the bilinear form

$$(v_t + 6vv_x + v_{xxx})_x + 3v_{yy} = 0 \tag{1}$$

$$(D_x D_t + D_x^4 + 3D_y^2) f \dot{f} = 0. \tag{2}$$

The generalised B-type KP (g-BKP) equation in $(3+1)$ dimensions [24–27] can be furnished as

$$v_{ty} - v_{xxxy} - 3(v_x v_y)_x + 3v_{xz} = 0, \tag{3}$$

where $v = v(x, y, z, t)$ is the wave amplitude along with three spatial coordinates and one temporal coordinate and subscripts denote the partial derivatives of v with respect to the respective variables. The g-BKP equation describes the evolution of quasi-one-dimensional shallow water waves, when the effects of viscosity and surface tension are taken to be negligible [27]. The g-BKP equation has a wide range of applications in various fields of mathematical physics such as non-linear optics, oceanography, nonlinear waves, string theory, Bose–Einstein condensation, etc. Wazwaz [24] obtained multiple soliton solutions and multiple singular soliton solutions of the generalised KP equation by utilising the simplified form of Hirota''s method. Wazwaz and El-Tantawy [25] achieved multiple soliton solutions for the g-KP equation via the Hirota method. Ma and Fan [26] constructed N-soliton solutions of the g-BKP equation (3) by using the linear superposition principle of linear exponential travelling waves. Ma and Zhu [27] gained multiple wave solutions of (3) by employing the multiple exp-function algorithms via Hirota's perturbation scheme. In this paper, we focus on studying a new form of the $(3+1)$-dimensional generalised B-type KP–Boussinesq (gBKP-B)equation [28–31] which describes the severe effect on dispersion relation as well as phase shift and enhanced by adding an extra term (v_{tt}) to eq. (3) and this is introduced by Wazwaz and El-Tantawy [29]. The gBKP-B equation has the form

$$v_{ty} - v_{xxxy} - 3(v_x v_y)_x + v_{tt} + 3v_{xz} = 0. \tag{4}$$

Deng *et al* [28] constructed the rational solution including the semi-rational solutions and breather-type kink soliton solutions of the $(3+1)$-dimensional B-type KP–Boussinesq equation by using the bilinear method and fusion and fission between lump waves and solitons were also observed. Wazwaz and El-Tantawy [29] applied the simplified Hirota technique and established 1- and 2-soliton solutions, where the coefficients of spatial variables were arbitrary, for the generalised BKP-B equation (4). Gao and Zhang [30] obtained the Lie symmetry reduction with the help of a one-dimensional optimal system. Besides, they solved the reduced equation via the tanh method and established some exact explicit solutions of the gBKP-B equation (4). Recently, Khalique and Moleleki [31] obtained symmetry reductions via the Lie symmetry technique and then they solved the reduced equation through the (G'/G)-expansion method. Besides, conversation laws were derived by applying the multiplier method via the Ibragimov approach.

Lie symmetry technique [32–35] was pioneered by Sophus Lie (1842–1899), which is one of the best techniques for obtaining exact analytic solutions of nonlinear PDEs Lie symmetry technique is effective, systematic and has been applied to many physical models and nonlinear PDEs [36–48]. This technique is effective to get group-invariant solutions and dynamics of localised solitary wave solutions of nonlinear PDEs.

The prime objective of this study is to obtain localised solitary wave solutions and exact analytic solutions of the $(3+1)$-dimensional generalised B-type KP–Boussinesq (gBKP-B) equation by employing the Lie group method. It is remarkable that our newly formed solutions are completely new and never have been reported in the literature. In [31], a few exact solutions were derived with the help of symmetry reductions, direct integration and (G'/G)-expansion method whereas in this work, we obtained abundant exact closed-form solutions under ten symmetry subalgebras via one-dimensional optimal system approach. Therefore, in this article, we attained numerous explicit solutions compared to the solutions obtained in [30,31]. The generated exact solutions are expressed explicitly including arbitrary independent functional and free parameters which are useful and helpful to describe the internal mechanism of complex nonlinear phenomena. Furthermore, the dynamical analysis of soliton solutions of the gBKP-B equation is discussed physically using their 3D graphics via numerical simulation.

The remaining paper is organised as follows: In §2, we obtain the Lie point symmetries of the $(3+1)$-dimensional gBKP-B equation. In §3, a one-dimensional optimal system of the governing equation is derived. We obtain numerous closed-form invariant solutions with the aid of symmetry reductions in §4. The dynamical analysis of the gained exact solutions based on numerical simulation is given in §5. Finally, §6 is devoted to the concluding remarks.

2. Lie point symmetries

In this section, we derive Lie point symmetries, infinitesimal generator, commutative table, adjoint table and closed-form invariant solution of the (3+1)-dimensional gBKP-B equation (4). Assume one-parameter Lie group transformations as follows and defined in [32,33]

$$\begin{aligned}
x^* &= x + \epsilon\,\xi(x,y,t,z,v) + O(\epsilon^2),\\
y^* &= y + \epsilon\,\phi(x,y,t,z,v) + O(\epsilon^2),\\
z^* &= z + \epsilon\,\psi(x,y,t,z,v) + O(\epsilon^2),\\
t^* &= t + \epsilon\,\tau(x,y,t,z,v) + O(\epsilon^2),\\
v^* &= v + \epsilon\,\eta(x,y,t,z,v) + O(\epsilon^2),
\end{aligned} \tag{5}$$

where ξ, ϕ, ψ, τ and η are infinitesimal generators. Therefore, the associated infinitesimal generator is

$$\mathbf{V} = \xi\,\partial_x + \phi\,\partial_y + \psi\,\partial_z + \tau\,\partial_t + \eta\,\partial_v. \tag{6}$$

The fourth prolongation Pr^4 of $\mathbf{V}$ to the gBKP-B (4) equation is

$$\begin{aligned}
Pr^4\mathbf{V} = \mathbf{V} &+ \eta^x\frac{\partial}{\partial v_x} + \eta^y\frac{\partial}{\partial v_y} + \eta^{xx}\frac{\partial}{\partial v_{xx}}\\
&+ \eta^{xy}\frac{\partial}{\partial v_{xy}} + \eta^{yt}\frac{\partial}{\partial v_{yt}} + \cdots .
\end{aligned} \tag{7}$$

Utilising this prolongation including invariant conditions to the gBKP-B equation (4), one obtains

$$\begin{aligned}
&\eta^{yt} - \eta^{xxxy} - 3\eta^x v_{xy} - 3\eta^{xy} v_x\\
&\quad - 3\eta^y v_{xx} - 3\eta^{xx} v_y + \eta^{tt} + 3\eta^{xz} = 0,
\end{aligned} \tag{8}$$

where the extended coefficients η^x, η^y, η^{tt}, η^{xx}, η^{xy}, η^{xz}, η^{yt}, η^{xxxy} and the total derivative operators D_x, D_y, D_z and D_t are described in detail in [32,33].

We substitute the values of extended coefficients and total derivatives into eq. (8), to obtain the desired determining equation as

$$\begin{aligned}
&(\eta)_v = -\frac{1}{5}(\psi)_z,\\
&(\eta)_x = -(\phi)_z + \frac{1}{2}(\tau)_z,\ (\eta)_y = -(\xi)_z,\\
&(\eta)_{tt} = 3(\phi)_{zz} - 3(\tau)_{zz},\\
&(\tau)_t = \frac{3}{5}(\psi)_z,\ (\tau)_y = 0,\ (\tau)_x = 0,\\
&(\xi)_t = -\frac{3}{2}(\tau)_z,\ (\xi)_v = 0,\ (\xi)_x = \frac{1}{5}(\psi)_z,\\
&(\xi)_y = 0,\ (\phi)_t = 0,\ (\phi)_v = 0,\ (\phi)_x = 0,\\
&(\phi)_y = \frac{3}{5}(\psi)_z,\ (\psi)_t = 0,\ (\psi)_v = 0,\ (\psi)_x = 0,\\
&(\psi)_y = 0,\ (\psi)_{zz} = 0.
\end{aligned} \tag{9}$$

Afterwards, we solve the determining equations, the desired infinitesimals generators of gBKP-B equation (4) as follows:

$$\begin{aligned}
&\xi = \frac{c_1}{3}x - \frac{3t}{2}f_5'(z) + f_3(z),\quad \phi = c_1 y + f_4(z),\\
&\psi = \frac{5c_1}{3}z + c_2,\quad \tau = c_1 t + f_5(z),\\
&\eta = -\frac{c_1}{3}v + f_1(z) + t f_2(z) - y f_3'(z) - x f_4'(z)\\
&\quad + \frac{3}{2}t^2 f_4''(z) + \frac{x}{2}f_5'(z) - \frac{3}{2}t(t-y)f_5''(z).
\end{aligned} \tag{10}$$

Consequently, we obtain the following vector fields of gBKP-B equation (4) with the aid of (10):

$$\begin{aligned}
\mathbf{V}_1 &= \frac{x}{3}\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{5z}{3}\frac{\partial}{\partial z}\\
&\quad + t\frac{\partial}{\partial t} - \frac{v}{3}\frac{\partial}{\partial v},\\
\mathbf{V}_2 &= \frac{\partial}{\partial z},\quad \mathbf{V}_3(f_1) = f_1(z)\frac{\partial}{\partial v},\\
\mathbf{V}_4(f_2) &= t f_2(z)\frac{\partial}{\partial v},\\
\mathbf{V}_5(f_3) &= f_3(z)\frac{\partial}{\partial x} - y f_3'(z)\frac{\partial}{\partial v},\\
\mathbf{V}_6(f_4) &= f_4(z)\frac{\partial}{\partial y} + \frac{3}{2}t^2 f_4''(z)\frac{\partial}{\partial v} - x f_4'(z)\frac{\partial}{\partial v},\\
\mathbf{V}_7(f_5) &= -\frac{3}{2}t f_5'(z)\frac{\partial}{\partial x} + f_5(z)\frac{\partial}{\partial t}\\
&\quad - \frac{3}{2}t(t-y)f_5''(z)\frac{\partial}{\partial v} + \frac{x}{2}f_5'(z)\frac{\partial}{\partial v}.
\end{aligned} \tag{11}$$

3. A one-dimensional optimal system of subalgebras

We follow the same procedure to construct the one-dimensional optimal system of symmetry subalgebras as described in detail in [4,5,33,42]. We construct a one-dimensional optimal system of symmetry subalgebras in this section. By means of commutation relations between seven infinitesimal generators given in table 1, these infinitesimals given in (11) can be written as a linear combination of $\mathbf{V}_i$ as follows:

$$\begin{aligned}
\mathbf{V} &= \alpha_1\mathbf{V}_1 + \alpha_2\mathbf{V}_2 + \alpha_3\mathbf{V}_3 + \alpha_4\mathbf{V}_4\\
&\quad + \alpha_5\mathbf{V}_5 + \alpha_6\mathbf{V}_6 + \alpha_7\mathbf{V}_7.
\end{aligned} \tag{12}$$

Moreover, we derive the adjoint relations as provided in table 2. Using the Olver technique [33], the adjoint relations of a (3 + 1)-dimensional gBKP-B equation in table 2 are determined via computerised symbolic computation for the commutator relations of those vector fields.

Table 1. Commutator table.

*	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$	$\mathbf{V}_4$	$\mathbf{V}_5$	$\mathbf{V}_6$	$\mathbf{V}_7$
$\mathbf{V}_1$	0	$-\frac{5}{3}\mathbf{V}_2$	$\frac{1}{3}\mathbf{V}_3(5zf_1'+f_1)$	$\frac{5}{3}\mathbf{V}_4(zf_2')$	$\frac{1}{3}\mathbf{V}_5(5zf_3'-f_3)$	$\mathbf{V}_6(\frac{5}{3}zf_4'-f_4)$	$\mathbf{V}_7(\frac{5}{3}zf_5'-f_5)$
$\mathbf{V}_2$	$\frac{5}{3}\mathbf{V}_2$	0	$\mathbf{V}_3(f_1')$	$\mathbf{V}_4(f_2')$	$\mathbf{V}_5(f_3')$	$\mathbf{V}_6(f_4')$	$\mathbf{V}_7(f_5')$
$\mathbf{V}_3$	$-\frac{1}{3}\mathbf{V}_3(5zf_1'+f_1)$	$-\mathbf{V}_3(f_1')$	0	0	0	0	0
$\mathbf{V}_4$	$-\frac{5}{3}\mathbf{V}_4(zf_2')$	$-\mathbf{V}_4(f_2')$	0	0	0	0	$\mathbf{V}_3(-f_5f_2)$
$\mathbf{V}_5$	$-\frac{1}{3}\mathbf{V}_5(5zf_3'-f_3)$	$-\mathbf{V}_5(f_3')$	0	0	0	$\mathbf{V}_3(-f_3f_4')$	$\frac{1}{2}\mathbf{V}_3(f_3f_5')$
$\mathbf{V}_6$	$-\mathbf{V}_6(\frac{5}{3}zf_4'-f_4)$	$-\mathbf{V}_6(f_4')$	0	0	$-\mathbf{V}_3(-f_3f_4')$	0	$\frac{3}{2}\mathbf{V}_4(f_4f_5''-2f_5f_4''-f_4'f_5')$
$\mathbf{V}_7$	$-\mathbf{V}_7(\frac{5}{3}zf_5'-f_5)$	$-\mathbf{V}_7(f_5')$	0	$-\mathbf{V}_3(-f_5f_2)$	$-\frac{1}{2}\mathbf{V}_3(f_3f_5')$	$-\frac{3}{2}\mathbf{V}_4(f_4f_5''-2f_5f_4''-f_4'f_5')$	0

Table 2. Adjoint table.

