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A dynamical study of certain nonlinear diffusion–reaction equations with a nonlinear convective flux term

ANAND MALIK[1], HITENDER KUMAR[2,*], RISHI PAL CHAHAL[1] and FAKIR CHAND[3]

[1]Department of Physics, Chaudhary Bansi Lal University, Bhiwani 127 021, India
[2]Department of Physics, Pt. Neki Ram Sharma Government College, Rohtak 124 001, India
[3]Department of Physics, Kurukshetra University, Kurukshetra 136 119, India
*Corresponding author. E-mail: hkkhatri24@gmail.com

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Abstract. We explore the dynamics of quadratic and quartic nonlinear diffusion–reaction equations with nonlinear convective flux term, which arise in well-known physical and biological problems such as population dynamics of the species. Three integration techniques, namely the (G'/G)-expansion method, its generalised version and Kudryashov method, are adopted to solve these equations. We attain new travelling and solitary wave solutions in the form of Jacobi elliptic functions, hyperbolic functions, trigonometric functions and rational solutions with some constraint relations that naturally appear from the structure of these solutions. The travelling population fronts, which are the general solutions of nonlinear diffusion–reaction equations, describe the species invasion if higher population density corresponds to the species invasion. This effort highlights the significant features of the employed algebraic approaches and shows the diversity in the constructed solutions.

Keywords. Nonlinear diffusion–reaction equation; Jacobi elliptic functions; kink–antikink soliton.

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1. Introduction

The nonlinear partial differential equations (PDEs) are widely used to model several natural and fundamental problems in the fields of physics, chemistry, biology, neurology, ecology, plasma and mathematical sciences [1,2]. In the scientific investigation of a problem, the system in hand is generally modelled in the form of mathematical equations which is often nonlinear. The exact analytic solutions of such nonlinear mathematical equations can provide much physical information of the system concerned.

Therefore, in the recent times, substantial efforts have been made to acquire exact analytical solutions of such nonlinear PDEs and a large number of pronounced and efficient techniques have been developed for obtaining explicit travelling wave solutions [3–41].

In the last few years, a novel sophisticated technique called (G'/G)-expansion method [42] was presented for a reliable treatment of nonlinear wave equations. Subsequently, some other significant applications of this method have also been reported in the literature [43–52]. Very recently, a simplified and generalised version of the (G'/G)-expansion method and Kudryashov method is also reported with their applications for nonlinear PDEs [53–56].

In the contemporary work, we have considered the nonlinear diffusion–reaction (DR) equations, which have numerous applications in various branches of biological, chemical and physical systems. A large number of simplified versions of the nonlinear DR equations have been studied in ancient times. Triki *et al* [57] studied the three variants of nonlinear DR equations with derivative-type and algebraic-type nonlinearities with short-range and long-range diffusion terms in inhomogeneous media using the auxiliary equation method. Bhardwaj *et al* [58] obtained periodic, double-kink, bell and antikink-type solutions of the cubic–quintic nonlinear reaction–diffusion equation with variable convection coefficients. Recently, we have studied the DR equation for a particular case when the diffusion coefficient D was independent of concentration or density [59], but in several circumstances such as insect dispersal models and small rodents D becomes density-dependent [1,60]. Consequently, in such circumstances, growth in the concentration of a species will increase D. Hence, here, we explore the dynamics by solving certain types of nonlinear DR equations arising from such physical considerations. Here, we go beyond the previous study and specifically seek exact solutions of the following DR equations which comprise quadratic and quartic nonlinearities with a nonlinear convective flux term:

$$C_t + kCC_x = DC_{xx} + \alpha C - \beta C^2, \tag{1}$$

$$C_t + kC^2C_x = DC_{xx} + \alpha C - \beta C^4, \tag{2}$$

where $C = C(x,t)$ has diverse meanings depending on the types of phenomena under study, D is the diffusion coefficient, and k, α and β are physical constants which are to be determined. Equations (1) and (2) define transport phenomena, such as ion-exchange column chromatography and population of the species, in which both diffusion and convection processes have equal importance and the nonlinear diffusion is supposed to be equivalent to the nonlinear convection effects [1].

