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Modified multiple scale technique for the stability of the fractional delayed nonlinear oscillator

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Abstract. In the present proposal, the familiar method of the parameter expansion is combined with the multiple scales to study the stability behaviour of the Riemann–Liouville fractional derivative applied to the cubic delayed Duffing oscillator. The analysis of the modified multiple scale perturbation leads to a system of nonlinear differential-algebraic equations governing the solvability conditions. The nonlinear differential equation was reduced to the linear differential equation with the help of the algebraic one. The stability attitude of the periodic motion is determined by the steady-state analysis. Such a periodic motion is needed to better understand the dynamics of the fractional cubic delayed Duffing oscillator.

Keywords. Stability analysis; Duffing oscillator; delay differential equation; fraction derivative; homotopy perturbation; frequency expansion; multiple scales method.

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1. Introduction

Fractional calculus was introduced in Newton's time, and it has become a very hot topic in various fields, especially in mathematics and engineering, where classic mechanics was of no use to describe any phenomenon on the porous size scale. The term 'fractional calculus' refers to integration and differentiation to an arbitrary order. A complex analytic version of the fractional differentiation/integration has been discussed by Srivastava and Owa [1]. It is well known that the fractional derivatives/integrals have been defined in a variety of ways as [2,3] given by Riemann, Liouville, Grüunwald, Weyl and others. There are too many definitions on fractional derivative and new ones arise every day [4–7]. Among all the fractional derivatives, He's fractional derivative [8,9] and the local fractional derivative [10] are mathematically correct, has physical foundation and practically relevant. Kolwankar and Gangal [11,12] have proposed local fractional derivative operators through the renormalisation of Riemann–Liouville definition. Local fractional calculus has been implemented for solving non-differentiable equations in various physical events [13–16].

The effectiveness of the fractional-order derivative on the conduct of the nonlinear dynamical system is very motivating and it is handled by numerous researchers. Fractional derivatives appear in different field such as biology, fluid mechanics and viscoelasticity [17–20]. The homotopy analysis approach was successfully applied to determine the approximate solution of the Van der Pol equation with the fractional order. The fractional derivative is related to the Caputo sense [21]. The vibrations of the Duffing oscillator having quadratic and cubic nonlinearities have been discussed by Dal [22]. The motion described by the equation contains the fractional-order term. The analysis depends on the multiple time-scales method. The calculation of the fractional-order derivative has employed different analytical and numerical techniques including averaging method [23,24]. Arikoglu and Ozkol used the differential-transform technique for obtaining the solution of fractional differential equations [25].

The celebrated Duffing equation is a nonlinear differential equation, with its applications in numerous physical, engineering and biological problems [26–29]. The subharmonic resonance of the fractional Duffing oscillator was discussed by applying the averaging method [30]. The solution and the amplitude–frequency equation were obtained approximately. The existing condition of subharmonic resonance in the approximate solution was derived, and its stability condition was obtained.

Differential equations containing delayed terms have many applications such as in manufacturing processes, biology, chemical kinetics, economics, control systems, and other areas. At present, some researches had been done in fractional-order and time-delay systems. For example, Deng *et al* [31] calculated the stability of n-dimensional time-delayed systems for the linear fractional differential equation. Shi and Wang [32] gave the stability condition of a delayed system with the fractional order. Babakhani *et al* [33] investigated the presence of solutions near the equilibrium in fractional-order delayed differential equations and the Hopf bifurcations. Wahi and Chatterjee [34] used the method of averaging for conservative oscillators which may be strongly nonlinear, under small perturbations including delayed and fractional derivative terms.

Due to the rapid development of nonlinear science, different methods were used to solve nonlinear problems. Perturbation techniques are well instituted and utilised for over a century to find approximate analytical solutions for mathematical models. Differential equations, difference equations, integrodifferential equations and algebraic equations and integrals can be solved approximately using these perturbation techniques [35–39]. Recently, El-Dib [40] proposed a new perturbation method to handle strongly nonlinear systems. The method combines multiple scales and the homotopy perturbation method. The new method is applied to free vibrations of a linear damped oscillator, undamped, and damped Duffing oscillator. This new method can effectively solve numerous harmonic forced non-linear vibrations [40–44]. Ren *et al* [45] utilised some effective modifications of this method to improve it, making the method accessible to loose classes of nonlinear problems.