Ad	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$	$\mathbf{V}_4$	$\mathbf{V}_5$	$\mathbf{V}_6$	$\mathbf{V}_7$
$\mathbf{V}_1$	$\mathbf{V}_1$	$e^{\frac{5\epsilon}{3}}\mathbf{V}_2$	$e^{-\frac{\epsilon}{3}}\mathbf{V}_3$	$e^{-\frac{5\epsilon}{3}}\mathbf{V}_4$	$e^{-\frac{\epsilon}{3}}\mathbf{V}_5$	$e^{-\epsilon}\mathbf{V}_6$	$e^{-\epsilon}\mathbf{V}_7$
$\mathbf{V}_2$	$\mathbf{V}_1-\frac{5\epsilon}{3}\mathbf{V}_2$	$\mathbf{V}_2$	$e^{-\epsilon}\mathbf{V}_3$	$e^{-\epsilon}\mathbf{V}_4$	$e^{-\epsilon}\mathbf{V}_5$	$e^{-\epsilon}\mathbf{V}_6$	$e^{-\epsilon}\mathbf{V}_7$
$\mathbf{V}_3$	$\mathbf{V}_1+\frac{\epsilon}{3}\mathbf{V}_3$	$\mathbf{V}_2+\epsilon\mathbf{V}_3$	$\mathbf{V}_3$	$\mathbf{V}_4$	$\mathbf{V}_5$	$\mathbf{V}_6$	$\mathbf{V}_7$
$\mathbf{V}_4$	$\mathbf{V}_1+\frac{5\epsilon}{3}\mathbf{V}_4$	$\mathbf{V}_2+\epsilon\mathbf{V}_4$	$\mathbf{V}_3$	$\mathbf{V}_4$	$\mathbf{V}_5$	$\mathbf{V}_6$	$\mathbf{V}_7-\epsilon\mathbf{V}_3$
$\mathbf{V}_5$	$\mathbf{V}_1+\frac{\epsilon}{3}\mathbf{V}_5$	$\mathbf{V}_2+\epsilon\mathbf{V}_5$	$\mathbf{V}_3$	$\mathbf{V}_4$	$\mathbf{V}_5$	$\mathbf{V}_6-\epsilon\mathbf{V}_3$	$\mathbf{V}_7-\frac{\epsilon}{2}\mathbf{V}_3$
$\mathbf{V}_6$	$\mathbf{V}_1+\epsilon\mathbf{V}_6$	$\mathbf{V}_2+\epsilon\mathbf{V}_6$	$\mathbf{V}_3$	$\mathbf{V}_4$	$\mathbf{V}_5+\epsilon\mathbf{V}_3$	$\mathbf{V}_6$	$\mathbf{V}_7-\frac{3\epsilon}{2}\mathbf{V}_4$
$\mathbf{V}_7$	$\mathbf{V}_1+\epsilon\mathbf{V}_7$	$\mathbf{V}_2+\epsilon\mathbf{V}_7$	$\mathbf{V}_3$	$\mathbf{V}_4-\epsilon\mathbf{V}_3$	$\mathbf{V}_5+\frac{\epsilon}{2}\mathbf{V}_3$	$\mathbf{V}_6+\frac{3\epsilon}{2}\mathbf{V}_4$	$\mathbf{V}_7$

3.1 *Formation of invariants*

To attain one-dimensional optimal system of Lie algebra $\mathbb{R}^7$, thus there is a need to construct the invariant for the suitable selection of representative factors/elements. Thus, the desired matrix representations of $\mathrm{ad}(\mathbf{V}_i)$ can be furnished as

$$\begin{aligned}
&\mathrm{Ad}(\exp(\epsilon\mathbf{W}))(\mathbf{V})\\
&\quad= e^{-\epsilon\mathbf{W}}\mathbf{V}e^{\epsilon\mathbf{W}} = \mathbf{V}-\epsilon[\mathbf{W},\mathbf{V}]\\
&\qquad+\frac{1}{2!}\epsilon^2[\mathbf{W},[\mathbf{W},\mathbf{V}]]-\cdots\\
&\quad=(\alpha_1\mathbf{V}_1+\cdots+\alpha_n\mathbf{V}_n)\\
&\qquad-\epsilon[\beta_1\mathbf{V}_1+\cdots+\beta_n\mathbf{V}_n,\alpha_1\mathbf{V}_1+\ldots\\
&\qquad+\alpha_n\mathbf{V}_n]+O(\epsilon^2)\\
&\quad=(\alpha_1\mathbf{V}_1+\cdots+\alpha_n\mathbf{V}_n)\\
&\qquad-\epsilon(\Theta_1\mathbf{V}_1+\cdots+\Theta_n\mathbf{V}_n),
\end{aligned}\tag{13}$$

where $\Theta = \Theta(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n)$ are obtained with the help of symbolic calculations via the commutator table. The commutative relations of the seven-dimensional Lie algebra are expressed in table 1. Putting $\mathbf{V}=\sum_{i=1}^{7}\alpha_i\mathbf{V}_i$ and $\mathbf{W}=\sum_{j=1}^{7}\beta_j\mathbf{V}_j$ in (11)

$$\begin{aligned}
&\Theta_1 = 0,\quad \Theta_2 = -\frac{5}{3}\alpha_2\beta_1+\frac{5}{3}\alpha_1\beta_2,\\
&\Theta_3 = \frac{1}{3}\alpha_3\beta_1-\frac{1}{3}\alpha_1\beta_3+\alpha_3\beta_2-\alpha_2\beta_3\\
&\qquad+\alpha_7\beta_4-\alpha_4\beta_7+\alpha_6\beta_5-\alpha_5\beta_6\\
&\qquad+\frac{1}{2}\alpha_7\beta_5-\frac{1}{2}\alpha_5\beta_7,\\
&\Theta_4 = \frac{5}{3}\alpha_4\beta_1-\alpha_4\beta_2-\frac{5}{3}\alpha_1\beta_4+\frac{3}{2}\alpha_7\beta_6\\
&\qquad+\alpha_2\beta_4-\frac{3}{2}\alpha_6\beta_7,\\
&\Theta_5 = \frac{1}{3}\alpha_5\beta_1+\alpha_5\beta_2-\frac{1}{3}\alpha_1\beta_5-\alpha_2\beta_5,\\
&\Theta_6 = \alpha_6\beta_1-\alpha_2\beta_6-\alpha_1\beta_6+\alpha_6\beta_2,\\
&\Theta_7 = \alpha_7\beta_1+\alpha_7\beta_2-\alpha_2\beta_7-\alpha_1\beta_7.
\end{aligned}\tag{14}$$

For any β_j, $1\le j\le 7$, it imposes

$$\begin{aligned}
&\Theta_1\frac{\partial\phi}{\partial\alpha_1}+\Theta_2\frac{\partial\phi}{\partial\alpha_2}+\Theta_3\frac{\partial\phi}{\partial\alpha_3}+\Theta_4\frac{\partial\phi}{\partial\alpha_4}\\
&\quad+\Theta_5\frac{\partial\phi}{\partial\alpha_5}+\Theta_6\frac{\partial\phi}{\partial\alpha_6}+\Theta_7\frac{\partial\phi}{\partial\alpha_7}=0.
\end{aligned}\tag{15}$$

Equating the various coefficients of different powers of β_j in eq. (15), we obtain seven differential equations with $\phi(\alpha_1,\alpha_2,\ldots,\alpha_7)$ as

$$
\begin{aligned}
&\beta_1 : -\frac{5\alpha_2}{2}\frac{\partial\phi}{\partial\alpha_2}\\
&\quad +\frac{\alpha_3}{3}\frac{\partial\phi}{\partial\alpha_3}+\frac{5\alpha_4}{3}\frac{\partial\phi}{\partial\alpha_4}+\frac{\alpha_5}{3}\frac{\partial\phi}{\partial\alpha_5}\\
&\quad +\alpha_6\frac{\partial\phi}{\partial\alpha_6}+\alpha_7\frac{\partial\phi}{\partial\alpha_7}=0,\\
&\beta_2 : \frac{5\alpha_1}{3}\frac{\partial\phi}{\partial\alpha_2}+\alpha_3\frac{\partial\phi}{\partial\alpha_3}+\alpha_4\frac{\partial\phi}{\partial\alpha_4}\\
&\quad +\alpha_5\frac{\partial\phi}{\partial\alpha_5}+\alpha_6\frac{\partial\phi}{\partial\alpha_6}+\alpha_7\frac{\partial\phi}{\partial\alpha_7}=0,\\
&\beta_3 : -\left(\frac{\alpha_1}{3}+\alpha_2\right)\frac{\partial\phi}{\partial\alpha_3}=0,\\
&\beta_4 : \alpha_7\frac{\partial\phi}{\partial\alpha_3}-\left(\frac{5\alpha_1}{3}+\alpha_2\right)\frac{\partial\phi}{\partial\alpha_4}=0,\\
&\beta_5 : \left(\frac{\alpha_7}{2}+\alpha_6\right)\frac{\partial\phi}{\partial\alpha_3}-\left(\frac{\alpha_1}{3}+\alpha_2\right)\frac{\partial\phi}{\partial\alpha_5}=0,\\
&\beta_6 : -\alpha_5\frac{\partial\phi}{\partial\alpha_3}+\frac{3\alpha_7}{2}\frac{\partial\phi}{\partial\alpha_7}\\
&\quad -(\alpha_1+\alpha_2)\frac{\partial\phi}{\partial\alpha_6}=0,\\
&\beta_7 : -(\frac{\alpha_5}{2}+\alpha_4)\frac{\partial\phi}{\partial\alpha_3}\\
&\quad +\frac{3\alpha_6}{2}\frac{\partial\phi}{\partial\alpha_4}-\alpha_1\frac{\partial\phi}{\partial\alpha_7}=0.
\end{aligned}
\tag{16}
$$

Solving system (16), one obtains $\phi(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7)=F(\alpha_1)$ which is also called the general invariant function of Lie algebra $\mathbb{R}^7$, where F is an arbitrary function of α_1. As a result, the governing equation (4) has one basic invariant only.

3.2 *Calculation of the adjoint transformation matrix*

For $F_i^s : g \to g$ defined by $\mathbf{V} \to \mathrm{Ad}(\exp(\epsilon_i\mathbf{V}_i)\cdot\mathbf{V})$ is a linear map, for $i=1,2,\ldots,7$. The matrix A_i^ϵ of $F_i^\epsilon, i=1,2,\ldots,7$ with respect to basis $\{\mathbf{V}_1,...,\mathbf{V}_7\}$ are given and defined in [33,42] as follows:

$$
A_1^\epsilon=\begin{pmatrix}
1&0&0&0&0&0&0\\
0&\mathrm{e}^{\frac{5}{3}\epsilon_1}&0&0&0&0&0\\
0&0&\mathrm{e}^{-\frac{\epsilon_1}{3}}&0&0&0&0\\
0&0&0&\mathrm{e}^{-\frac{5}{3}\epsilon_1}&0&0&0\\
0&0&0&0&\mathrm{e}^{-\frac{\epsilon_1}{3}}&0&0\\
0&0&0&0&0&\mathrm{e}^{-\epsilon_1}&0\\
0&0&0&0&0&0&\mathrm{e}^{-\epsilon_1}
\end{pmatrix},
$$

$$
A_2^\epsilon=\begin{pmatrix}
1&-\frac{5}{3}\epsilon_2&0&0&0&0&0\\
0&1&0&0&0&0&0\\
0&0&\mathrm{e}^{-\epsilon_2}&0&0&0&0\\
0&0&0&\mathrm{e}^{-\epsilon_2}&0&0&0\\
0&0&0&0&\mathrm{e}^{-\epsilon_2}&0&0\\
0&0&0&0&0&\mathrm{e}^{-\epsilon_2}&0\\
0&0&0&0&0&0&\mathrm{e}^{-\epsilon_2}
\end{pmatrix},
$$

$$
A_3^\epsilon=\begin{pmatrix}
1&0&\frac{\epsilon_3}{3}&0&0&0&0\\
0&1&\epsilon_3&0&0&0&0\\
0&0&1&0&0&0&0\\
0&0&0&1&0&0&0\\
0&0&0&0&1&0&0\\
0&0&0&0&0&1&0\\
0&0&0&0&0&0&1
\end{pmatrix},
$$

$$
A_4^\epsilon=\begin{pmatrix}
1&0&0&\frac{5}{3}\epsilon_4&0&0&0\\
0&1&0&\epsilon_4&0&0&0\\
0&0&1&0&0&0&0\\
0&0&0&1&0&0&0\\
0&0&0&0&1&0&0\\
0&0&0&0&0&1&0\\
0&0&\epsilon_4&0&0&0&1
\end{pmatrix},
$$

$$
A_5^\epsilon=\begin{pmatrix}
1&0&0&0&\frac{\epsilon_5}{3}&0&0\\
0&1&0&0&\epsilon_5&0&0\\
0&0&1&0&0&0&0\\
0&0&0&1&0&0&0\\
0&0&0&0&1&0&0\\
0&0&-\epsilon_5&0&0&1&0\\
0&0&-\frac{\epsilon_5}{2}&0&0&0&1
\end{pmatrix},
$$

$$
A_6^\epsilon=\begin{pmatrix}
1&0&0&0&0&\epsilon_6&0\\
0&1&0&0&0&\epsilon_6&0\\
0&0&1&0&0&0&0\\
0&0&0&1&0&0&0\\
0&0&\epsilon_6&0&1&0&0\\
0&0&0&0&0&1&0\\
0&0&0&\frac{-3\epsilon_6}{2}&0&0&1
\end{pmatrix},
$$

$$
A_7^\epsilon=\begin{pmatrix}
1&0&0&0&0&0&\epsilon_7\\
0&1&0&0&0&0&\epsilon_7\\
0&0&1&0&0&0&0\\
0&0&\epsilon_7&1&0&0&0\\
0&0&\frac{\epsilon_7}{2}&0&1&0&0\\
0&0&0&\frac{3\epsilon_7}{2}&0&1&0\\
0&0&0&0&0&0&1
\end{pmatrix}
$$