The existence of solutions of nonlinear DR equations may be well understood by first turning PDE into an ordinary differential equation (ODE) with the assumption of a travelling wave solution. The general travelling wavefront solutions of nonlinear DR equations found substantial mathematical and physical interests as they arise in a wide area of natural sciences such as biology and ecology [1].

The flux or current density of species $J(r,t)$, which can be cells, the amount of chemicals, the number of animals and so on, are proportional to the gradient of the concentration of the species. In one dimension, the current density $J(x,t) = -D \text{ grad } C(x,t)$, where the $C(x,t)$ is the concentration of the species, i.e. concentration of chemicals, number of cells or animals at point x and time t. D is the diffusion coefficient or diffusivity of the species and the minus sign simply indicates that diffusion transports matter from a high to a low concentration. It is a measure of how well the particles disperse from a high- to a low-density region. For example, in blood, haemoglobin molecules have a diffusion coefficient of the order of 10^{-7} cm^2 s^{-1} while that for oxygen in blood is of the order of 10^{-5} cm^2 s^{-1}.

The structure of the paper is as follows: in §2, a short description of the methods for finding travelling wave solutions of nonlinear DR equations is given. In §3 and 4, eqs (1) and (2) are studied by the (G'/G)-expansion method and its generalised version. In §5, we found travelling wave solutions of DR equations using the Kudryashov method. The graphical and physical interpretation of the obtained results is given in §6. Finally, concluding remarks are presented in §7.

2. Description of the algebraic methods

2.1 (G'/G)-*expansion method*

We concisely describe key steps of the (G'/G)-expansion method. Assume that a nonlinear PDE is of the form

$$P(q, q_t, q_x, q_{tt}, q_{xt}, q_{xx}, \ldots) = 0, \tag{3}$$

where $q = q(x,t)$ is an unknown function and P is a polynomial in $q = q(x,t)$ and its partial derivatives, in which higher order derivatives and nonlinear terms are involved.

Step 1: In order to find the travelling wave solution of eq. (3), introduce the wave variable $\xi = (x - vt)$, so that $q(x,t) = q(\xi)$.

Based on this, we make the following changes:

$$\frac{\partial}{\partial t} = -v\frac{\partial}{\partial \xi}, \quad \frac{\partial^2}{\partial t^2} = v^2\frac{\partial^2}{\partial \xi^2},$$
$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}, \quad \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2} \tag{4}$$

and subsequently for other derivatives. With the aid of eq. (4), PDE (3) changes to an ODE as

$$P(q, q_\xi, q_{\xi\xi}, q_{\xi\xi\xi}, \ldots) = 0, \tag{5}$$

where q_ξ, $q_{\xi\xi}$, etc. designate the derivative of q with respect to ξ.

Here, one can integrate ODE (5) as many times as possible and fix the constant of integration as zero for simplicity.

Step 2: The solution of eq. (5) can be specified by a polynomial in (G'/G), i.e.

$$q(\xi) = \sum_{i=0}^{n} a_i \left(\frac{G'}{G}\right)^i, \tag{6}$$

where $i = 0$ to n, a_i, λ and μ are constants to be determined later and $a_n \neq 0$. Here, $G = G(\xi)$ satisfies the second-order linear ODE of the form

$$G'' + \lambda G' + \mu G = 0, \tag{7}$$

where prime denotes the derivative of $G(\xi)$ with respect to ξ. Considering the homogeneous balance among the highest-order derivatives and nonlinear terms appearing in ODE (5), the positive integer n can be determined.

Step 3: Replacing eq. (6) into eq. (5) and using eq. (7), bring together all terms with the same order of (G'/G), and then comparing each coefficient of the resulting polynomial to zero yields a set of algebraic equations for a_i, v, λ and μ.