Due to the complexities of fractional calculus, most of the fractional-order differential equations do not have the exact solutions, and as an alternative, the approximate methods are extensively used for solving these types of equations. Some of the recent methods for approximate solutions of fractional-order differential equations are the Adomian decomposition method, the homotopy perturbation method, the variational iteration method, homotopy analysis method, etc. In this paper, the parameter expansion approach is adsorbed in the multiple scales method to make the results more accurate. One of the most significant features of the multiple time-scale methods is to yield the amplitude of the wave solution as a function of time, unlike the frequency expansion method where the amplitude is assumed to be constant in time. Due to this combination of the two methods, the amplitude is controlled by a differential equation associated with the algebraic frequency equation. The procedure will be applied to the fractional nonlinear delayed Duffing oscillator.

2. The mathematical problem

The nonlinear oscillator has the following fractional-damping delayed effect:

$$\ddot{y}(t) + \omega_0^2 y(t) + Q y^3(t) = \eta D^\alpha y^3(t-\tau), \tag{1}$$

with the initial conditions: $y(0) = A,\ \dot{y}(0) = 0$, where the coefficients $\eta,\ Q$ and ω_0^2 are real constants.

One of the greatest frequently utilised tools in the theory of fractional calculus is prepared by the Riemann–Liouville operators [2]. The Riemann–Liouville fractional derivative can provide the physical interpretation of the initial conditions needed for the initial value problems involving fractional differential equations. Moreover, this operator has the advantages of quick convergence, better stability and best accuracy.

The Riemann–Liouville time-fractional derivative of the function f of order $0 \le \alpha < 1$ [2] is defined by

$$D_a^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t \frac{f(\gamma)}{(t-\gamma)^\alpha} \mathrm{d}\gamma;\ \ 0 < \alpha < 1. \tag{2}$$

Observe the fact that the Riemann–Liouville fractional derivative of $f(t) = \mathrm{e}^{ist}$, $t \in \mathbb{R}$, $a \to -\infty$ [2,46] is

$$I_{-\infty}^\alpha \mathrm{e}^{ist} = (is)^{-\alpha} \mathrm{e}^{ist}$$

and

$$D_{-\infty}^\alpha \mathrm{e}^{ist} = (is)^\alpha \mathrm{e}^{ist}, \quad \alpha \in \mathbb{R}.$$

2.1 *The modified multiple scale method*

In the proposal adopted here, the modification is made by adsorbing the parameter expansion technique with the multiple-scale technology. This modification will improve the analysis to solve the given problem. According to the homotopy method, the following equation can be established:

$$\ddot{y}(t;\rho) + \omega_0^2 y(t;\rho) + \rho\big(Q y^3(t;\rho) - \eta D^\alpha y^3(t-\tau;\rho)\big) = 0;\quad \rho \in [0,\ 1]. \tag{3}$$

Consider the fast and the slow time scales which are defined as

$$T_0 = t,\ \ T_1 = \rho t,\ \ T_2 = \rho^2 t, \ldots, T_n = \rho^n t.$$

The time derivatives and fractional derivative are expanded as the multiple scale method [35,40]

$$\frac{\mathrm{d}}{\mathrm{d}t} = D_0 + \rho D_1 + \rho^2 D_2 + \cdots \tag{4}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = D_0^2 + 2\rho D_0 D_1 + \rho^2\big(D_1^2 + 2D_0 D_2\big) + \cdots$$

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} = \big(D_0 + \rho D_1 + \rho^2 D_2 + \cdots\big)^\alpha. \tag{5}$$

Applying Taylor expansion, we have

$$D^\alpha = D_0^\alpha + \rho\alpha D_0^{\alpha-1} D_1 + \tfrac{1}{2}\alpha\rho^2((\alpha-1)D_0^{\alpha-2}D_1^2 + 2D_2 D_0^{\alpha-1}) + \cdots, \tag{6}$$

where $D_n = \partial/\partial T_n$.

