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Asymptotic iteration method applied to new confining potentials

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Abstract. This work intends to evaluate the energy spectrum of a particle influenced by the new type of confined interactions introduced in our previous work [Assi and Sous, *Eur. Phys. J. Plus* **133(5)**, 175 (2018); Assi *et al, Mod. Phys. Lett.* **33(32)**, 1850128 (2018)]. We have used the asymptotic iteration method (AIM) to carry out numerical computations and our results agree to a high degree of accuracy with those obtained by other researchers using different methods as shown in the tables.

Keywords. Schrödinger equation; bound states; confined potentials; hyperbolic Pöschl–Teller potential; asymptotic iteration method.

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1. Introduction

Obtaining analytical and numerical solutions of the wave equations is a very important task in quantum mechanics to understand different physical systems at the atomic level [1–4]. Different analytical and numerical methods have been used in the past to obtain solutions to the associated eigenvalue problems such as the supersymmetry method (SUSY) [5,6], the exact quantisation method (EQM) [7–14], the tridiagonal representation approach (TRA) [15–17], the asymptotic iteration method (AIM) [18–23] and many other methods [24–27].

Very recently, Assi *et al* [17] introduced the following one-dimensional (1D) and three-dimensional (3D) potentials:

$$V(x) = \frac{B}{\cosh^2(\alpha x)} + C\cosh(2\alpha x), \tag{1.1}$$

$$V(r) = \frac{A}{\sinh^2(\alpha r)} + \frac{B}{\cosh^2(\alpha r)} + C\cosh(2\alpha r), \tag{1.2}$$

where $A,\ C > 0$, $B < 0$ for bound states and $\alpha > 0$ is just a scaling parameter. We shall solve the corresponding Schrödinger equation and obtain the energy eigenvalues for both potentials using the AIM.

The flow of this work is organised as follows. In §2 and §3, we apply the AIM to solve the wave equation for the 1D and the 3D potentials, respectively. In addition, §4 contains our results and discussions for both the problems. Finally, we conclude our work in §5.

2. The 1D potential well

The non-relativistic stationary wave equation (in units $\hbar = m = 1$) for the potential in (1.1) reads [28,29]

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{B}{\cosh^2(\alpha x)} + C\cosh(2\alpha x)\right]\psi(x) = E\psi(x), \tag{2.1}$$

where $x \in (-\infty, +\infty)$. Using the change of variable $y = 2\tanh^2(\alpha x) - 1$, eq. (2.1) becomes

$$\left\{-\alpha^2(1+y)(1-y)^2\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{\alpha^2}{2}(1-y)(1+3y)\frac{\mathrm{d}}{\mathrm{d}y} + \frac{B}{2}(1-y) + \frac{4C}{1-y}\right\}\psi(y) = (E+C)\psi(y), \tag{2.2}$$

where $y \in [-1, +1]$. The above equation can be written in the following AIM form:

$$\frac{\mathrm{d}^2\psi(y)}{\mathrm{d}y^2} = \lambda_0(y)\frac{\mathrm{d}\psi(y)}{\mathrm{d}y} + s_0(y)\psi(y), \tag{2.3}$$

where

$$\lambda_0(y) = \frac{1}{2}\frac{1+3y}{1-y^2}, \tag{2.4a}$$

$$s_0(y) = \frac{\tilde{B}}{1-y^2} + \frac{\tilde{C}}{(1-y^2)(1-y)^2} - \frac{\tilde{E}}{(1-y^2)(1-y)} \tag{2.4b}$$

and

$$\tilde{B} = \frac{B}{2\alpha^2}, \quad \tilde{C} = \frac{4C}{\alpha^2}, \quad \tilde{E} = \frac{E+C}{\alpha^2}, \tag{2.4c}$$