Step 4: Meanwhile, the general solutions of eq. (7) depending on whether $\lambda^2 - 4\mu > 0$, $\lambda^2 - 4\mu < 0$, $\lambda^2 - 4\mu = 0$ are given as

$$\left(\frac{G'}{G}\right)=\begin{cases}\frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{A_1\sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi+A_2\cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi}{A_1\cosh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi+A_2\sinh(\frac{1}{2}\sqrt{\lambda^2-4\mu})\xi}\right)-\frac{\lambda}{2}, & \lambda^2-4\mu>0,\\ \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-A_1\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2})\xi+A_2\cos(d\frac{1}{2}\sqrt{4\mu-\lambda^2})\xi}{A_1\cos(\frac{1}{2}\sqrt{4\mu-\lambda^2})\xi+A_2\sin(\frac{1}{2}\sqrt{4\mu-\lambda^2})\xi}\right)-\frac{\lambda}{2}, & \lambda^2-4\mu<0,\\ \frac{A_2}{A_1+A_2\xi}-\frac{\lambda}{2}, \quad \lambda^2-4\mu=0. \end{cases} \tag{8}$$

The above results can be written in a more simplified form as

$$\left(\frac{G'}{G}\right)=\begin{cases}\frac{\sqrt{\lambda^2-4\mu}}{2}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+\xi_0\right)-\frac{\lambda}{2}, \quad \lambda^2-4\mu>0, \quad \tanh\xi_0=\frac{A_1}{A_2}, \left|\frac{A_1}{A_2}\right|>1,\\ \frac{\sqrt{\lambda^2-4\mu}}{2}\coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi+\xi_0\right)-\frac{\lambda}{2}, \quad \lambda^2-4\mu>0, \quad \coth\xi_0=\frac{A_1}{A_2}, |\frac{A_1}{A_2}|<1,\\ \frac{\sqrt{4\mu-\lambda^2}}{2}\cot\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi+\xi_0\right)-\frac{\lambda}{2}, \quad \lambda^2-4\mu>0, \quad \cot\xi_0=\frac{A_2}{A_1},\\ \frac{A_2}{A_1+A_2\xi}-\frac{\lambda}{2}, \quad \lambda^2-4\mu=0. \end{cases} \tag{9}$$

Therefore, these results are the simplified forms of the result obtained by the (G'/G) method. Hence, we call this method the simplified (G'/G)-expansion method.

Now substituting a_i, v and the general solutions of eq. (7) which are from eqs (8) and (9) into eq. (6), we obtain travelling wave solutions of nonlinear eq. (3).

2.2 *Generalised (G'/G)-expansion method*

Step 1: Similar to that in §2.1.

Step 2: The solution of eq. (5) in terms of a polynomial in (G'/G) can be expressed as

$$q(\xi)=\sum_{i=0}^{n}a_i\left(\frac{G'}{G}\right)^i, \tag{10}$$

where $G=G(\xi)$ satisfies the subsequent Jacobi elliptic equation

$$[G']^2=q_2G^4(\xi)+q_1G^2(\xi)+q_0, \tag{11}$$

where $i=0$ to n, a_i, q_2, q_1, q_0 are the arbitrary constants to be determined later and $a_n\neq 0$.

Step 3: On exchange of eq. (10) into eq. (5) and using eq. (11), we get polynomial in $G^j(\xi)$, $G'(\xi)G^j(\xi)$ ($j=\pm 1,\pm 2\ldots$). Equating each coefficient of the resulted polynomials to zero yields a system of algebraic equations for a_i, q_2, q_1 and q_0.

Step 4: As general solutions of eq. (11) are well known for us (see Appendix A), then substituting a_i and the general solutions of eq. (11) into eq. (10), we obtain several new travelling wave solutions in terms of Jacobi elliptic functions of nonlinear PDE (3).

2.3 *Kudryashov method*

To make the discussion more coherent, in this paper, we highlight briefly the main features of the Kudryashov method.