Hence, we obtain the global adjoint matrix using these seven matrices as

$$A = \begin{pmatrix} 1 & -\frac{5}{3}\epsilon_2 & A_{13} & A_{14} & \frac{\epsilon_5}{3} - \frac{5}{3}\epsilon_2\epsilon_5 & \epsilon_6 - \frac{5}{3}\epsilon_2\epsilon_6 & \epsilon_7 - \frac{5}{3}\epsilon_2\epsilon_7 \\ 0 & \mathrm{e}^{\frac{5\epsilon_1}{3}} & A_{23} & A_{24} & \mathrm{e}^{\frac{5\epsilon_1}{3}}\epsilon_5 & \mathrm{e}^{\frac{5\epsilon_1}{3}}\epsilon_6 & \mathrm{e}^{\frac{5\epsilon_1}{3}}\epsilon_7 \\ 0 & 0 & \mathrm{e}^{-\frac{\epsilon_1}{3}-\epsilon_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-\frac{5\epsilon_1}{3}-\epsilon_2}\epsilon_7 & \mathrm{e}^{-\frac{5\epsilon_1}{3}-\epsilon_2} & 0 & 0 & 0 \\ 0 & 0 & A_{53} & 0 & \mathrm{e}^{-\frac{\epsilon_1}{3}-\epsilon_2} & 0 & 0 \\ 0 & 0 & -\mathrm{e}^{-\epsilon_1-\epsilon_2}\epsilon_5 & A_{64} & 0 & \mathrm{e}^{-\epsilon_1-\epsilon_2} & 0 \\ 0 & 0 & A_{73} & -\frac{3}{2}\mathrm{e}^{-\epsilon_1-\epsilon_2}\epsilon_6 & 0 & 0 & \mathrm{e}^{-\epsilon_1-\epsilon_2} \end{pmatrix}, \tag{17}$$

where

$$A_{13} = -\frac{1}{3}5\epsilon_2\epsilon_3 + \frac{\epsilon_3}{3} + \left(\frac{\epsilon_5}{3} - \frac{5\epsilon_2\epsilon_5}{3}\right)\epsilon_6 + \left(\frac{5\epsilon_4}{3} - \frac{5\epsilon_2\epsilon_4}{3}\right)\epsilon_7 + \frac{1}{2}\left(\frac{\epsilon_5}{3} - \frac{5\epsilon_2\epsilon_5}{3}\right)\epsilon_7,$$

$$A_{23} = \mathrm{e}^{\frac{5\epsilon_1}{3}}\epsilon_3 + \mathrm{e}^{\frac{5\epsilon_1}{3}}\epsilon_5\epsilon_6 + \mathrm{e}^{\frac{5\epsilon_1}{3}}\epsilon_4\epsilon_7 + \frac{1}{2}\mathrm{e}^{\frac{5\epsilon_1}{3}}\epsilon_5\epsilon_7,$$

$$A_{53} = \mathrm{e}^{-\frac{\epsilon_1}{3}-\epsilon_2}\epsilon_6 + \frac{1}{2}\mathrm{e}^{-\frac{\epsilon_1}{3}-\epsilon_2}\epsilon_7,$$

$$A_{73} = \mathrm{e}^{-\epsilon_1-\epsilon_2}\epsilon_4 - \frac{1}{2}\mathrm{e}^{-\epsilon_1-\epsilon_2}\epsilon_5 - \frac{3}{2}\mathrm{e}^{-\epsilon_1-\epsilon_2}\epsilon_6\epsilon_7,$$

$$A_{14} = -\frac{1}{3}5\epsilon_2\epsilon_4 + \frac{5\epsilon_4}{3} + \frac{3}{2}\left(\epsilon_6 - \frac{5\epsilon_2\epsilon_6}{3}\right)\epsilon_7,$$

$$A_{24} = \mathrm{e}^{\frac{5\epsilon_1}{3}}\epsilon_4 + \frac{3}{2}\mathrm{e}^{\frac{5\epsilon_1}{3}}\epsilon_6\epsilon_7,$$

$$A_{64} = \frac{3}{2}\mathrm{e}^{-\epsilon_1-\epsilon_2}\epsilon_7.$$

3.3 *One-dimensional optimal system for the gBKP-B equation*

The general transformation equation to the generalised gBKP-B equation (4) is

$$(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7) = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) \cdot A, \tag{18}$$

where A is the global matrix which is already derived above.

$$\gamma_1 = \alpha_1, \quad \gamma_2 = \alpha_2 - \epsilon_2, \quad \gamma_3 = \alpha_3 + \frac{2\epsilon_3}{5},$$

$$\gamma_4 = \alpha_4 - \alpha_3\epsilon_2 + \alpha_2\epsilon_3 - \frac{3}{5}\epsilon_4,$$

$$\gamma_5 = \alpha_5 + \frac{5}{2}\alpha_4\epsilon_3 + \epsilon_3\epsilon_4 + \left(-\frac{5}{2}\epsilon_2\epsilon_3 + \frac{5}{2}\epsilon_4\right)\alpha_3 - \frac{\epsilon_5}{5},$$

$$\gamma_6 = \alpha_6 + \frac{1}{8}\alpha_5\epsilon_3 + \frac{6}{5}\epsilon_6,$$

$$\gamma_7 = \alpha_7 + \alpha_6\epsilon_2 - \frac{1}{8}\alpha_3\epsilon_5 - \frac{1}{20}\alpha_3\epsilon_5 - \epsilon_2\epsilon_6 + \frac{\epsilon_7}{5}, \tag{19}$$

which must have solutions for ϵ_i's for $i = 1, 2, \ldots, 7$ (assuming $\epsilon_1 = 0$).

Case 1: For $\alpha_1 = 1$, the representative element $\tilde{\mathbf{V}} = \mathbf{V}_1$. Substituting $\gamma_1 = 1$ into eq. (19) we get

$$\epsilon_2 = \alpha_2, \ \epsilon_3 = -\frac{1}{2}(5\alpha_3), \ \epsilon_4 = -\frac{5}{3}(\alpha_2\alpha_3 - \alpha_4),$$

$$\epsilon_5 = \frac{5}{4}\left(25\alpha_2\alpha_3^2 - 25\alpha_3\alpha_4 + 4\alpha_5\right),$$

$$\epsilon_6 = \frac{5}{96}(5\alpha_3\alpha_5 - 16\alpha_6), \ \epsilon_7 = 5(\alpha_2\alpha_6 - \alpha_7).$$

Case 2: Let us consider the representative element $\tilde{\mathbf{V}} = \mathbf{V}_1 + \mathbf{V}_3$. We substitute $\alpha_1 = \alpha_3 = 1$ and $\gamma_1 = \gamma_3 = 1$ into eq. (19), to get

$$\epsilon_2 = \alpha_2, \ \epsilon_3 = 0, \ \epsilon_4 = -\frac{5}{3}(\alpha_2 - \alpha_4),$$

$$\epsilon_5 = -\frac{5}{6}\left(25\alpha_2 - 25\alpha_4 - 6\alpha_5\right), \ \ \epsilon_6 = -\frac{5}{6}\alpha_6,$$

$$\epsilon_7 = \frac{5}{48}\left(48\alpha_2\alpha_6 - 125\alpha_2 + 125\alpha_4 + 30\alpha_5 - 48\alpha_7\right).$$

Case 3: Consider the representative element $\tilde{\mathbf{V}} = \mathbf{V}_1 + \mathbf{V}_7$. Substituting $\alpha_1 = \alpha_7 = 1$ and $\gamma_1 = \gamma_7 = 1$ into eq. (19), we get

$$\epsilon_2 = \alpha_2, \ \epsilon_3 = -\frac{5\alpha_3}{2}, \ \epsilon_4 = -\frac{5}{3}(\alpha_2\alpha_3 - \alpha_4),$$

$$\epsilon_5 = \frac{5}{4}\left(25\alpha_2\alpha_3^2 - 25\alpha_3\alpha_4 + 4\alpha_5\right),$$

$$\epsilon_6 = -\frac{5}{96}\left(5\alpha_3\alpha_5 - 16\alpha_6\right), \quad \epsilon_7 = 5\alpha_2\alpha_6.$$

Case 4: We take representative element $\tilde{\mathbf{V}} = \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_4$. Substituting $\alpha_1 = \alpha_2 = \alpha_4 = 1$ and $\gamma_1 = \gamma_2 =$

$\gamma_4 = 1$ into eq. (19), we get

$$\epsilon_2 = 0,\ \epsilon_3 = -\frac{5\alpha_3}{2},\ \epsilon_4 = -\frac{25}{3}\alpha_3,$$
$$\epsilon_5 = -\frac{5}{4}\left(25\alpha_3 - 4\alpha_5\right),$$
$$\epsilon_6 = \frac{5}{96}\left(5\alpha_3\alpha_5 - 16\alpha_6\right),$$
$$\epsilon_7 = -\frac{5}{96}\left(25\alpha_3\alpha_5 - 80\alpha_6 + 96\alpha_7\right).$$

Case 5: For $\alpha_2 = 1$, the representative element $\tilde{\mathbf{V}} = \mathbf{V}_2$. By substituting $\gamma_2 = 1$ into eq. (19), we get

$$\epsilon_3 = 0, \quad \epsilon_6 = -\alpha_7.$$

Case 6: Select a representative element $\tilde{\mathbf{V}} = \mathbf{V}_2 + \mathbf{V}_3$. Substituting $\gamma_2 = \gamma_3 = 1, \gamma_i = 0, i = 1, 4, 5, 6, 7$ and $\alpha_2 = \alpha_3 = 1$ into eq. (19), we obtain the solution

$$\epsilon_2 = \epsilon_3 = -8\frac{\alpha_6}{\alpha_5},\ \epsilon_4 = -2\left(\alpha_5 - 32\frac{\alpha_6^2}{\alpha_5^2}\right),$$
$$\epsilon_5 = 0,\ \epsilon_6 = \frac{-8\alpha_6^2 - \alpha_5\alpha_7}{\alpha_5}.$$

Case 7: For $\alpha_2 = \alpha_4 = 1$, the representative element $\tilde{\mathbf{V}} = \mathbf{V}_2 + \mathbf{V}_4$. By substituting $\gamma_2 = \gamma_4 = 1$ into eq. (19) we get

$$\epsilon_3 = 0,\ \epsilon_6 = -\alpha_7.$$

Case 8: Select a representative element $\tilde{\mathbf{V}} = \mathbf{V}_2 + \mathbf{V}_3 + \mathbf{V}_5$. Substituting $\gamma_1 = 0, \gamma_2 = \gamma_3 = \gamma_5 = 1, \gamma_i = 0, i = 4,\ 6,\ 7$ and $\alpha_2 = \alpha_3 = \alpha_5 = 1$ into eq. (19), we obtain the solution

$$\epsilon_2 = \alpha_4 - 8\alpha_6,\ \epsilon_3 = -8\alpha_6,\ \epsilon_4 = 64\alpha_6^2,$$
$$\epsilon_5 = 0,\ \epsilon_6 = (\alpha_4 - 8\alpha_6)\alpha_6 - \alpha_7.$$

Case 9: Select a representative element $\tilde{\mathbf{V}} = \mathbf{V}_2 + \mathbf{V}_5 + \mathbf{V}_6$. Substituting $\gamma_2 = \gamma_5 = \gamma_6 = 1, \gamma_i = 0, i = 1,\ 3,\ 4,\ 7$ and $\alpha_2 = \alpha_5 = \alpha_6 = 1$ into eq. (19), we obtain the solution

$$\epsilon_3 = 0,\ \epsilon_6 = -\alpha_7 + \epsilon_2.$$

Proceeding as above, we can find the value of ϵ_i's for certain members of optimal system.