The expansion of the dependent variable $y(t;\rho)$ and the variable $y(t-\tau;\rho)$ are

$$y(t;\rho) = \sum_n \rho^n y_n(T_0, T_1, T_2) \tag{7}$$

and

$$y(t-\tau;\rho) = \sum_n \rho^n Y_n(T_0-\tau, T_1-\rho\tau, T_2-\rho^2\tau). \tag{8}$$

Applying Taylor expansion, we obtain

$$Y_n(T_0-\tau, T_1-\rho\tau, T_2-\rho^2\tau) = \big[1 - \rho\tau D_1 + \tfrac{1}{2}\rho^2\tau\big(\tau D_1^2 - 2D_2\big) + \cdots\big] \times Y_n(T_0-\tau, T_1, T_2). \tag{9}$$

Substituting (4)–(9) into (3), and using the frequency expansion

$$\omega^2 = \omega_0^2 + \rho\omega_1 + \rho^2\omega_2 + \cdots, \tag{10}$$

we get the equations in the zero order, the first order and the second order in the form

$$D_0^2 y_0 + \omega^2 y_0 = 0; \qquad y_0(0, T_1, T_2) = A(T_1, T_2),$$
$$D_0 y_0(0, T_1, T_2) = 0, \tag{11}$$

$$D_0^2 y_1 + \omega^2 y_1 = \omega_1 y_0 - 2D_1 D_0 y_0 + \eta D_0^\alpha Y_0^3 - Q y_0^3;$$
$$y_1(0, T_1, T_2) = 0,$$
$$D_0 y_1(0, T_1, T_2) + D_1 y_0(0, T_1, T_2) = 0, \tag{12}$$

$$D_0^2 y_2 + \omega^2 y_2 = \omega_2 y_0 + \omega_1 y_1 - 2D_1 D_0 y_1 - \big(D_1^2 + 2D_0 D_2\big) y_0 - 3Q y_0^2 y_1 + 3\eta D_0^\alpha Y_0^2 Y_1 + \eta(\alpha D_0^{\alpha-1} - \tau D_0^\alpha) D_1 Y_0^3;$$
$$y_2(0, T_1, T_2) = 0,$$
$$D_0 y_2(0, T_1, T_2) + D_1 y_1(0, T_1, T_2) + D_2 y_0(0, T_1, T_2) = 0, \tag{13}$$

The solution of eq. (11) is

$$y_0(T_0, T_1, T_2) = A(T_1, T_2)\cos\omega T_0. \tag{14}$$

Accordingly, we have

$$Y_0(T_0-\tau, T_1, T_2) = A(T_1, T_2)\cos\omega(T_0-\tau). \tag{15}$$

Inserting (14) and (15) into (12), removing the secular terms of the non-resonance case, yields

$$D_1 A + a_1(\omega) A^3 = 0, \tag{16}$$

$$\omega_1 = b_1(\omega) A^2. \tag{17}$$

The two solvability conditions (16) and (17) represent a differential-algebraic system having two unknowns, the amplitude function $A(T_1, T_2)$ and the correction in the frequency ω_1, where the coefficients a_1 and b_1 are

$$a_1(\omega) = \tfrac{3}{8}\eta\omega^{\alpha-1}\sin\big(\omega\tau - \tfrac{1}{2}\pi\alpha\big), \tag{18}$$

$$b_1(\omega) = \tfrac{3}{4}Q - \tfrac{3}{4}\eta\omega^\alpha\cos\big(\omega\tau - \tfrac{1}{2}\pi\alpha\big). \tag{19}$$

The solution of eq. (12) is derived as

$$y_1(T_0, T_1, T_2) = -\frac{1}{\omega}D_1 A\sin\omega T_0 - \frac{Q}{16\omega^2}A^3\sin 2\omega T_0\sin\omega T_0 + \frac{3^\alpha}{16}A^3\eta\omega^{\alpha-2}\big[\sin\big(2\omega T_0 - (3\omega\tau - \tfrac{1}{2}\pi\alpha)\big) + \sin(3\omega\tau - \tfrac{1}{2}\pi\alpha)\big]\sin\omega T_0. \tag{20}$$

Consequently, the function $y_1(T_0-\tau, T_1, T_2)$ is given by

$$Y_1(T_0-\tau, T_1, T_2) = -\frac{1}{\omega}D_1 A\sin\omega(T_0-\tau) - \frac{Q}{16\omega^2}A^3\sin 2\omega(T_0-\tau)\sin\omega(T_0-\tau) + \frac{3^\alpha}{16}A^3\eta\omega^{\alpha-2}\big[\sin\big(2\omega T_0 - (5\omega\tau - \tfrac{1}{2}\pi\alpha)\big) + \sin(3\omega\tau - \tfrac{1}{2}\pi\alpha)\big]\sin\omega(T_0-\tau). \tag{21}$$