To improve the accuracy and fast convergence of the AIM, in particular, we would like to reduce the order of the regular singularity at $y = 1$, and we use the following transformation in (2.3):

$$\psi(y) = (1+y)^a (1-y)^b g(y), \tag{2.5}$$

where a and b are some non-negative parameters selected so as to improve the convergence of our AIM algorithm, and $g(y)$ is some function of y. Using (2.5) back in eq. (2.3), we get

$$\frac{\mathrm{d}^2 g(y)}{\mathrm{d}y^2} = \tilde{\lambda}_0(y)\frac{\mathrm{d}g(y)}{\mathrm{d}y} + \tilde{s}_0(y)g(y), \tag{2.6}$$

where

$$\tilde{\lambda}_0(y) = \lambda_0(y) - 2\left(\frac{a}{1+y} - \frac{b}{1-y}\right) \tag{2.7a}$$

and

$$\tilde{s}_0(y) = s_0(y) + \lambda_0(y)\left\{\frac{a}{1+y} - \frac{b}{1-y}\right\} - \frac{a(a-1)}{(1+y)^2} + \frac{b(b-1)}{(1-y)^2} - \frac{2ab}{1-y^2}. \tag{2.7b}$$

Next, we should find, iteratively, a set of functions $\{\tilde{\lambda}_n(y), \tilde{s}_n(y)\}_{n=1}$ using the following recursion relations [18]:

$$\tilde{\lambda}_n = \tilde{\lambda}'_{n-1} + \tilde{s}_{n-1} + \tilde{\lambda}_0\tilde{\lambda}_{n-1}, \quad \tilde{s}_n = \tilde{s}'_{n-1} + \tilde{s}_0\tilde{\lambda}_{n-1}, \tag{2.8}$$

where the primes stand for the derivatives with respect to y. This process is done using a suitable programming language. The next step in the computation will be finding the eigenvalues of the original problem by solving the following quantisation condition [18]:

$$\begin{vmatrix} \tilde{\lambda}_n & \tilde{s}_n \\ \tilde{\lambda}_{n-1} & \tilde{s}_{n-1} \end{vmatrix} = 0. \tag{2.9}$$

The evaluation of the eigenenergies using eq. (2.9) will force us to use a suitable seed value for y to obtain the actual energy eigenvalues, and we use $y = 0$ for our case. The wave function, however, is obtained using the following recursive relation [18]:

$$\psi(y) = A(1+y)^a(1-y)^b \exp\left(-\int_0^y \frac{\tilde{s}_n(t)}{\tilde{\lambda}_n(t)}\mathrm{d}t\right), \tag{2.10}$$

where A is the normalisation constant. The above solution must match with that which was obtained earlier [17]. Note that because our potential is symmetric around the origin, the physical solutions of the wave equation must have definite parity [28,29]. Thus, the physical solutions of eq. (2.1) are written in a compact form as follows:

$$\varphi_\pm(y) = \frac{1}{\sqrt{2}}[\psi(y) \pm \psi(-y)], \tag{2.11}$$

where the plus sign corresponds to the even states and the minus sign corresponds to the odd states. In the next section, we apply the AIM to the 3D problem.

3. Three-dimensional spherically symmetric confined potential

The radial stationary wave equation for the second potential in (1.2) reads as

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{A}{\sinh^2(\alpha r)} + \frac{B}{\cosh^2(\alpha r)} + C\cosh(2\alpha r) + \frac{\ell(\ell+1)}{2r^2}\right]\psi(r) = E\psi(r), \tag{3.1}$$

where ℓ is the angular momentum quantum number. To simplify the calculations, we use the following approximation for the centrifugal term suggested recently by Assi *et al* [17]:

$$\frac{1}{r^2} \approx \alpha^2\left[\frac{1}{\sinh^2(\alpha r)} + \frac{31}{945}\frac{1}{\cosh^2(\alpha r)} - \frac{16}{945}\cosh(2\alpha r) + \frac{20}{63}\right]. \tag{3.2}$$

The reader should refer to other approximations used before [30,31]. Substituting eq. (3.2) back in (3.1), we get [17]