Eventually, an optimal system of one-dimensional symmetry subalgebras for a gBKP-B equation is furnished in the following way:

$$\begin{aligned}
&\text{(i)} & \mathfrak{T}_1 &= \mathbf{V}_1\\
&\text{(ii)} & \mathfrak{T}_2 &= \mathbf{V}_1 + \mathbf{V}_3\\
&\text{(iii)} & \mathfrak{T}_3 &= \mathbf{V}_1 + \mathbf{V}_7\\
&\text{(iv)} & \mathfrak{T}_4 &= \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_4\\
&\text{(v)} & \mathfrak{T}_5 &= \mathbf{V}_2\\
&\text{(vi)} & \mathfrak{T}_6 &= \mathbf{V}_2 + \mathbf{V}_3\\
&\text{(vii)} & \mathfrak{T}_7 &= \mathbf{V}_2 + \mathbf{V}_4\\
&\text{(viii)} & \mathfrak{T}_8 &= \mathbf{V}_2 + \mathbf{V}_3 + \mathbf{V}_5\\
&\text{(ix)} & \mathfrak{T}_9 &= \mathbf{V}_2 + \mathbf{V}_5 + \mathbf{V}_6\\
&\text{(x)} & \mathfrak{T}_{10} &= \mathbf{V}_2 + \mathbf{V}_4 + \mathbf{V}_5
\end{aligned} \tag{20}$$

4. Exact invariant solutions

This section constructs a variety of closed-form invariant solutions for the corresponding symmetry subalgebras by solving the Lagrange's characteristic equation [33]

$$\frac{\mathrm{d}x}{\xi} = \frac{\mathrm{d}y}{\phi} = \frac{\mathrm{d}z}{\psi} = \frac{\mathrm{d}t}{\tau} = \frac{\mathrm{d}v}{\eta}. \tag{21}$$

4.1 *Subalgebra* $\mathfrak{T}_1 := \mathbf{V}_1 = \frac{x}{3}\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{5z}{3}\frac{\partial}{\partial z} + t\frac{\partial}{\partial t} - \frac{v}{3}\frac{\partial}{\partial v}$

The Langrange's system (21) becomes

$$\frac{\mathrm{d}x}{x/3} = \frac{\mathrm{d}y}{y} = \frac{dz}{5z/3} = \frac{\mathrm{d}t}{t} = \frac{\mathrm{d}v}{-v/3} \tag{22}$$

which gives the similarity solution

$$v(x, y, z, t) = \frac{V(X, Y, T)}{z^{\frac{1}{5}}} \tag{23}$$

with

$$X = \frac{x}{z^{\frac{1}{5}}},\ Y = \frac{y}{z^{\frac{3}{5}}} \ \text{ and } \ T = \frac{t}{z^{\frac{3}{5}}}.$$

Putting (23) into (4), we get

$$V_{XXXY} + 3V_XV_{XY} + 3V_YV_{XX} + \frac{3}{5}XV_{XX} + \frac{9}{5}(TV_{XT} + YV_{XY}) + \frac{6}{5}V_X - (V_{TT} + V_{YT}) = 0. \tag{24}$$

To get the group-invariant solution, apply the Lie group method again which results in new infinitesimal generators:

$$\xi_X = -\frac{9}{10}a_1T + a_3, \quad \xi_Y = a_2, \quad \xi_T = a_1$$

and

$$\eta_V = \frac{18}{50}(a_1 - a_2)T^2 + \frac{1}{50}(50a_4 - 18a_1Y)T + \frac{3}{10}a_1X - \frac{3}{5}a_2X - \frac{1}{5}a_3Y + a_5, \tag{25}$$

where a_i's $(1 \le i \le 5)$ are arbitrary constants.

Case (i): When $a_1 \neq 0$ and other constants are zero

From eq. (25), the associated characteristic equation becomes

$$\frac{\mathrm{d}X}{-(9/10)T} = \frac{\mathrm{d}Y}{0} = \frac{\mathrm{d}T}{1} = \frac{\mathrm{d}V}{(18/50)T^2 - (18/50)YT + (3/10)X}. \quad (26)$$

Solving eq. (26), we obtain

$$V(X, Y, T) = G(R, S) + \frac{3}{100}T\left(7T^2 - 6TY + 10X\right), \quad (27)$$

with

$$R = X + \frac{9T^2}{20}, \quad S = Y.$$

Using eq. (27) into (24), we get the following (1 + 1) nonlinear partial differential equation:

$$G_{RRRS} + 3G_RG_{RS} + 3G_SG_{RR} + \frac{3}{5}RG_{RR} + \frac{9}{5}SG_{RS} + \frac{3}{10}G_R + \frac{9}{25}S = 0. \quad (28)$$

Infinitesimals of (28) are

$$\xi_R = -\frac{1}{2}b_1R + b_2, \quad \xi_S = b_1S$$

and

$$\eta_G = -\frac{1}{5}Sb_2 + \frac{1}{2}b_1G + b_3. \quad (29)$$

On simplifying (29), we obtain

$$G(R, S) = S^{\frac{1}{2}}H(w) - \frac{2}{5}(A_2S + 5A_3) \quad (30)$$

with $w = S^{\frac{1}{2}}(R - 2A_2)$, and $A_2 = b_2/b_1$ and $A_3 = b_3/b_1$ are constants. Using eq. (30) into (28), we get

$$25wH^{(4)} + 100H^{(3)} + 15H'(7 + 10H' + 10wH'') + 45wH'' + 15(2w + 5H)H'' + 18 = 0 \quad (31)$$

which gives the solutions

$$H(w) = \delta_1 - \frac{2}{5}w \quad \text{and} \quad H(w) = \frac{\delta_2}{w} - \frac{3}{10}w, \quad (32)$$

where δ_1 and δ_2 are arbitrary constants.

Accordingly, we derive exact-invariant solution of gBKP-B (4)

$$v(x, y, z, t) = \delta_1\sqrt{\frac{y}{z}} + \frac{3tx}{10z} + \frac{21t^3 - 36t^2y}{100z^2} + \frac{2A_2y}{5z^{\frac{4}{5}}} - \frac{2}{5}\left(\frac{5A_3}{\sqrt[5]{z}} + \frac{xy}{z}\right), \quad (33)$$

$$v(x, y, z, t) = \frac{21t^2(2t - 3y)}{200z^2} + \frac{3x(t - y)}{10z} + \frac{yA_2}{5z^{\frac{4}{5}}} + \frac{20\delta_2z}{(xz - 40A_2z^{\frac{6}{5}} + 9t^2)} - \frac{2A_3}{5z^{\frac{1}{5}}}. \quad (34)$$

Case (ii): When $a_1 = a_2 \neq 0$ and other constants are zero

From eq. (25), the associated characteristic equation becomes

$$\frac{\mathrm{d}X}{-(9/10)T} = \frac{\mathrm{d}Y}{1} = \frac{\mathrm{d}T}{1} = \frac{\mathrm{d}V}{-(18/50)YT - (3/10)X}. \quad (35)$$

Solving eq. (35), we obtain

$$V(X, Y, T) = G(R, S) - \frac{3}{100}T\left(T^2 + 6TY + 10X\right) \quad (36)$$

with

$$R = X + \frac{9T^2}{20}, \quad S = Y - T.$$

Using eq. (36) into (24), we get the following (1 + 1) nonlinear partial differential equation:

$$G_{RRRS} + 3G_RG_{RS} + 3G_SG_{RR} + \frac{3}{5}RG_{RR} + \frac{9}{5}SG_{RS} + \frac{3}{10}G_R + \frac{9}{25}S = 0. \quad (37)$$

Infinitesimals of (37) are

$$\xi_R = -\frac{1}{2}b_1R + b_2, \quad \xi_S = b_1S$$

and

$$\eta_G = -\frac{1}{5}Sb_2 + \frac{1}{2}b_1G + b_3. \quad (38)$$

On simplifying (38), we get

$$G(R, S) = S^{\frac{1}{2}}H(w) - \frac{2}{5}(A_2S + 5A_3), \quad (39)$$

with $w = S^{\frac{1}{2}}(R - 2A_2)$, and $A_2 = b_2/b_1$ and $A_3 = b_3/b_1$ are constants.

Using eq. (39) into (37), we get

$$25wH^{(4)} + 100H^{(3)} + 15H'(7 + 10H' + 10wH'') + 45wH'' + 15(2w + 5H)H'' + 18 = 0 \quad (40)$$

which gives the solutions

$$H(w) = \delta_3 - \frac{3}{10}w \quad \text{and} \quad H(w) = \frac{\delta_4}{w} - \frac{3}{10}w, \quad (41)$$

where δ_3 and δ_4 are arbitrary constants.

Accordingly, we derive exact invariant solutions of gBKP-B (4) as

$$v(x,y,z,t) = \frac{21t^3}{200z^2} - \frac{63yt^2}{200z^2} - \frac{3xy}{10z} + \delta_3\sqrt{\frac{y-t}{z}} + \frac{(y-t)}{5z^{\frac{4}{5}}}A_2 - \frac{2A_3}{z^{\frac{1}{5}}}, \tag{42}$$

$$v(x,y,z,t) = \frac{3t}{100z^2}\left(t^2+6ty+10xz\right) + \frac{20\delta_4 z}{9t^2+20xz-40A_2z^{\frac{4}{5}}} - \frac{2(y-t)}{5z^{\frac{4}{5}}}A_2 - \frac{3(y-t)}{200z^2} \times \left(9t^2+20xz-40A_2z^{\frac{6}{5}}\right) - \frac{2A_3}{z^{\frac{1}{5}}}. \tag{43}$$

4.2 *Subalgebra* $\mathfrak{T}_2 := \mathbf{V}_1 + \mathbf{V}_3$

Lagrange's equation for subalgebra $\mathfrak{T}_2$ is

$$\frac{\mathrm{d}x}{x/3} = \frac{\mathrm{d}y}{y} = \frac{\mathrm{d}z}{5z/3} = \frac{\mathrm{d}t}{t} = \frac{\mathrm{d}v}{f_1(z)-(v/3)}. \tag{44}$$

On that account, eq. (44) furnishes the similarity form

$$v(x,y,z,t) = \frac{V(X,Y,T)}{z^{\frac{1}{5}}} + \frac{3}{5z^{\frac{1}{5}}}\int \frac{f_1(z)}{z^{\frac{4}{5}}}\mathrm{d}z \tag{45}$$

with

$$X = \frac{x}{z^{\frac{1}{5}}},\ Y = \frac{y}{z^{\frac{3}{5}}} \quad \text{and} \quad T = \frac{t}{z^{\frac{3}{5}}}.$$

Taking the similarity solution v from (45) into (4), we acquire the newly diminished equation

$$5(V_{TT}+V_{YT}) - 6V_X - 15(V_XV_Y)_X - 9(TV_{XT}+YV_{XY}) - 3XV_{XX} - 5V_{XXXY} = 0. \tag{46}$$

Again, employing LST on eq. (46), new infinitesimals are given as

$$\xi_X = -\frac{9a_1}{10}T + a_3,\quad \xi_Y = a_2,\quad \xi_T = a_1,$$
$$\eta_V = \frac{1}{50}(18a_1-18a_2)T^2 + \frac{1}{50}(-18Ya_1+50a_4)T + \frac{3}{10}Xa_1 - \frac{3}{5}Xa_2 - \frac{1}{5}a_3Y + a_5, \tag{47}$$

where a_i's $(1 \le i \le 5)$ are arbitrary constants.

For $a_1 = a_2 \neq 0$ and all other constants are zero. From eq. (47), we get the characteristic system as

$$\frac{\mathrm{d}X}{-(9/10)T} = \frac{\mathrm{d}Y}{1} = \frac{\mathrm{d}T}{1} = \frac{\mathrm{d}V}{-(18/50)YT-(3/10)X} \tag{48}$$

which gives

$$V(X,Y,T) = F(R,S) - \frac{3}{100}T(T^2+10X+6TY) \tag{49}$$

with

$$R = X + \frac{9T^2}{20},\ S = Y - T.$$

Using (49) and (46), we have the reduced equation

$$50F_{RRRS} + 150(F_SF_R)_R + 30RF_{RR} + 15F_R + 90SF_{RS} + 18S = 0. \tag{50}$$

Again, applying the Lie symmetry method on eq. (50), the new infinitesimals are

$$\xi_R = -\frac{1}{2}b_1R + b_2,\quad \xi_S = b_1S,$$
$$\eta_F = -\frac{1}{5}Sb_2 + \frac{1}{2}b_1F + b_3, \tag{51}$$

where b_i's $(1 \le i \le 3)$ are arbitrary constants. The characteristic system for (51) is

$$\frac{\mathrm{d}R}{-(1/2)b_1R+b_2} = \frac{\mathrm{d}S}{b_1S} = \frac{\mathrm{d}F}{-(1/5)Sb_2+(1/2)b_1F+b_3} \tag{52}$$

that gives the similarity form

$$F(R,S) = \sqrt{S}H(w) - \frac{2}{5}(SB_2+5B_3) \tag{53}$$

with $w = \sqrt{S}(R-2B_2)$, and $B_2 = b_2/b_1$ and $B_3 = b_3/b_1$ are constants. Using (53) and (50), we get an ODE

$$25wH^{(4)} + 100H^{(3)} + 15H'(7+10H'+10wH'') + 45wH'' + 15(2w+5H)H'' + 18 = 0. \tag{54}$$

On solving (54), we have

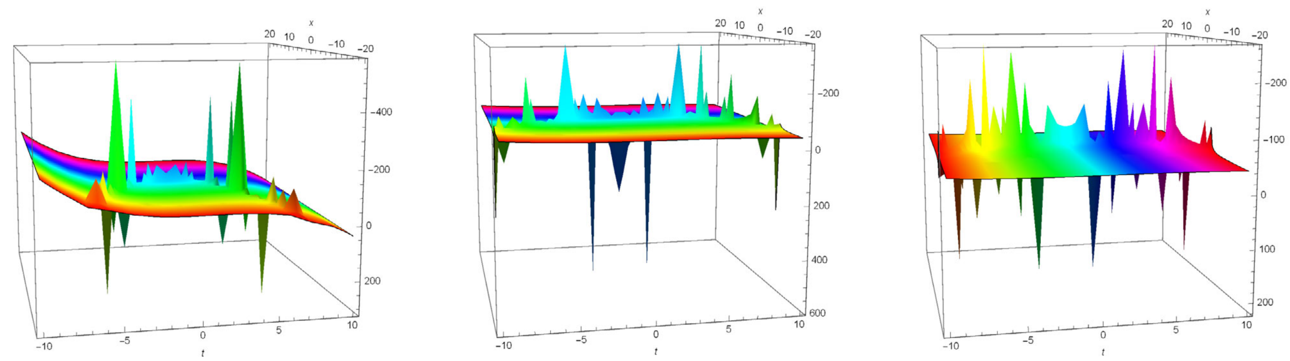
$$H(w) = \delta_5 - \frac{3}{10}w$$

and

$$H(w) = \frac{\delta_6}{w} - \frac{3}{10}w, \tag{55}$$

where δ_5 and δ_6 are any two arbitrary constants.