Substituting (14)–(17), (20) and (21) into (13) and eliminate the source of the secular terms, after some trigonometric simplification, we have

$$D_2 A + a_2(\omega) A^5 = 0, \tag{22}$$

$$\omega_2 = b_2(\omega) A^4, \tag{23}$$

where the coefficients a_2 and b_2 are given below:

$$a_2(\omega) = \frac{3}{128}\omega^{2\alpha-3}\eta^2\left\{9[-\omega\tau\cos(2\omega\tau - \pi\alpha) + \omega\tau - \alpha\sin(2\omega\tau - \pi\alpha)] - \frac{3^\alpha}{2}\sin(4\omega\tau - \pi\alpha)\right\} + \frac{3}{256}\omega^{\alpha-3}Q\eta\big[3^\alpha\sin(3\omega\tau - \tfrac{1}{2}\pi\alpha) + \sin(\omega\tau - \tfrac{1}{2}\pi\alpha)\big], \tag{24}$$

$$b_2(\omega) = \frac{3\eta^2\omega^{2\alpha-2}}{128}\big[3(6\alpha+1)(-1+\cos(2\omega\tau-\pi\alpha)) -18\omega\tau\sin(2\omega\tau-\pi\alpha) -\tfrac{1}{2}3^\alpha(3\cos(4\omega\tau-\pi\alpha)-\cos 2\omega\tau)\big] +\frac{3Q\eta\omega^{\alpha-2}}{128}\big(3^\alpha\cos(3\omega\tau-\tfrac{1}{2}\pi\alpha) +\cos(\omega\tau-\tfrac{1}{2}\pi\alpha)\big) - \frac{3Q^2}{128\omega^2}. \quad (25)$$

The uniform solution of eq. (13) is formulated as

$$y_2(T_0,T_1,T_2) = -\frac{1}{\omega}D_2 A\sin\omega T_0 - \frac{1}{512\omega^4}Q^2A^5(\sin 4\omega T_0+\sin 2\omega T_0)\sin\omega T_0 + \frac{1}{256}\omega^{\alpha-4}Q\eta A^5\sin^2\omega T_0$$
$$\times\begin{bmatrix} 3^{\alpha+1}\cos(\omega T_0+3\omega\tau-\frac{1}{2}\pi\alpha)+2\times 3^\alpha\cos(\omega T_0-3\omega\tau+\frac{1}{2}\pi\alpha)+2\times 5^\alpha\cos(\omega T_0-5\omega\tau+\frac{1}{2}\pi\alpha) \\ +3^\alpha(3\omega T_0-3\omega\tau+\frac{1}{2}\pi\alpha)+5^\alpha\cos(3\omega T_0-5\omega\tau+\frac{1}{2}\pi\alpha)+30\cos(\omega T_0+\omega\tau-\frac{1}{2}\pi\alpha)-33(\omega T_0-\omega\tau+\frac{1}{2}\pi\alpha)\end{bmatrix}$$
$$+\frac{3^\alpha}{128}\omega^{2\alpha-4}\eta^2A^5\sin^2\omega T_0\begin{Bmatrix} -5^\alpha\cos(\omega T_0-8\omega\tau+\pi\alpha)+\frac{1}{2}\times 5^\alpha\cos(3\omega T_0-8\omega\tau+\pi\alpha) \\ +3(\alpha-5)\cos(\omega T_0-2\omega\tau)+\left(\frac{33}{2}-3\alpha\right)\cos(\omega T_0-4\omega\tau+\pi\alpha) \\ -9\omega\tau\sin(\omega T_0-4\omega\tau+\pi\alpha)+9\omega\tau\sin(\omega T_0-2\omega\tau)-\frac{1}{2}\times 3^{\alpha+1}\cos\omega T_0\end{Bmatrix}. \quad (26)$$

If the two iterations are enough, we substitute (14), (20) and (26) into (7) and setting $\rho\to 1$, yields