We consider a nonlinear PDE, with a physical field q, and independent variables x, y, t as

$$P(q_x, q_t, q_{xx}, q_{xt}, q_y, q_{yy}\ldots)=0, \tag{12}$$

where P is a polynomial in $q(x,y,t)$ and its partial derivatives in which the highest-order derivatives and nonlinear terms are involved. In the following, we define the main steps of this method:

Step 1: Using the wave transformation $q(x, y, t) = q(\xi)$, where $\xi = k(x + y - vt)$, to transform eq. (12) to the following ODE:

$$F(q, q_{\xi}, q_{\xi\xi}, q_{\xi\xi\xi}, \ldots) = 0, \tag{13}$$

where F is a polynomial in $q(\xi)$ and its total derivatives, and the subscript denotes the derivative with respect to ξ.

Step 2: The solution of eq. (13) can be expressed in the more general form

$$q(\xi) = \sum_{n=0}^{N} a_n[\Upsilon(\xi)]^n, \tag{14}$$

where a_n $(n = 0, 1, 2, 3, \ldots, N)$ are constants to be evaluated later, such that $a_N \neq 0$, while $\Upsilon(\xi)$ satisfies the new equation

$$\Upsilon(\xi) = \frac{1}{1 + \exp(\xi + \xi_0)}, \tag{15}$$

which denotes the solution to the ODE, which is a special kind of Riccati equation

$$\frac{\mathrm{d}\Upsilon}{\mathrm{d}\xi} = \Upsilon^2(\xi) - \Upsilon(\xi). \tag{16}$$

Step 3: Find the positive integer N by matching the highest-order partial derivative with the highest-order nonlinear term in eq. (13). If N is not an integer, then a transformation formula should be used to overcome this difficulty.

Step 4: Substitute eq. (14) into eq. (13), and calculate all the necessary derivatives $q_{\xi}, q_{\xi\xi}, q_{\xi\xi\xi} \ldots$ of the unknown function $q(\xi)$ as follows:

$$q_{\xi} = \sum_{n=0}^{N} a_n n \Upsilon^n(\Upsilon - 1), \tag{17}$$

$$q_{\xi\xi} = \sum_{n=0}^{N} n\Upsilon^n(\Upsilon - 1)[(1 + n)\Upsilon - n]a_n \tag{18}$$

and so on. Replacing eqs (14), (17) and (18) into eq. (13), we acquire the polynomial

$$F[\Upsilon(\xi)] = 0. \tag{19}$$

Step 5: Assembling all the terms of the same powers of the function $\Upsilon(\xi)$ in the polynomial eq. (19) and equating them to zero, we can develop a set of algebraic equations which can be solved by MATHEMATICA to get the unknown parameters a_n, k and v. Consequently, we obtain the analytical exact solutions of eq. (12).

After a brief depiction of the methods, we now solve DR equations with quadratic and quartic nonlinearities using the aforementioned methods.

3. Solutions by the (G'/G)-expansion method

Now by using the travelling wave variable $\xi = x - \omega t$, eqs (1) and (2) can be written as

$$DC'' + \omega C' - kCC' + \alpha C - \beta C^2 = 0, \tag{20}$$

$$DC'' + \omega C' - kC^2C' + \alpha C - \beta C^4 = 0. \tag{21}$$

Balancing the highest-order derivative with nonlinear term in eq. (20), we get $n = 1$. Thus, for this value of n, ansatz (6) takes the following form:

$$C(\xi) = a_0 + a_1\left(\frac{G'}{G}\right), \quad a_1 \neq 0. \tag{22}$$

The use of eq. (22) in eq. (20) and the rationalisation of the resultant expression with respect to the powers of (G'/G) yields the following set of algebraic equations:

$$2Da_1 + ka_1^2 = 0, \tag{23a}$$

$$3D\lambda a_1 - \omega a_1 + k(\lambda a_1^2 + a_0a_1) - \beta a_1^2 = 0, \tag{23b}$$

$$D(2\mu + \lambda^2)a_1 - \omega\lambda a_1 + k(\mu a_1^2 + \lambda a_0a_1) + \alpha a_1 - 2\beta a_0a_1 = 0, \tag{23c}$$

$$D\mu\lambda a_1 - \omega\mu a_1 + k\mu a_0a_1 + \alpha a_0 - \beta a_0^2 = 0. \tag{23d}$$