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\tilde{A}}{\sinh^2(\alpha r)} + \frac{\tilde{B}}{\cosh^2(\alpha r)} + \tilde{C}\cosh(2\alpha r)\right]\psi(r) = \tilde{E}\psi(r), \tag{3.3}$$

where

$$\tilde{A} = A + \frac{\alpha^2\ell(\ell+1)}{2}, \tag{3.4a}$$

$$\tilde{B} = B + \frac{31}{945}\frac{\alpha^2\ell(\ell+1)}{2}, \tag{3.4b}$$

Table 1. Lowest energy eigenvalues of the 1D potential (1.1) for $B = -100$, $C = 5$ and $\alpha = 1/\sqrt{2}$ obtained using AIM in this work against the results obtained in [17].

n	Even states (this work)	Even states [17]	Odd states (this work)	Odd states [17]
0	−89.8612818110	−89.8612818109	−79.7931821718	−79.7931821718
1	−70.1359637166	−70.1359637166	−60.8725193053	−60.8725193053
2	−51.9825141858	−51.9825141858	−43.4420839706	−43.4420839706
3	−35.2237088862	−35.2237088863	−27.2963654106	−27.2963654123
4	−19.6260386829	−19.6260386805	−12.1766206821	−12.1766206450
5	−4.91112636640	−4.9111262992	2.20693532829	2.2069354803
6	9.21225692283	9.2122537047	16.1366869670	16.1366691263
7	23.0084559604	23.008508563	29.8520161410	29.8523145060
8	36.6893107656	36.6889029437	43.5378311374	43.5356440685
9	50.4064368096	50.4068617266	57.3125644558	57.3142707495

$$\tilde{C} = C - \frac{16}{945}\frac{\alpha^2\ell(\ell+1)}{2} \tag{3.4c}$$

and

$$\tilde{E} = E - \frac{20}{63}\frac{\alpha^2\ell(\ell+1)}{2}. \tag{3.4d}$$

Now, by making the change in the variable $y = 2\tanh^2(\alpha r) - 1$ back in eq. (3.3), we get

$$\frac{\mathrm{d}^2\psi(y)}{\mathrm{d}y^2} = \lambda_0(y)\frac{\mathrm{d}\psi(y)}{\mathrm{d}y} + s_0(y)\psi(y), \tag{3.5}$$

where

$$\lambda_0(y) = \frac{1}{2}\frac{1+3y}{1-y^2}, \tag{3.6a}$$

$$s_0(y) = \frac{\bar{A}}{(1-y^2)^2} + \frac{\bar{B}}{1-y^2} + \frac{\bar{C}}{(1-y^2)(1-y)} - \frac{\varepsilon}{(1-y^2)(1-y)} \tag{3.6b}$$

and

$$\bar{A} = \frac{2\tilde{A}}{\alpha^2}, \quad \bar{B} = \frac{\tilde{B}}{2\alpha^2},$$
$$\bar{C} = \frac{4\tilde{C}}{\alpha^2}, \quad \varepsilon = \frac{\tilde{E}+\tilde{C}+\tilde{A}}{\alpha^2}. \tag{3.7}$$

The next steps are mathematically similar to those followed in the previous section from eq. (2.5) up to eq. (2.10) but with the new form of $s_0(y)$ given in eq. (3.6b). So, we are not going to rewrite them again here. In the next section, we shall present our numerical results for the eigenenergies of both potentials which agree with what have been obtained in [17].

4. Results and discussion

4.1 *Lowest energy eigenvalues of the 1D potential*

Here, we have numerically calculated the lowest energy eigenvalues for two different cases with the same values of the potential parameters taken in ref. [17]. In the first case, we took $B = -100, C = 5$ and $\alpha = 1/\sqrt{2}$. Our results are tabulated in table 1 against the results obtained using different methods [17], where the stability of the results obtained using $a = b = 0$ for the even state's eigenvalues and the odd state's energies obtained using $a = b = 1/2$. Comparing columns two with three, and four with five, we notice that the results are very close to a few decimal points which verify the results obtained in [17] using the AIM. Similarly, we have also considered another case with $B = -10000$, $C = 5$ and $\alpha = 1/\sqrt{2}$ as given in tables 2 and 3. The results presented in table 2 using the two methods agree with high accuracy up to 12 decimal digits.