$$y(t) = A\cos\omega t - \frac{1}{\omega}\frac{\mathrm{d}A}{\mathrm{d}t}\sin\omega t - \frac{Q}{16\omega^2}A^3\sin 2\omega T_0\sin\omega T_0 + \frac{3^\alpha}{16}A^3\eta\omega^{\alpha-2}\big[\sin\big(2\omega T_0-(3\omega\tau-\tfrac{1}{2}\pi\alpha)\big)$$
$$+\sin(3\omega\tau-\tfrac{1}{2}\pi\alpha)\big]\sin\omega T_0 - \frac{1}{512\omega^4}Q^2A^5(\sin 4\omega T_0+\sin 2\omega T_0)\sin\omega T_0 + \frac{1}{256}\omega^{\alpha-4}Q\eta A^5\sin^2\omega T_0$$
$$\times\begin{bmatrix} 3^{\alpha+1}\cos(\omega T_0+3\omega\tau-\frac{1}{2}\pi\alpha)+2\times 3^\alpha\cos(\omega T_0-3\omega\tau+\frac{1}{2}\pi\alpha)+2\times 5^\alpha\cos(\omega T_0-5\omega\tau+\frac{1}{2}\pi\alpha) \\ +3^\alpha(3\omega T_0-3\omega\tau+\frac{1}{2}\pi\alpha)+5^\alpha\cos(3\omega T_0-5\omega\tau+\frac{1}{2}\pi\alpha)+30\cos(\omega T_0+\omega\tau-\frac{1}{2}\pi\alpha)-33(\omega T_0-\omega\tau+\frac{1}{2}\pi\alpha)\end{bmatrix}$$
$$+\frac{3^\alpha}{128}\omega^{2\alpha-4}\eta^2A^5\sin^2\omega T_0$$
$$\times\begin{Bmatrix} -5^\alpha\cos(\omega T_0-8\omega\tau+\pi\alpha)+\frac{1}{2}\times 5^\alpha\cos(3\omega T_0-8\omega\tau+\pi\alpha)+3(\alpha-5)\cos(\omega T_0-2\omega\tau) \\ +\left(\frac{33}{2}-3\alpha\right)\cos(\omega T_0-4\omega\tau+\pi\alpha)-9\omega\tau\sin(\omega T_0-4\omega\tau+\pi\alpha)+9\omega\tau\sin(\omega T_0-2\omega\tau) \\ -\frac{1}{2}\times 3^{\alpha+1}\cos\omega T_0\end{Bmatrix}. \quad (27)$$

In formulating the above approximate solution, the amplification of the derivative $(\rho D_1+\rho^2 D_2)A(T_1,T_2)$ becomes $\mathrm{d}A/\mathrm{d}t$.

2.2 *Construction of the amplitude and the frequency equations*

To construct the amplitude equation, the solvability conditions (16) and (22), that are obtained in the first-order and the second-order perturbations, need to be combined into a one-amplitude equation. These equations enable us to set the unknown function A in terms of the slow time T_1 and the slower time T_2. Besides, the stability behaviour depends mainly on the solutions of these equations. For this purpose, integrate eq. (16) partially with respect to the variable T_1. Then integrate eq. (22) partially with respect to the variable T_2. In other words, simply multiply eqs (16) and (22) by ρ and ρ^2, respectively. It follows that the partial differentiation in these amplitude equations may be transformed into $\mathrm{d}A/\mathrm{d}t$, by letting $\rho\to 1$. Finally, one may establish the following amplitude equation:

$$\frac{\mathrm{d}A}{\mathrm{d}t} + a_1(\omega)A^3 + a_2(\omega)A^5 = 0. \quad (28)$$

The same procedure can be used to construct the frequency equation from the other solvability conditions (17) and (23) obtained from the first and the second-order perturbations. Inserting into expansion (10) produces

$$\omega^2 = \omega_0^2 + b_1A^2(t) + b_2A^4(t). \quad (29)$$

At this end, the expanded frequency ω^2 is composed of the amplitude function A which depends on the variable t and is given by the nonlinear first-order equation (28).

To relax this nonlinearity, one can remove A^5 from eq. (28) with the help of eq. (29) to yield

$$\frac{\mathrm{d}A}{\mathrm{d}t} + \frac{a_2}{b_2}(\omega^2 - \omega_0^2)A + \left(a_1 - a_2\frac{b_1}{b_2}\right)A^3 = 0. \tag{30}$$

Again, remove A^3 from (30) with the help of (29) yields the following non-homogeneous linear first-order differential equation in A^2:

$$\frac{\mathrm{d}A^2}{\mathrm{d}t} + 2P(\omega)A^2 = 2K(\omega), \tag{31}$$

where the coefficients $P(\omega)$ and $K(\omega)$ are presented as

$$P(\omega) = \frac{a_2}{b_2}(\omega^2 - \omega_0^2) - \frac{b_1}{b_2}\left(\frac{a_1b_2 - a_2b_1}{b_2}\right), \tag{32}$$

$$K(\omega) = -\left(\frac{a_1b_2 - a_2b_1}{b_2}\right)\frac{(\omega^2 - \omega_0^2)}{b_2}. \tag{33}$$