On solving the above set of algebraic equations, we obtain

$$a_0 = \frac{1}{k^2}(\omega k - \lambda kD - 2D\beta), \quad a_1 = -\frac{2D}{k},$$
$$\omega = \frac{1}{2k\beta}[\alpha k^2 + 4D\beta^2], \quad (\lambda^2 - 4\mu) = \frac{\alpha^2k^2}{4D^2\beta^2}. \tag{24}$$

By using eq. (24) in expression (22), we obtain

$$C(\xi) = \frac{1}{k^2}(\omega k - \lambda kD - 2D\beta) - \frac{2D}{k}\left(\frac{G'}{G}\right). \tag{25}$$

Substituting the general solution of eq. (7) into eq. (25), we have the following types of travelling wave solutions of eq. (1):

Case (i): When $(\lambda^2 - 4\mu) > 0$, we have hyperbolic function solution as

$$C(\xi) = \frac{\alpha}{2\beta}\left[1 \mp \left\{\frac{A_1\sinh\left(\pm\frac{\alpha k}{4\beta D}\xi\right) + A_2\cosh\left(\pm\frac{\alpha k}{4\beta D}\xi\right)}{A_1\cosh\left(\pm\frac{\alpha k}{4\beta D}\xi\right) + A_2\sinh\left(\pm\frac{\alpha k}{4\beta D}\xi\right)}\right\}\right]. \tag{26}$$

In particular, if $A_1 \neq 0$, $A_2 = 0$, then $C(\xi)$ becomes

$$C(\xi) = \frac{\alpha}{2\beta}\left[1 \mp \tanh\left(\pm\frac{\alpha k}{4\beta D}\xi\right)\right]. \tag{27}$$

But when $A_2 \neq 0$, $A_1 = 0$, then $C(\xi)$ becomes

$$C(\xi) = \frac{\alpha}{2\beta}\left[1 \mp \coth\left(\pm\frac{\alpha k}{4\beta D}\xi\right)\right]. \tag{28}$$

These solutions of eq. (1) are similar to the results obtained in [61] which are known as diffusive kink and antikink soliton solutions. The diffusive kink wave solution (antikink) adopts the '+' ('−') sign in eq. (27) and has asymptotic states $\mp 1(\pm 1)$ as $\xi \to \mp\infty$. Both kinks and antikinks here can move in either a positive or a negative direction with wave speed ω. Now $C(\xi)$ which is given in eq. (26) can be written in the simplified form as

$$C(\xi) = \frac{\alpha}{2\beta}\left[1 \mp \tanh\left(\pm\frac{\alpha k}{4\beta D}\xi + \xi_0\right)\right], \tag{29}$$

where $\xi_0 = \tanh^{-1}(A_1/A_2)$, $|A_1/A_2| > 1$ and

$$C(\xi) = \frac{\alpha}{2\beta}\left[1 \mp \coth\left(\pm\frac{\alpha k}{4\beta D}\xi + \xi_0\right)\right], \tag{30}$$

where $\xi_0 = \coth^{-1}(A_1/A_2)$, $|A_1/A_2| < 1$.

Solutions (28) and (30) are singular solutions and less acceptable in physical terms.

Case (ii): When $(\lambda^2 - 4\mu) < 0$, we have the complex hyperbolic function solution as

$$C(\xi) = \frac{\alpha}{2\beta}\left[1 \mp \left\{\frac{-\mathrm{i}A_1 \sinh\left(\pm\frac{\alpha k}{4\beta D}\xi\right) + A_2 \cosh\left(\pm\frac{\alpha k}{4\beta D}\xi\right)}{A_1 \cosh\left(\pm\frac{\alpha k}{4\beta D}\xi\right) + \mathrm{i}A_2 \sinh\left(\pm\frac{\alpha k}{4\beta D}\xi\right)}\right\}\right]. \tag{31}$$

In particular, if $A_1 \neq 0$, $A_2 = 0$, then $C(\xi)$ becomes

$$C(\xi) = \frac{\alpha}{2\beta}\left[1 \pm \mathrm{i}\tanh\left(\pm\frac{\alpha k}{4\beta D}\xi\right)\right]. \tag{32}$$

But when $A_2 \neq 0$, $A_1 = 0$, then $C(\xi)$ becomes

$$C(\xi) = \frac{\alpha}{2\beta}\left[1 \pm \mathrm{i}\coth\left(\pm\frac{\alpha k}{4\beta D}\xi\right)\right]. \tag{33}$$

These solutions of eq. (1) are complex travelling wave solutions.