Table 2. Lowest 10 eigenvalues of the 1D eq. (2.1) for $B = -10000$, $C = 5$ and $\alpha = 1/\sqrt{2}$ corresponding to the even states obtained in this work vs. those found using different approaches in [17].

n	Even states (this work)	Even states [17]
0	−9945.09966135	−9945.09966135
1	−9746.49677011	−9746.49677011
2	−9549.89072670	−9549.8907267
3	−9355.28140035	−9355.28140035
4	−9162.66865346	−9162.66865346
5	−8972.05234112	−8972.05234112
6	−8783.43231070	−8783.4323107
7	−8596.80840134	−8596.80840134
8	−8412.18044336	−8412.18044336
9	−8229.54825775	−8229.54825775

Table 3. Lowest 10 eigenvalues of the 1D eq. (2.1) for $B = -10000$, $C = 5$ and $\alpha = 1/\sqrt{2}$ associated with the odd states obtained in this work using the AIM.

n	Odd states (using AIM)
0	−9845.54860183
1	−9647.94415044
2	−9452.33648233
3	−9258.72546335
4	−9067.11095232
5	−8877.49280053
6	−8689.87085128
7	−8504.24493939
8	−8320.61489058
9	−8138.98052093

Table 4. Lowest 10 energies of the 3D potential for the s-wave case obtained using the AIM against those obtained by the HDM for the choice of parameters $A = 10$, $B = -200$, $C = 10$ and $\alpha = 1/\sqrt{2}$.

n	E_n (AIM, this work)	E_n (HDM, [17])
0	−92.7542191059	−92.7542191071
1	−69.5274752236	−69.5274752304
2	−47.4599081738	−47.4599081966
3	−26.3521411333	−26.3521411922
4	−5.99208981885	−5.9920899479
5	13.8293065397	13.8293062858
6	33.3030740081	33.3030735452
7	52.5923168605	52.5923160320
8	71.8289102768	71.8289091921
9	91.1153421519	91.1153398126

4.2 *Energy spectrum of the 3D potential*

The computations obtained for the 3D potential are divided into two tables for the angular momentum values $\ell = 0$ and 5, respectively, with the potential parameters taken as $A = 10$, $B = -200$, $C = 10$ and $\alpha = 1/\sqrt{2}$. In table 4, the energies obtained using the AIM agree to a high degree of accuracy with those calculated using the Hamiltonian diagonalisation method (HDM) for the zero angular momentum quantum number. In addition, for $\ell = 5$, our results also agree with those obtained using the HDM in [17] as shown in table 5. Note that in both cases, and for numerical purposes, we have taken $a = b = 0$.

5. Conclusion

In this work, we have applied the AIM for two new confining potentials that were introduced in [17]. The numerical computation of the energy spectrum agrees to a high degree of accuracy with those obtained using the HDM in [17]. This reflects the power of the AIM for solving a wider class of eigenvalue problems. Those potentials are new, and we are not aware of any direct physical application that might be suitably modelled using these interactions.

Table 5. Comparison between the lowest energies obtained using the AIM and the HDM for $\ell = 5$ the state with the parameters $A = 10$, $B = -200$, $C = 10$ and $\alpha = 1/\sqrt{2}$.

n	E_n (AIM, this work)	E_n (HDM, [17])
0	−67.4183920149	−67.4183920149
1	−45.5096792463	−45.5096792463
2	−24.5678690265	−24.5678690265
3	−4.37734190490	−4.3773419048
4	15.2748216808	15.2748216809
5	34.5830153201	34.5830153204
6	53.7126959917	53.7126959924
7	72.7969867796	72.7969867879
8	91.9387273925	91.9387272103
9	111.215271559	111.2152735964

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