It is suitable to recall the unknown function $A^2(t)$ as $B(t)$ in eqs (31) and (29). Thus, we have

$$\frac{\mathrm{d}B}{\mathrm{d}t} + 2P(\omega)B = 2K(\omega), \tag{34}$$

$$\omega^2 = \omega_0^2 + b_1B(t) + b_2B^2(t). \tag{35}$$

The above equations represent a differential-algebraic system in two unknowns $B(t)$ and the frequency ω. Equation (34) is a linear first-order equation which has an integrating factor in the form of $I = \mathrm{e}^{2\int P(\omega)\mathrm{d}t}$. Therefore, it is an exact solution which has the form

$$B(t) = \mathrm{e}^{-2\int P(\omega)\mathrm{d}t}\int_0^t 2K(\omega)\mathrm{e}^{2\int P(\omega)\mathrm{d}\gamma}\,\mathrm{d}\gamma\,. \tag{36}$$

It is difficult to evaluate the above integration until the transcendental functions $P(\omega)$ and $K(\omega)$ are relaxed.

Assuming that the frequency ω is very close to ω_0 so that $\omega^2 - \omega_0^2 = \Delta$, then the above integration will lead to

$$B(t) = \frac{-\Delta(a_1b_2 - a_2b_1)}{a_2b_2\Delta - b_1(a_1b_2 - a_2b_1)}(1 - \mathrm{e}^{-2P(\omega)t}). \tag{37}$$

Clearly, the amplitude function $A(t)$ will be bound when $P > 0$, i.e.

$$b_2a_2\Delta - b_1(a_1b_2 - a_2a_1) > 0. \tag{38}$$

The small value Δ can be formulated by inserting (37) into (35), to yield

$$\begin{aligned}\Delta = &-\frac{1}{2a_2^2b_2^2}[a_2b_1b_2(a_2b_1 - a_1b_2)(1 + \mathrm{e}^{-2Pt}) \\ &-(a_1b_2 - a_2b_1)^2(1 - \mathrm{e}^{-2Pt})^2] \\ &\pm\frac{1}{2a_2^2b_2^2}\{[a_2b_1b_2(a_2b_1 - a_1b_2)(1 + \mathrm{e}^{-2Pt}) \\ &-(a_2b_1 - a_1b_2)^2(1 - \mathrm{e}^{-2Pt})^2]^2 \\ &-4a_2^2b_1^2b_2^2(a_2b_1 - a_1b_2)^2\mathrm{e}^{-2Pt}\}^{1/2}.\end{aligned} \tag{39}$$

This is a very complicated transcendental frequency equation which depends on the damping term e^{-2Pt}.

In other words, if the solution of the quadratic equation (35) is inserted into eq. (34) the result will not depend on time. This refers to the steady-state solution.

2.3 *Steady-state response*

The frequency–amplitude equation (35) is transcendental and very complicated. Furthermore, the integration (36) cannot be, in general, obtained analytically. We shall study the steady case, which is more serious and significant in vibration engineering. Letting $\mathrm{d}B/\mathrm{d}t = 0$ into eq. (34), yields the steady-state solution B_s in the form

$$B_s = K_s/P_s, \tag{40}$$

where the suffix s denotes the steady-state response, and K_s and P_s are given as

$$P_s = \frac{a_2}{b_2}(b_1 + b_2B_s)B_s - (a_1b_2 - a_2b_1)\frac{b_1}{b_2^2}, \tag{41}$$

$$K_s = -\frac{1}{b_2^2}(a_1b_2 - a_2b_1)(b_1 + b_2B_s)B_s, \tag{42}$$

where the following relation is used:

$$\omega_s^2 = \omega_0^2 + b_1B_s + b_2B_s^2. \tag{43}$$

The above relation is derived from (35) corresponding to the steady-state response. Inserting (41) and (42) into (40) gives

$$B_s = -\frac{a_1}{a_2}. \tag{44}$$

Substituting (44) into (41)–(43) yields

$$P_s(\omega_s) = \frac{1}{a_2b_2^2}(a_2b_1 - a_1b_2)^2, \tag{45}$$

$$K_s(\omega_s) = -\frac{a_1(a_2b_1 - a_1b_2)^2}{a_2^2b_2^2}, \tag{46}$$

$$\omega_s^2 = \omega_0^2 - \frac{a_1b_1}{a_2} + \frac{a_1^2b_2}{a_2^2}. \tag{47}$$