Accordingly, we obtain the following solutions of the gBKP-B (4):



$z = 0.961$ $z = 3$ $z = 4.81$

Figure 1. Three distinct complex structures of elastic interactions between curve-shaped lumps and oscillating multisolitons for solution (43) with parameters $A_2 = 1.7$, $A_3 = 31$, $\delta_4 = 15.7$ and $y = 0.7$.

$$v(x, y, z, t) = \frac{21t^3}{200z^2} - \frac{63t^2 y}{200z^2} - \frac{3xy}{10z} + \delta_5 \sqrt{\frac{-t+y}{z}} + \frac{(-t+y)B_2}{5z^{\frac{4}{5}}} + \frac{3}{5z^{\frac{1}{5}}} \int \frac{f_1(z)}{z^{\frac{4}{5}}} \mathrm{d}z - \frac{2B_3}{z^{\frac{1}{5}}}, \tag{56}$$

$$v(x, y, z, t) = \frac{3}{5z^{\frac{1}{5}}} \int \frac{f_1(z)}{z^{\frac{4}{5}}} \mathrm{d}z - \frac{3t}{100z^2}(t^2 + 6ty + 10xz) + \frac{20\delta_6 z}{(9t^2 + 20xz - 40z^{\frac{6}{5}} B_2)} - \frac{3(y-t)}{200z^2}(9t^2 + 20xz - 40z^{\frac{6}{5}} B_2) - \frac{2B_2(y-t)}{5z^{\frac{4}{5}}} - \frac{2B_3}{z^{\frac{1}{5}}}. \tag{57}$$

4.3 *Subalgebra* $\mathfrak{T}_3 := \mathbf{V}_1 + \mathbf{V}_7$ *when* $f_5(z) = Az$

The related Lagrange's system is interpreted as

$$\frac{\mathrm{d}x}{\frac{x}{3} - \frac{3A}{2}t} = \frac{\mathrm{d}y}{y} = \frac{\mathrm{d}z}{\frac{5z}{3}} = \frac{\mathrm{d}t}{t + Az} = \frac{\mathrm{d}v}{\frac{A}{2}x - \frac{v}{3}}. \tag{58}$$

Equation (58) produces

$$v(x, y, z, t) = \frac{V(X, Y, T)}{z^{\frac{1}{5}}} + \frac{3}{64}A(18At + 16x - 9A^2 z), \tag{59}$$

with

$$X = \frac{16x + 9A(4t - 3Az)}{16z^{\frac{1}{5}}}, \quad Y = \frac{y}{z^{\frac{3}{5}}} \quad \text{and} \quad T = \frac{2t - 3Az}{2z^{\frac{3}{5}}}.$$

We obtain a new reduction equation on solving (59) and (4)

$$5(V_{TT} + V_{YT}) - 6V_X - 15(V_X V_Y)_X - 9(TV_{XT} + YV_{XY}) - 3XV_{XX} - 5V_{XXXY} = 0. \tag{60}$$

By the application of Lie symmetry method on eq. (60), the desired infinitesimals are

$$\xi_X = \frac{-9a_1}{10}T + a_3, \quad \xi_Y = a_2, \quad \xi_T = a_1,$$

$$\eta_V = \frac{1}{50}(18a_1 - 18a_2)T^2 + \frac{1}{50}(-18Ya_1 + 50a_4)T + \frac{3}{10}Xa_1 - \frac{3}{5}Xa_2 - \frac{1}{5}a_3 Y + a_5, \tag{61}$$

where a_i's $(1 \le i \le 5)$ are arbitrary constants.

Suppose $a_1 = a_2 \neq 0$ and all other constants are zero. By eq. (61), the characteristic equation becomes

$$\frac{\mathrm{d}X}{-(9/10)T} = \frac{\mathrm{d}Y}{1} = \frac{\mathrm{d}T}{1} = \frac{\mathrm{d}V}{-(18/50)YT - (3/10)X}, \tag{62}$$

which provides

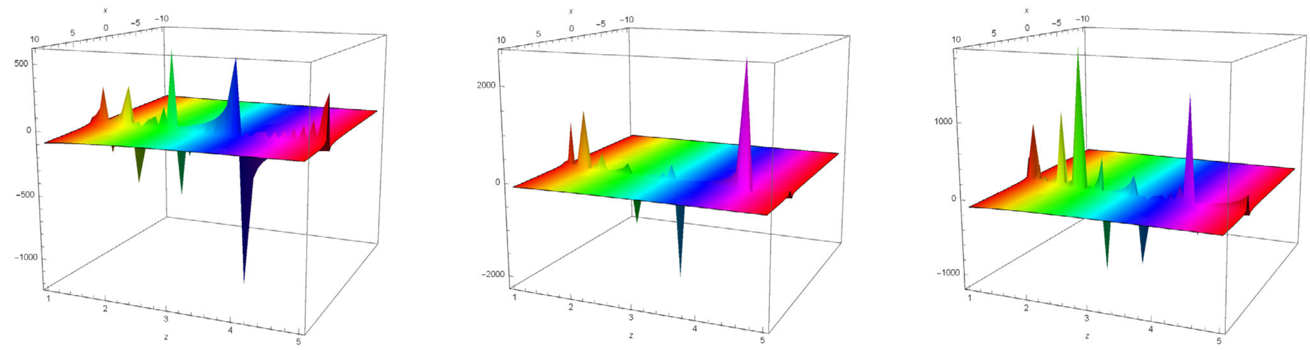
$$V(X, Y, T) = F(R, S) - \frac{3}{100}T(T^2 + 10X + 6TY), \tag{63}$$

with

$$R = X + \frac{9T^2}{20}, \quad S = Y - T.$$

Using (63) and (60), we get the following equation:

$$50F_{RRRS} + 150(F_S F_R)_R + 30RF_{RR} + 15F_R + 90SF_{RS} + 18S = 0. \tag{64}$$



$t = 1$ $t = 1.6$ $t = 2.3$

Figure 2. Three distinct complex structures of lump wave solitons for solution (57) with parameters $a_0 = 4$, $B_2 = 1.7$, $B_3 = 31$, $\delta_6 = 15.7$, $y = 11$ and $f_1(z) = a_0 z$.

Applying Lie symmetry method on eq. (64) again, we get the set of infinitesimals as

$$\xi_R = \frac{-1}{2} b_1 R + b_2, \quad \xi_S = b_1 S,$$
$$\eta_F = \frac{-1}{5} S b_2 + \frac{1}{2} b_1 F + b_3, \tag{65}$$

where b_i's ($1 \le i \le 3$) are arbitrary constants. Characteristic equation for (65) is

$$\frac{\mathrm{d}R}{-(1/2) b_1 R + b_2} = \frac{\mathrm{d}S}{b_1 S} = \frac{\mathrm{d}F}{-(1/5) S b_2 + \frac{1}{2} b_1 F + b_3} \tag{66}$$

which derives the similarity form

$$F(R, S) = \sqrt{S} H(w) - \frac{2}{5}(S B_2 + 5 B_3) \tag{67}$$

with $w = \sqrt{S}(R - 2B_2)$, and $B_2 = b_2/b_1$ and $B_3 = b_3/b_1$ are constants. Using (67) and (64), we get an ODE

$$25 w H^{(4)} + 100 H^{(3)} + 15 H'(7 + 10H' + 10 w H'') + 45 w H'' + 15(2w + 5H) H'' + 18 = 0. \tag{68}$$

We solve (68), to get

$$H(w) = \delta_7 - \frac{3}{10} w$$

and

$$H(w) = \frac{\delta_8}{w} - \frac{3}{10} w, \tag{69}$$

where δ_7 and δ_8 are any two arbitrary constants.

Group-invariant solutions of the gBKP-B equation (4) with the help of back substitution are:

$$v(x, y, z, t) = \frac{9t^3}{400z^2} - \frac{63t^2 y}{200z^2} - \frac{189t^2 A}{400z} + \delta_7 \sqrt{\frac{2(y - t) + 3Az}{2z}} + \frac{27At}{400z}(4y + 23Az) + \frac{(2(y - t) + 3Az) B_2}{10 z^{\frac{4}{5}}} - \frac{3x}{20}(2y - 5Az) - \frac{27A^2}{800}(6y + 23Az) - \frac{2B_3}{z^{\frac{1}{5}}}, \tag{70}$$

$$v(x, y, z, t) = -\frac{27}{32} A^2 (t - Az) - \frac{1}{320z}(16x - 27A^2 z + 36At)(6t + 5z - 144Az) - \frac{(2y - t + 3Az) B_2}{5 z^{\frac{4}{5}}} + \frac{20 \delta_8 z}{(18t^2 - 27A^2 z^2 + 36Atz + 40xz - 80 B_2 z^{\frac{6}{5}})} - \frac{3}{800z^2}(2t - 3Az)^2 (2t + 6y - 3Az) - \frac{2B_3}{z^{\frac{1}{5}}} - \frac{3}{20} \frac{(2y - t + 3Az)}{z^{\frac{4}{5}}} \times (18t^2 - 27A^2 z^2 + 36Atz + 40xz - 80 B_2 z^{\frac{6}{5}}). \tag{71}$$

4.4 *Subalgebra* $\mathfrak{T}_4 := \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_4$

The associated Lagrange's system of $\mathfrak{T}_4$ is

$$\frac{\mathrm{d}x}{x/3} = \frac{\mathrm{d}y}{y} = \frac{\mathrm{d}z}{(5z/3) + 1} = \frac{\mathrm{d}t}{t} = \frac{\mathrm{d}v}{t f_2(z) - (v/3)}. \tag{72}$$

The similarity form of eq. (72) is

$$v(x, y, z, t) = \frac{V(X, Y, T)}{(3 + 5z)^{\frac{1}{5}}}$$

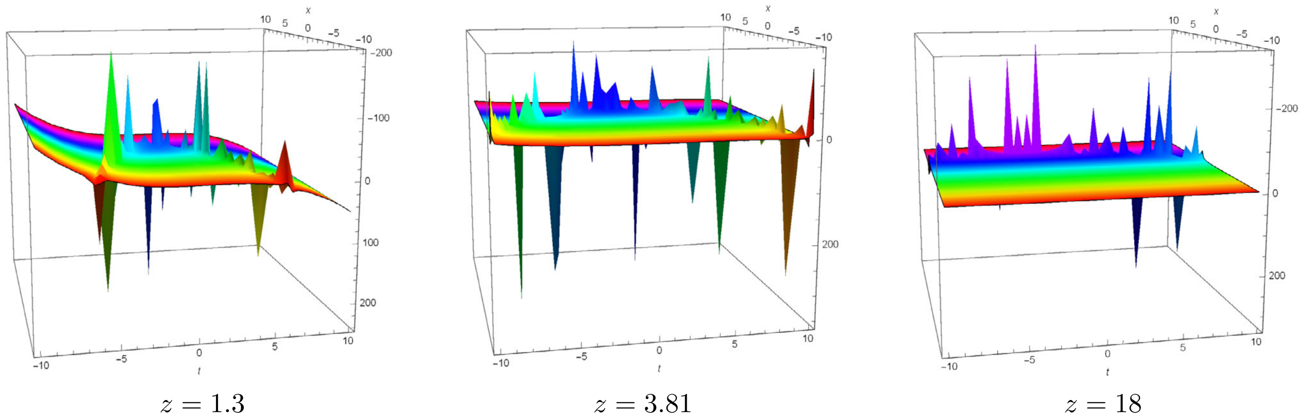


Figure 3. Three distinct complex structures of elastic interactions between curve-shaped lump wave solitons and oscillating multisolitons for solution (71) with parameters $A = 0.16$, $B_2 = 1.37$, $B_3 = 1.2$, $\delta_8 = 5$ and $y = 0.11$.

$$+\frac{3t}{(3+5z)^{\frac{1}{5}}}\int\frac{f_2(z)}{(3+5z)^{\frac{4}{5}}}\mathrm{d}z$$

with

$$X=\frac{x}{(3+5z)^{\frac{1}{5}}},\quad Y=\frac{y}{(3+5z)^{\frac{3}{5}}}$$

and

$$T=\frac{t}{(3+5z)^{\frac{3}{5}}}. \tag{73}$$

On substitution of v from (73) in (4), we get

$$V_{XXXY}+6V_X+3(V_XV_Y)_X+9(TV_{XT}+YV_{XY})$$
$$+3XV_{XX}-(V_{TT}+V_{YT})=0. \tag{74}$$

For eq. (74), the set of infinitesimals can be provided as

$$\xi_X=\frac{-9a_1}{2}T+a_3,\quad \xi_Y=a_2,\quad \xi_T=a_1,$$
$$\eta_V=\frac{3}{2}(a_1-2a_2)X-(9Ta_1+a_3)Y$$
$$+9T^2a_1-9T^2a_2+Ta_4+a_5, \tag{75}$$

where a_i's ($1 \le i \le 5$) are arbitrary constants. Let $a_1 \neq 0$ and the remaining constants are zero.

For eq. (75), the characteristic equation becomes

$$\frac{\mathrm{d}X}{-\frac{9}{2}T}=\frac{\mathrm{d}Y}{0}=\frac{\mathrm{d}T}{1}=\frac{\mathrm{d}V}{(3/2)X-9TY+9T^2}. \tag{76}$$

Similarity form for eq. (76) is

$$V(X,Y,T)=F(R,S)+\frac{3RT}{2}-\frac{9ST^2}{2}+\frac{15T^3}{8} \tag{77}$$

with

$$R=X+\frac{9T^2}{4},\quad S=Y.$$

With the help of (77) and (74), we find a PDE

$$F_{RRRS}+3(F_SF_R)_R+3RF_{RR}+\frac{3}{2}F_R$$
$$+9SF_{RS}+9S=0. \tag{78}$$

Now, apply LSM on eq. (78), then we derive the appropriate infinitesimals

$$\xi_R=-\frac{1}{2}b_1R+b_2,\quad \xi_S=b_1S,$$
$$\eta_F=-Sb_2+\frac{1}{2}b_1F+b_3, \tag{79}$$

where b_i's ($1 \le i \le 3$) are arbitrary constants.

For eq. (79), the characteristic equation becomes

$$\frac{\mathrm{d}R}{-(1/2)b_1R+b_2}=\frac{\mathrm{d}S}{b_1S}=\frac{\mathrm{d}F}{-Sb_2+(1/2)b_1F+b_3}. \tag{80}$$

Similarity solution for eq. (80) is presented as follows:

$$F(R,S)=\sqrt{S}H(w)-\frac{2}{b_1}(Sb_2+b_3) \tag{81}$$

with

$$w=\sqrt{S}\left(R-2\frac{b_2}{b_1}\right).$$

On combining (81) and (78), we can promptly obtain

$$wH^{(4)}+4H^{(3)}+6wH'H''+3HH''$$
$$+15wH''+6(H')^2+21H'+18=0. \tag{82}$$

On solving (82), we have

$$H(w)=\delta_9-\frac{3}{2}w$$

and

$$H(w)=\frac{\delta_{10}}{w}-\frac{3}{2}w, \tag{83}$$

where δ_9 and δ_{10} are arbitrary constants.

On solving by substitution, we get the general solutions of the gBKP-B (4):

$$\begin{aligned}v(x,y,z,t) &= \frac{3t}{(3+5z)^{\frac{1}{5}}}\int\frac{f_2(z)}{(3+5z)^{\frac{4}{5}}}\mathrm{d}z\\&+\frac{6t(7t^2-6ty+2t(3+5z)x)}{8(3+5z)^2}+\sqrt{\frac{y}{3+5z}}\delta_9\\&-\frac{3y}{8(3+5z)^2}\left(9t^2+4(3+5z)x+3\frac{b_2}{b_1}(3+5z)^{\frac{6}{5}}\right)\\&-\frac{b_2}{b_1}\frac{2y}{(3+5z)^{\frac{4}{5}}}-\frac{b_3}{b_1}\frac{2}{(3+5z)^{\frac{1}{5}}},\end{aligned}\tag{84}$$

$$\begin{aligned}v(x,y,z,t) &= \frac{3t}{(3+5z)^{\frac{1}{5}}}\int\frac{f_2(z)}{(3+5z)^{\frac{4}{5}}}\mathrm{d}z\\&+\frac{(15t^3-36t^2y)}{8(3+5z)^2}+\frac{3t(9t^2+4(3+5z)x)}{8(3+5z)^{\frac{4}{5}}}\\&-\frac{b_3}{b_1}\frac{2}{(3+5z)^{\frac{1}{5}}}-\frac{3y}{8(3+5z)^2}\\&\times\left(9t^2+4(3+5z)x-8\frac{b_2}{b_1}(3+5z)^{\frac{6}{5}}\right)\\&-\frac{b_2}{b_1}\frac{2y}{(3+5z)^{\frac{4}{5}}}\\&+\frac{4\delta_{10}(3+5z)}{\left(9t^2+4(3+5z)x-8\frac{b_2}{b_1}(3+5z)^{\frac{6}{5}}\right)}.\end{aligned}\tag{85}$$

4.5 *Subalgebra* $\mathfrak{T}_5 := \mathbf{V}_2 = \frac{\partial}{\partial z}$

The associated Lagrange's system reads as

$$\frac{\mathrm{d}x}{0}=\frac{\mathrm{d}y}{0}=\frac{\mathrm{d}z}{1}=\frac{\mathrm{d}t}{0}=\frac{\mathrm{d}v}{0}\tag{86}$$

which gives

$$v(x,y,z,t)=V(X,Y,T)\tag{87}$$

with invariants

$$X=x,\quad Y=y,\quad T=t.$$

Using (87) into (4), we thus obtain the (2 + 1)-dimensional reduced nonlinear PDE:

$$V_{XXXY}+3V_XV_{XY}+3V_YV_{XX}-V_{TT}-V_{YT}=0\tag{88}$$

which has the general solution

$$\begin{aligned}V(X,Y,T)&=c_3\\&+2c_2\tanh\left(c_1T+c_2X-\frac{c_1^2Y}{c_1-4c_2^3}+c_4\right),\end{aligned}\tag{89}$$

where $c_1,\ c_2,\ c_3$ and c_4 are constants of integration.

Therefore, the resulting solution of the gBKP-B equation (4) is

$$\begin{aligned}v(x,y,z,t)&=c_3\\&+2c_2\tanh\left(c_4+c_1t+c_2x-\frac{c_1^2y}{c_1-4c_2^3}\right).\end{aligned}\tag{90}$$

Further, Let us take

$$V(X,Y,T)=H(w),\tag{91}$$

where $w=aX+bY+cT$.

Here $a,\ b$ and c are arbitrary constants. Taking (91) into (88), we have

$$a^3bH^{(4)}+6a^2bH'H''-c(b+c)H''=0.\tag{92}$$

The primitives are

$$H(w)=\delta_{11}+\delta_{12}w,\quad H(w)=\delta_{13}+\frac{c(b+c)}{6a^2b}w$$

and

$$H(w)=\delta_{14}+\frac{2a}{w}+\frac{c(b+c)}{6a^2b}w,\tag{93}$$

where δ_{11}, δ_{12}, δ_{13} and δ_{14} are arbitrary constants.

Hence, we acquire the following exact invariant solutions of gBKP-B (4):

$$v(x,y,z,t)=\delta_{11}+\delta_{12}\left(ax+by+ct\right),\tag{94}$$

$$v(x,y,z,t)=\delta_{13}+\frac{c(b+c)}{6a^2b}\left(ax+by+ct\right),\tag{95}$$

$$\begin{aligned}v(x,y,z,t)&=\delta_{14}+\frac{2a}{(ax+by+ct)}\\&\quad+\frac{c(b+c)}{6a^2b}\left(ax+by+ct\right).\end{aligned}\tag{96}$$

4.6 *Subalgebra* $\mathfrak{T}_6 := \mathbf{V}_2+\mathbf{V}_3 = \frac{\partial}{\partial z}+f_1(z)\frac{\partial}{\partial v}$

Lagrange's system reads as

$$\frac{\mathrm{d}x}{0}=\frac{\mathrm{d}y}{0}=\frac{\mathrm{d}z}{1}=\frac{\mathrm{d}t}{0}=\frac{\mathrm{d}v}{f_1(z)}\tag{97}$$

which gives

$$v(x,y,z,t)=V(X,Y,T)+\int f_1(z)\,\mathrm{d}z\tag{98}$$

with invariants $X=x,\ Y=y,\ T=t$.

Substituting (98) into (4), we thus obtain the $(2+1)$-dimensional reduced nonlinear PDE:

$$V_{XXXY}+3V_XV_{XY}+3V_YV_{XX}-V_{TT}-V_{YT}=0 \tag{99}$$

which has the general solution

$$V(X,Y,T)=c_3 + 2c_2\tanh\left(c_1T+c_2X-\frac{c_1^2Y}{c_1-4c_2^3}+c_4\right), \tag{100}$$

where $c_1,\ c_2,\ c_3$ and c_4 are constants of integration.

Therefore, the resulting solution of the gBKP-B equation (4) is

$$v(x,y,z,t)=c_3+2c_2\tanh \times\left(c_4+c_1t+c_2x-\frac{c_1^2y}{c_1-4c_2^3}\right) + \int f_1(z)\,\mathrm{d}z. \tag{101}$$

Further, we consider that

$$V(X,Y,T)=H(w), \tag{102}$$

where $w=aX+bY+cT$.

Here, $a,\ b$ and c are arbitrary constant parameters. We substitute (102) into (99), to obtain

$$a^3bH^{(4)}+6a^2bH'H''-c(b+c)H''=0. \tag{103}$$

The primitives are

$$H(w)=\delta_{15}+\delta_{16}w$$

and

$$H(w)=\delta_{17}+\frac{2a}{w}+\frac{c(b+c)}{6a^2b}w, \tag{104}$$

where δ_{15}, δ_{16} and δ_{17} are arbitrary constants.

Hence, we acquire the following exact-invariant solutions of gBKP-B equation (4):

$$v(x,y,z,t)=\delta_{15}+\delta_{16}\,(ax+by+ct)+\int f_1(z)\,\mathrm{d}z, \tag{105}$$

$$v(x,y,z,t)=\delta_{17}+\frac{2a}{(ax+by+ct)} + \frac{c(b+c)}{6a^2b}\,(ax+by+ct)+\int f_1(z)\,\mathrm{d}z. \tag{106}$$

4.7 *Subalgebra* $\mathfrak{T}_7 := \mathbf{V}_2+\mathbf{V}_4$

Lagrange's system for $\mathfrak{T}_7$ is

$$\frac{\mathrm{d}x}{0}=\frac{\mathrm{d}y}{0}=\frac{\mathrm{d}z}{1}=\frac{\mathrm{d}t}{0}=\frac{\mathrm{d}v}{tf_2(z)}. \tag{107}$$

Equation (107) provides the following similarity solution:

$$v(x,y,z,t)=V(X,Y,T)+t\int f_2(z)\mathrm{d}z \tag{108}$$

with invariants $X=x,\ Y=y$ and $T=t$.

After substitution of (108) into (4), we find a new reduction equation as

$$V_{TT}+V_{YT}-3V_XV_{XY}-3V_YV_{XX}-V_{XXXY}=0 \tag{109}$$

that provides the general solution

$$V(X,Y,T)=c_3 + 2c_2\tanh\left(c_2X+c_1T-\frac{c_1^2Y}{c_1-4c_2^3}+c_4\right), \tag{110}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants of integration.

Therefore, the resultant solution of the gBKP-B equation (4) is

$$v(x,y,z,t)=c_3 + 2c_2\tanh\left(c_2x+c_1t-\frac{c_1^2y}{c_1-4c_2^3}+c_4\right) + t\int f_2(z)\mathrm{d}z. \tag{111}$$

Moreover, let

$$V(X,Y,T)=H(w), \tag{112}$$

where $w=aX+bY+cT$, and $a,\ b$ and c are arbitrary constant parameters.

Taking (112) into (109), we acquire the following reduced ODE:

$$a^3bH^{(4)}+6a^2bH'H''-c(c+b)H''=0. \tag{113}$$

The primitives are

$$H(w)=\delta_{18}+\frac{c(c+b)}{6a^2b}w$$

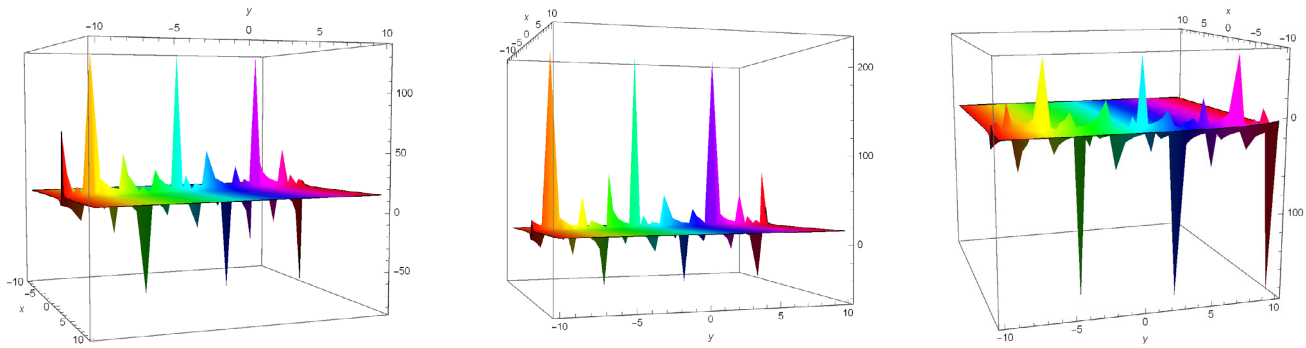
and

$$H(w)=\frac{2a}{w}+\delta_{19}+\frac{c(c+b)}{6a^2b}w, \tag{114}$$

where δ_{18} and δ_{19} are arbitrary constants.

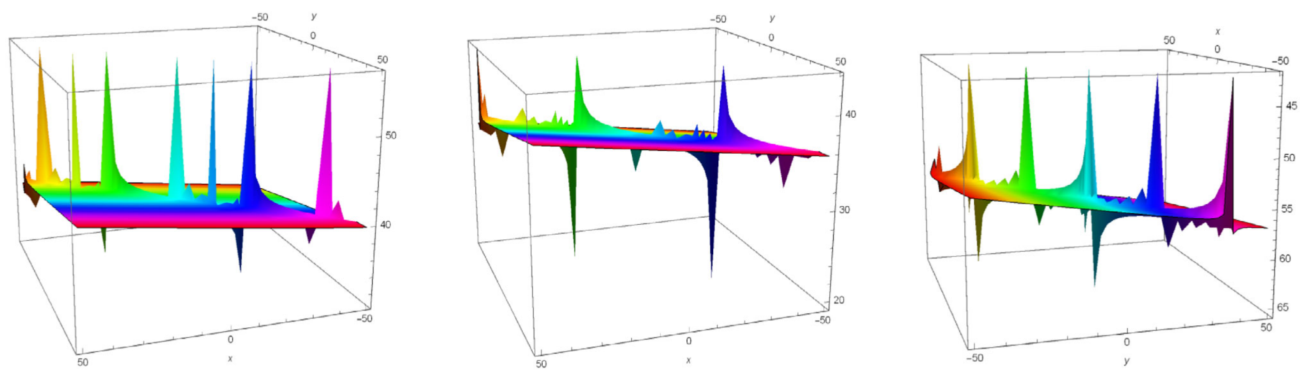
Finally, we obtain the following solutions of the gBKP-B equation (4):

$$v(x,y,z,t)=\delta_{18}+\frac{c(c+b)}{6a^2b}(ax+by+ct) + t\int f_2(z)\mathrm{d}z, \tag{115}$$



$t = 1$ $t = 3.831$ $t = 15$

Figure 4. Three distinct complex structures of lump waves for solution (106) with parameters $a = 191$, $b = 3$, $c = 10.3$, $\delta_{17} = 10$, $z = 0.019$ and $f_1(z) = z$.