Now, return to the integration of (36) we have

$$B(t) = \frac{a_1(\omega_s)}{a_2(\omega_s)}(\mathrm{e}^{-2P(\omega_s)t} - 1). \tag{48}$$

The steady-state frequency ω_s is given by the transcendental equation (47). Therefore, it is worthwhile to obtain an approximate value for it. We shall apply the homotopy perturbation technique to evaluate an approximate value for ω_s. For this, we bring the homotopy parameter $\varepsilon \in [0, 1]$ such that the algebraic homotopy equation is established as

$$\omega_s^2 = \omega_0^2 + \varepsilon\left(-\frac{a_1 b_1}{a_2} + \frac{a_1^2 b_2}{a_2^2}\right). \tag{49}$$

Accordingly, the frequency ω_s can be perturbed as

$$\omega_s = \omega_0 + \varepsilon\omega_d, \tag{50}$$

where ω_d represents a small deviation from ω_0. Utilising (50), the homotopy equation (49) is presented in the form

$$\begin{aligned}(\omega_0 + \varepsilon\omega_d)^2 &= \omega_0^2 + \varepsilon\left(-b_1(\omega_0 + \varepsilon\omega_d)\frac{a_1(\omega_0 + \varepsilon\omega_d)}{a_2(\omega_0 + \varepsilon\omega_d)}\right.\\ &\quad \left. + b_2(\omega_0 + \varepsilon\omega_d)\frac{a_1^2(\omega_0 + \varepsilon\omega_d)}{a_2^2(\omega_0 + \varepsilon\omega_d)}\right).\end{aligned} \tag{51}$$

Applying Taylor expansion and linearising in ε, yields

$$\omega_d = \frac{1}{2\omega_0}\left(-b_1(\omega_0)\frac{a_1(\omega_0)}{a_2(\omega_0)} + b_2(\omega_0)\frac{a_1^2(\omega_0)}{a_2^2(\omega_0)}\right). \tag{52}$$

Substituting (52) into (50) and letting $\varepsilon \to 1$, we obtain

$$\omega_s = \omega_0 - \frac{1}{2\omega_0}\left(b_1(\omega_0)\frac{a_1(\omega_0)}{a_2(\omega_0)} - b_2(\omega_0)\frac{a_1^2(\omega_0)}{a_2^2(\omega_0)}\right). \tag{53}$$

The above approximate frequency is in the autonomous case. It is seen from (48) that the function $B(t)$ is a damping function when $P > 0$, which requires that $a_2 > 0$, i.e.

$$\begin{aligned}&\eta^2\omega_0\{18[-\omega_0\tau\cos(2\omega_0\tau - \pi\alpha) + \omega_0\tau\\ &\quad -\alpha\sin(2\omega_0\tau - \pi\alpha)] - 3^\alpha\sin(4\omega_0\tau - \pi\alpha)\}\\ &\quad + Q\eta\omega_0^{1-\alpha}\left[3^\alpha\sin(3\omega_0\tau - \tfrac{1}{2}\pi\alpha)\right.\\ &\quad \left. + \sin(\omega_0\tau - \tfrac{1}{2}\pi\alpha)\right] > 0.\end{aligned} \tag{54}$$

3. Conclusion

In this article, we discussed the combined concept of the multiple scales method with the frequency expansion approach. This technique is applied to solve the fractional of the nonlinearity delayed Duffing oscillator which is presented in the three time-scales domain. In this approach, three perturbation levels are presented which yield two groups of solvability conditions. Each solvability condition consists of a nonlinear differential equation of the amplitude associated with an algebraic form for the frequency correction. These two groups are converted into a single nonlinear differential equation in the amplitude function. Moreover, the collected frequency expansion, which depends on the powers of the amplitude function, is presented. This is a very complicated system. The nonlinearity of the amplitude equation was converted into a linear amplitude equation with the help of the equation of nonlinear frequency. The stability criteria have been discussed using the application of the steady-state response. The steady-state solution has employed the nonlinear frequency equation which is solved using a homotopy perturbation technique. The attractive property of the modified multiple scales method is that it is implemented directly in a straightforward manner for discussing very complicated problems.

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