$t = 1$ $t = 19$ $t = 35$

Figure 5. Three distinct complex structures of multiple lumps and kinky solitons for solution (116) with parameters $\delta_{19} = 10,\ a = 10,\ b = 12,\ c = 2,\ z = 1.3$ and $f_2(z) = z$.

$$v(x, y, z, t) = \delta_{19} + \frac{2a}{(ax + by + ct)} + \frac{c(c+b)}{6a^2 b}(ax + by + ct) + t\int f_2(z)\mathrm{d}z. \tag{116}$$

4.8 *Subalgebra* $\mathfrak{T}_8 := \mathbf{V}_2 + \mathbf{V}_3 + \mathbf{V}_5 = f_3(z)\frac{\partial}{\partial x} + \frac{\partial}{\partial z} + (f_1(z) - y f_3'(z))\frac{\partial}{\partial v}$

For simplification, we assume $f_1(z) = -a_0 f_3'(z)$.

In this case, the related Lagrange's equation reads as

$$\frac{\mathrm{d}x}{f_3(z)} = \frac{\mathrm{d}y}{0} = \frac{\mathrm{d}z}{1} = \frac{\mathrm{d}t}{0} = \frac{\mathrm{d}v}{-(y + a_0) f_3'(z)} \tag{117}$$

which gives

$$v(x, y, z, t) = V(X, Y, T) - (y + a_0)\, f_3(z) \tag{118}$$

with invariants $X = x - \int f_3(z)\,\mathrm{d}z,\quad Y = y,\quad T = t$.

Substituting (118) into (4), we thus obtain the $(2+1)$-dimensional reduced nonlinear PDE:

$$V_{XXXY} + 3V_X V_{XY} + 3V_Y V_{XX} - V_{TT} - V_{YT} = 0 \tag{119}$$

which has the general solution

$$V(X, Y, T) = c_3 + 2c_2 \tanh\left(c_1 T + c_2 X - \frac{c_1^2 Y}{c_1 - 4c_2^3} + c_4\right), \tag{120}$$

where $c_1,\ c_2,\ c_3$ and c_4 are integration constants.

Therefore, the resulting solution of gBKP-B equation (4) is

$$v(x, y, z, t) = c_3 + 2c_2 \tanh \times \left(c_4 + c_1 t + c_2 x - c_2 \int f_3(z)\,\mathrm{d}z - \frac{c_1^2 y}{c_1 - 4c_2^3}\right) - (y + a_0)\, f_3(z). \tag{121}$$

Further, we consider that

$$V(X, Y, T) = H(w), \tag{122}$$

where $w = aX + bY + cT$ and $a,\ b$ and c are arbitrary constant parameters. Putting (122) into (119), we have

$$a^3bH^{(4)} + 6a^2bH'H'' - c(c+b)H'' = 0. \tag{123}$$

Primitives are

$$H(w) = \delta_{20} + \delta_{21}w, \quad H(w) = \delta_{22} + \frac{c(b+c)}{6a^2b}w$$

and

$$H(w) = \delta_{23} + \frac{2a}{w} + \frac{c(b+c)}{6a^2b}w, \tag{124}$$

where δ_{20}, δ_{21}, δ_{22} and δ_{23} are arbitrary constants.

Hence, we acquire the following exact invariant solutions of gBKP-B equation (4):

$$v(x,y,z,t) = \delta_{20} + \delta_{21}\left(ax + by + ct - a\int f_3(z)\,\mathrm{d}z\right) - (y + a_0)\, f_3(z), \tag{125}$$

$$v(x,y,z,t) = \delta_{22} + \frac{c(b+c)}{6a^2b}\left(ax + by + ct - a\int f_3(z)\,\mathrm{d}z\right) - (y + a_0)\, f_3(z), \tag{126}$$

$$v(x,y,z,t) = \delta_{23} + \frac{2a}{\left(ax + by + ct - a\int f_3(z)\,dz\right)} + \frac{c(b+c)}{6a^2b}\left(ax + by + ct - a\int f_3(z)\,\mathrm{d}z\right) - (y + a_0)\, f_3(z). \tag{127}$$

Again, we use the group-theoretic technique to obtain generators of (119)

$$\xi_X = \frac{a_1}{3}X + a_4, \quad \xi_Y = a_1Y + a_3, \quad \xi_T = a_1T + a_2, \quad \eta_V = -\frac{a_1}{3}V + a_5T + a_6, \tag{128}$$

where a_i's $(1 \le i \le 6)$ are arbitrary constants.

Then, the associated Lagrange's system is

$$\frac{\mathrm{d}X}{(a_1/3)X + a_4} = \frac{\mathrm{d}Y}{a_1Y + a_3} = \frac{\mathrm{d}T}{a_1T + a_2} = \frac{\mathrm{d}V}{-(a_1/3)V + a_5T + a_6}. \tag{129}$$

Let us suppose a_2 and a_5 are non-zero and others are zero. On solving eq. (129), we have the following similarity form:

$$V(X,Y,T) = \frac{a_5}{2a_2}T^2 + F(R,S) \tag{130}$$

with $R = X$ and $S = Y$.

Substituting (130) into (119), we thus obtain the (1 + 1)-dimensional PDE as

$$F_{RRRS} + 3F_RF_{RS} + 3F_SF_{RR} - \frac{a_5}{a_2} = 0. \tag{131}$$

Applying Lie symmetry method on eq. (131) again, the set of infinitesimals is

$$\xi_R = \frac{-1}{4}b_1R + b_3, \quad \xi_S = b_1S + b_2, \quad \eta_F = \frac{1}{4}Fb_1 + b_4, \tag{132}$$

where b_i's $(1 \le i \le 4)$ are arbitrary constants.

Suppose $b_1 \neq 0$ and all others are zero. Then, characteristic equation for (132) is

$$\frac{\mathrm{d}R}{-(1/4)R} = \frac{\mathrm{d}S}{S} = \frac{\mathrm{d}F}{(1/4)F} \tag{133}$$

which derives the similarity form

$$F(R,S) = S^{\frac{1}{4}}H(w) \quad \text{with} \quad w = S^{\frac{1}{4}}R. \tag{134}$$

Using (134) and (131), we get an ODE

$$wH^{(4)} + 4H^{(3)} + 6H'(H' + wH'') + 3HH'' - 4\frac{a_5}{a_2} = 0 \tag{135}$$

which gives

$$H(w) = \delta_{24} - \sqrt{\frac{2a_5}{3a_2}}w, \tag{136}$$

where δ_{24} is an arbitrary constant.

Exact invariant solutions of the gBKP-B equation (4) with the help of back substitution is

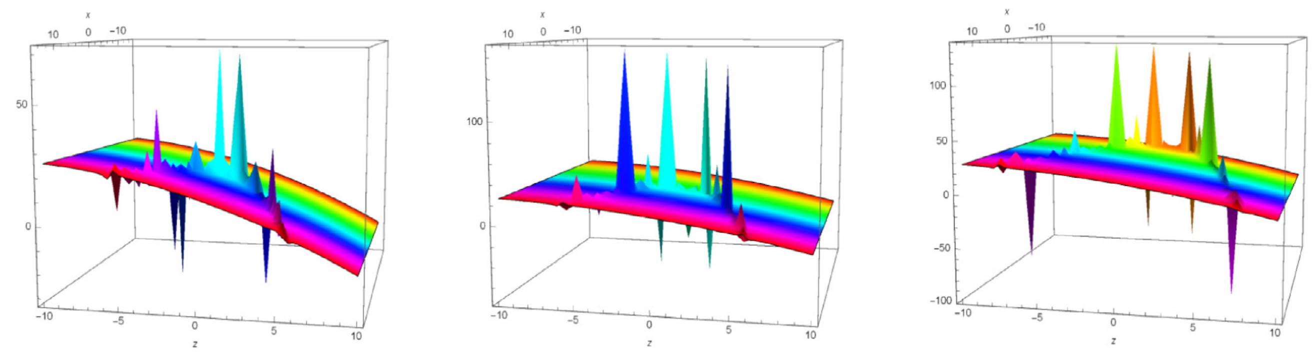
$$v(x,y,z,t) = \frac{a_5}{a_2}\frac{t^2}{2} + y^{\frac{1}{4}}\left[\delta_{24} + \sqrt{\frac{2a_5}{3a_2}}y^{\frac{1}{4}}\left(-x + \int f_3(z)\mathrm{d}z\right)\right] - (y + a_0)f_3(z). \tag{137}$$

4.9 *Subalgebra* $\mathfrak{T}_9 := \mathbf{V}_2 + \mathbf{V}_5 + \mathbf{V}_6$

For simplification, in this case, we assume $f_3(z) = f_4(z)$. Lagrange's system (21) for $\mathfrak{T}_9$ is

$$\frac{\mathrm{d}x}{f_4(z)} = \frac{\mathrm{d}y}{f_4(z)} = \frac{\mathrm{d}z}{1} = \frac{\mathrm{d}t}{0} = \frac{\mathrm{d}v}{(3/2)t^2 f_4''(z) - (x+y)f_4'(z)}. \tag{138}$$

Equation (138) provides the following similarity solution:



$t = 0.07$ $t = 5$ $t = 37$

Figure 6. Three distinct complex structures of elastic interactions between curve-shaped multiple lumps and oscillating multisolitons for solution (127) with parameters $\delta_{23} = 11,\ a_0 = 1,\ a = 19,\ b = 5,\ c = 7,\ y = 1$ and $f_3(z) = z$.

$$v(x, y, z, t) = V(X, Y, T) + \frac{3}{2}t^2 f_4'(z) - (x + y) f_4(z) \quad (139)$$

with $X = x - \int f_4(z)\mathrm{d}z$, $Y = y - \int f_4(z)\mathrm{d}z$ and $T = t$.

After substitution of (139) into (4), we find a new reduction equation as

$$V_{TT} + V_{YT} - 3V_X V_{XY} - 3V_Y V_{XX} - V_{XXXY} = 0 \quad (140)$$

that provides the general solution

$$V(X, Y, T) = c_3 + 2c_2 \tanh\left(c_2 X + c_1 T - \frac{c_1^2 Y}{c_1 - 4c_2^3} + c_4\right), \quad (141)$$

where c_1, c_2, c_3 and c_4 are arbitrary constants of integration. Therefore, the resultant solution of the gBKP-B equation (4) is

$$v(x, y, z, t) = c_3 + 2c_2 \tanh\left(c_2 x + c_1 t - \frac{c_1^2 y}{c_1 - 4c_2^3} + c_4\right) + \frac{3}{2}t^2 f_4'(z) - (x + y) f_4(z). \quad (142)$$

Moreover, we consider that

$$V(X, Y, T) = H(w), \quad (143)$$

where $w = aX + bY + cT$. Here, $a,\ b$ and c are arbitrary constants.

Putting (143) into (140), one obtains

$$a^3 b H^{(4)} + 6a^2 b H' H'' - c(c + b) H'' = 0 \quad (144)$$

which generates

$$H(w) = \delta_{25} + \delta_{26} w,$$

$$H(w) = \delta_{27} + \frac{c(c + b)}{6a^2 b} w$$

and

$$H(w) = \frac{2a}{w} + \delta_{28} + \frac{c(c + b)}{6a^2 b} w, \quad (145)$$

where δ_{25}, δ_{26}, δ_{27} and δ_{28} are arbitrary constants.

Finally, we derive exact solutions of the gBKP-B equation (4) as

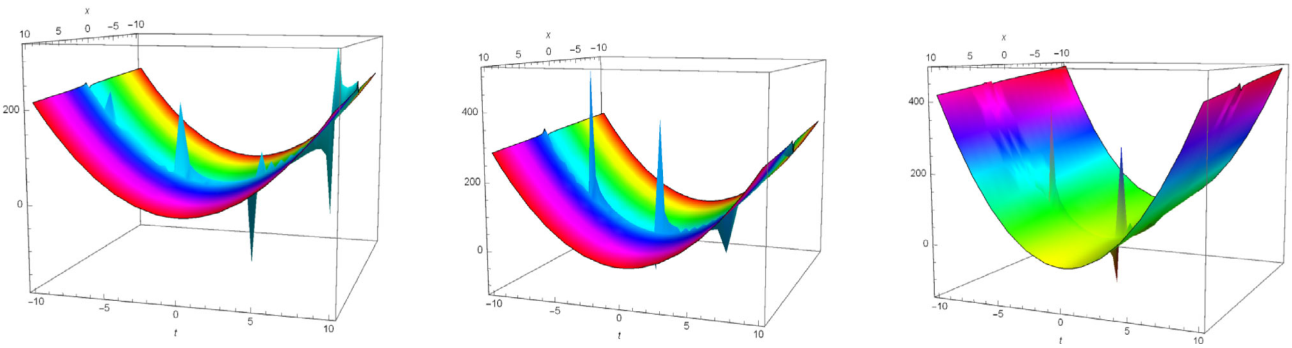
$$v(x, y, z, t) = \delta_{25} - (x + y) f_4(z) + \left[ct + ax + by - (a + b)\int f_4(z)\mathrm{d}z\right]\delta_{26} + \frac{3}{2}t^2 f_4'(z), \quad (146)$$

$$v(x, y, z, t) = \delta_{27} - (x + y) f_4(z) + \left[ct + ax + by - (a + b)\int f_4(z)\mathrm{d}z\right]\frac{c(b + c)}{6a^2 b} + \frac{3}{2}t^2 f_4'(z), \quad (147)$$

$$v(x, y, z, t) = \delta_{28} - (x + y) f_4(z) + \frac{2a}{\left[ct + ax + by - (a + b)\int f_4(z)\mathrm{d}z\right]} + \frac{3}{2}t^2 f_4'(z) + \left[ct + ax + by - (a + b)\int f_4(z)\mathrm{d}z\right]\frac{c(b + c)}{6a^2 b}. \quad (148)$$

4.10 *Subalgebra* $\mathfrak{T}_{10} := \mathbf{V}_2 + \mathbf{V}_4 + \mathbf{V}_5$

For simplification, we assume $f_2(z) = a_0 f_3'(z)$. As we have demonstrated already, we can find the exact solutions for $\mathfrak{T}_{10}$.



$z = 0.75$ $z = 1$ $z = 1.5$

Figure 7. Three distinct complex structures of elastic interactions between curve-shaped lumps and parabolic solitons for solution (148) with parameters $\delta_{28} = 10,\ a = 11.5,\ b = 2,\ c = 0.72,\ y = 3$ and $f_4(z) = 1 + z^2$.

The solutions are as follows:

$$v(x, y, z, t) = c_4 + 2c_2 \tanh\left(c_3 + c_2\left(x - \int f_3(z)\,\mathrm{d}z\right) - c_1\left(t - \frac{c_1 y}{c_1 - 4c_2^3}\right)\right) + (-y + ta_0)\, f_3(z), \tag{149}$$

$$v(x, y, z, t) = \delta_{29} + \frac{c(b + c)}{6a^2 b} \times \left(ax + by + ct - a\int f_3(z)\,\mathrm{d}z\right) - (y - ta_0)\, f_3(z), \tag{150}$$

$$v(x, y, z, t) = \delta_{30} + \frac{2a}{\left(ax + by + ct - a\int f_3(z)\,dz\right)} + \frac{c(b + c)}{6a^2 b}\left(ax + by + ct - a\int f_3(z)\,\mathrm{d}z\right) - (y - ta_0)\, f_3(z), \tag{151}$$

$$v(x, y, z, t) = \frac{a_5}{a_2}\frac{t^2}{2} + y^{\frac{1}{4}}\left[\delta_{31} + \sqrt{\frac{2a_5}{3a_2}}\, y^{\frac{1}{4}}\left(-x + \int f_3(z)\mathrm{d}z\right)\right] - (y - ta_0)\, f_3(z), \tag{152}$$

where $c_1, c_2, c_3, c_4, \delta_{29}, \delta_{30}$ and δ_{31} are constants.

5. Physical interpretation of soliton solutions

The nature of mathematical expressions can be made more predictable through their physical analysis. Graphical representation of the explicit solutions are much beneficial to explain the physically meaningful behaviour of the system. Also, it provides vital information to understand the phenomena physically. Numerical simulations have been performed to obtain the best view of graphical representations. Solitons are solitary wave packets and are known for their elastic scattering property that they do not change their shapes and amplitudes after mutual collision. Moreover, they play a prevalent role in the propagation of light in fibre, optical bistability and many other phenomena in plasma and fluid dynamics. In this section, we have analysed solutions (43), (57), (71), (106), (116), (127) and (148) of the g-BKPB equation (4) using their graphical structures. The graphical representations of the generated solutions describe the characteristics of multiple solitons. Various types of solitary wave solutions such as multiwave solitons, parabolic waves, quasiperiodic solitons and lump waves solitons have been exhibited. The choice of arbitrary constants and arbitrary functions contributes to the physically meaningful profiles.

Figure 1 describes curve-shaped elastic multisolitons observed for solution (43). This graphical representation is obtained by taking suitable values to the arbitrary constants as $A_2 = 1.7$, $A_3 = 31$, $\delta_4 = 15.7$ and $y = 0.7$ for $-20 \le x \le 20$ and $-10 \le t \le 10$.

Figure 2 depicts lump-type solitons for expression (57). The appropriate values of the introduced arbitrary constants are taken as $a_0 = 4$, $B_2 = 1.7$, $B_3 = 31$, $\delta_6 = 15.7$, $y = 11$ and arbitrary function as $f_1(z) = a_0 z$, for $-10 \le x \le 10$, $1 \le t \le 5$. The study of lump waves has a widespread application in many fields such as oceanographic engineering, non-linear optics, etc.

Figure 3 represents the elastic behaviour of curved-shaped multisoliton structure/characteristic of solution (71). The profile is plotted by considering suitable values of parameters as $y = 0.11$, for $-10 \le x \le 10$, $-10 \le t \le 10$, $A = 0.16$, $B_2 = 1.37$, $B_3 = 1.2$ and $\delta_8 = 5$.

Figure 4 reveals wave profile of the lump-type solitons in three-dimensional space of solution (106) that was observed via numerical simulation for $-10 \leq x \leq 10$, $-10 \leq y \leq 10$, $a = 191$, $b = 3$, $c = 10.3$, $\delta_{17} = 10$, $z = 0.019$ and $f_1(z) = z$.

Figure 5 is plotted by choosing suitable arbitrary constants as $a = 10$, $b = 12$, $c = 2$, $z = 1.3$, $\delta_{19} = 10$ and function as $f_2(z) = z$ in the particular solution (116). These three figures are traced at $t = 1$, 19 and 35 for $-50 \leq x \leq 50$, $-50 \leq y \leq 50$. These figures show lump-type soliton behaviour in the spatial profile. Lump-type solitons are localised in almost all directions in space.

Figure 6 shows elastic interactions between curved-shaped multisolitons and lumps are exhibited in this figure for the solution via eq. (127). The profile shows interaction between multisolitons and lumps by taking suitable values of arbitrary functional as $f_3(z) = z$ and remaining parameters as $a = 19$, $b = 5$, $c = 7$, $y = 1$, $\delta_{23} = 11$ and $a_0 = 1$. This profile is traced at $t = 0.07$, 5 and 37 for $-10 \leq x \leq 10$, $-10 \leq z \leq 10$.

Figure 7 represents the annihilation of parabolic curved-shaped profile of eq. (148) in 3D graphics. Interesting intersections of both lump-type solitons and parabolic solitons are observed for v in this figure at $z = 0.75$, $z = 1$ and $z = 1.5\ \forall -10 \leq x \leq 10$, $-10 \leq t \leq 10$. This profile is traced by taking the values of constants as $\delta_{28} = 10, a = 11.5, b = 2, c = 0.72$, $y = 3$ and arbitrary function as $f_4(z) = 1 + z^2$.

6. Conclusion

In summary, we have investigated the generalised BKP–Boussinesq (gBKP-B) equation using the symmetry reduction method, which is a robust, productive, impressive and strong mathematical tool for solving nonlinear PDEs. Lie point symmetries of the gBKP-B equation were considered and then used to derive a one-dimensional optimal system of symmetry subalgebras. Subsequently, three stages of symmetry reductions of the governing equation were carried out using the obtained symmetry subalgebras. The gBKP-B equation was transformed into various nonlinear ODEs which were then solved to attain the exact closed-form solutions of the equation. The solutions obtained have rich localised physical structures as there are five arbitrary independent functions and two parameters that are involved in the infinitesimal generators. The graphical analysis of the newly established solutions has been done by using MATHEMATICA codes via numerical simulation. The different dynamical features and characteristics of multiple solitons of the considered equation are especially analysed based on the suitable selection of arbitrary parameters and arbitrary independent functions. Some of the newly established solutions are more important and useful to explain various nonlinear complex physical phenomena, which makes this work more physically meaningful.

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References

[1] Sirendaoreji and J Sun, *Phys. Lett. A* **309**(**5–6**), 387 (2003)
[2] V B Matveev and M A Salle, *Darboux transformations and solitons* (Springer, Berlin, 1991)
[3] W Hong and Y D Jung, *Phys. Lett. A* **257**(**3–4**), 149 (1999)
[4] S Kumar and D Kumar, *Comput. Math. Appl.* **77**(**8**), 2096 (2019)
[5] S Kumar, A Kumar and W X Ma, *Chin. J. Phys*. **69**, 1 (2021)
[6] M Niwas, S Kumar and H Kharbanda, *J. Ocean Eng. Sci*., https://doi.org/10.1016/j.joes.2021.08.002 (2021)
[7] S Kumar, D Kumar and H Kharbanda, *Pramana – J. Phys.* **95**, 33 (2021)
[8] S Kumar, A Kumar and H Kharbanda, *Phys. Scr*. **95**, 065207 (2020)
[9] S Kumar and S Rani, *Pramana – J. Phys*. **95**, 51 (2021)
[10] I Aslan, *Comput. Math. Appl.* **61**(**6**), 1700 (2011)
[11] T Ozis and I Aslan, *Phys. Lett. A* **372**(**47**), 7011 (2008)
[12] W Hereman, P P Banerjee, A Korpel, G Assanto, A van Immerzeele and A Meerpoel, *J. Phys. A* **19**(**5**), 607 (1986)
[13] M J Ablowitz and P A Clarkson, *Solitons, nonlinear evolution equations and inverse scattering* (Cambridge University Press, Cambridge, 1991)
[14] M A Abdou, *Chaos Solitons Fractals* **31**(**1**), 95 (2007)
[15] P G Estevez, *J. Math. Phys.* **40**(**3**), 1406 (1999)
[16] R Hirota, *The direct method in soliton theory* (Cambridge University Press, New York, 2004)
[17] N A Kudryashov, *Chaos Solitons Fractals* **24**(**5**), 1217 (2005)
[18] N A Kudryashov and N B Loguinova, *Appl. Math. Comput*. **205**, 396 (2008)
[19] L. Zhang and C M Khalique, *Disc. Cont. Dynam. Syst. Ser. S* **11**(**4**), 759 (2018)
[20] M Wang, X Li and J Zhang, *Phys. Lett. A* **372**(**4**), 417 (2007)
[21] S Kumar, A Kumar and H Kharbanda, *Braz. J. Phys*. **51**, 1043 (2021)
[22] S Kumar and D Kumar, *Pramana – J. Phys.* **95**, 152 (2021)
[23] B B Kadomtsev and V I Petviashvili, *Sov. Phys. Dokl.* **15**, 539 (1970)

[24] A M Wazwaz, *Commun. Nonlinear Sci. Numer. Simul.* **17**(**2**), 491 (2012)
[25] A M Wazwaz and S A El-Tantawy, *Nonlinear Dyn*. **84**(**2**), 1107 (2016)
[26] W X Ma and E Fan, *Comput. Math. Appl*. **61**(**4**), 950 (2011)
[27] W X Ma and Z Zhu, *Appl. Math. Comput*. **218**(**24**), 11871 (2012)
[28] Y S Deng, B Tian, Y Sun, C R Zhang and C Hu, *Mod. Phys. Lett. B* **33**(**25**), 1950296 (2019)
[29] A M Wazwaz and S A El-Tantawy, *Nonlinear Dyn.* **88**(**4**), 3017 (2017)
[30] B Gao and Y Zhang, *Symmetry* **12**(**1**), 97 (2020)
[31] C M Khalique and L D Moleleki, *Results Phys.* **13**, 102239 (2019)
[32] G W Bluman and S Kumei, *Symmetries and differential equations* (Springer, Berlin, 1989)
[33] P Olver, *Applications of Lie groups to differential equations* (Springer, New York, 1993)
[34] L V Ovsiannikov, *Groups analysis of differential equation* (Academic Press, New York, 1982)
[35] B J Cantwell, *Introduction to symmetry analysis* (Cambridge University Press, Cambridge, 2002)
[36] S Kumar and A Kumar, *Nonlinear Dyn*. **98**(**3**), 1891 (2019)
[37] S Kumar and S Rani, *Pramana – J. Phys.* **94**, 116 (2020)
[38] S Kumar, M Niwas and A M Wazwaz, *Phys. Scr.* **95**(**9**), 095204 (2020)
[39] M Kumar and K Manju, *Int. J. Geom. Meth. Mod. Phys.* **18**(**2**), 2150028 (2021)
[40] M Kumar and K Manju, *Eur. Phys. J. Plus* **135**, 803 (2020)
[41] S Kumar, D Kumar and A M Wazwaz, *Eur. Phys. J. Plus* **136**, 531 (2021)
[42] X Hu, Y Li and Y Chen, *J. Math. Phys*. **56**(**5**), 053504 (2015)
[43] J-G Liu and W-P Xiong, *Results Phys*. **19**, 103532 (2020)
[44] J-G Liu and Q Ye, *Anal. Math. Phys.* **10**, 54 (2020)
[45] W-H Zhu and J-G Liu, *J. Math. Anal. Appl*. **502**(**1**), 125198 (2021)
[46] J-G Liu and W-H Zhu, *Nonlinear Dyn*. **103**, 1841 (2021)
[47] J-G Liu, W-H Zhu, M S Osman and W-X Ma, *Eur. Phys. J. Plus* **135**, 412 (2020)
[48] J-G Liu, W-H Zhu and Y He, *Z. Angew. Math. Phys*. **72**, 154 (2021)
[49] Y Tian and J-G Liu, *Nonlinear Dyn.* **104**, 1507 (2021)