Case (iii): When $(\lambda^2 - 4\mu) = 0$, the rational solution is obtained as

$$C(\xi) = \frac{\alpha}{2\beta} - \frac{2D}{k}\left(\frac{A_2}{A_1 + A_2\xi}\right). \tag{34}$$

The solutions found here might be worthwhile to interpret the population dynamics.

Likewise, for the analytical solutions of eq. (21), we get $n = 1$ in ansatz (6) by the balancing procedure and the form of $C(\xi)$ becomes the same as given in eq. (22). As before, using ansatz (22) along with eq. (7) in eq. (21), one obtains the following set of algebraic equations for unknowns, specifically a_0, a_1, ω and α as

$$ka_1^3 - \beta a_1^4 = 0, \tag{35a}$$

$$2Da_1 + k\lambda a_1^3 + 2ka_0a_1^2 - 4\beta a_0 a_1^3 = 0, \tag{35b}$$

$$3D\lambda a_1 - \omega a_1 + k\mu a_1^3 + 2k\lambda a_0 a_1^2 + ka_0^2 a_1 - 6\beta a_0^2 a_1^2 = 0, \tag{35c}$$

$$D(2\mu + \lambda^2)a_1 - \omega\lambda a_1 + 2k\mu a_0 a_1^2 + k\lambda a_0^2 a_1 + \alpha a_1 - 4\beta a_0^3 a_1 = 0, \tag{35d}$$

$$D\mu\lambda a_1 - \omega\mu a_1 + k\mu a_0^2 a_1 + \alpha a_0 - \beta a_0^4 = 0, \tag{35e}$$

which on solving by MATHEMATICA provide

$$a_0 = \frac{(2D\beta^2 + \lambda k^3)}{2\beta k^2}, \quad a_1 = \frac{k}{\beta},$$

$$\omega = -\frac{(\alpha k^6 + 16D^3\beta^4)}{4D\beta^2 k^3},$$

$$(\lambda^2 - 4\mu) = -\frac{4\beta^2(5D^2\beta^2 + \omega k^3)}{k^6}, \tag{36}$$

along with the constraint relation $\alpha = \pm(8D^3\beta^4/k^6)$ which gives two values of ω and $(\lambda^2 - 4\mu)$. Now inserting eq. (36) in eq. (22), we get the following kinds of travelling wave solutions of eq. (2).

Case (i): When $(\lambda^2 - 4\mu) > 0$ and $\alpha = ((8D^3\beta^4)/k^6)$, we have hyperbolic travelling wave solution as

$$C(\xi) = \frac{\beta D}{k^2}\left[1 \pm \left\{\frac{A_1 \sinh\left(\pm\frac{D\beta^2}{k^3}\xi\right) + A_2 \cosh\left(\pm\frac{D\beta^2}{k^3}\xi\right)}{A_1 \cosh\left(\pm\frac{D\beta^2}{k^3}\xi\right) + A_2 \sinh\left(\pm\frac{D\beta^2}{k^3}\xi\right)}\right\}\right], \tag{37}$$

where $\xi = x + (6D^2\beta^2/k^3)t$. In particular, if $A_1 \neq 0$, $A_2 = 0$, then $C(\xi)$ becomes

$$C(\xi)=\frac{\beta D}{k^2}\left[1\pm\tanh\left(\pm\frac{D\beta^2}{k^3}\xi\right)\right]. \tag{38}$$

But when $A_2\neq 0$, $A_1=0$, $C(\xi)$ becomes

$$C(\xi)=\frac{\beta D}{k^2}\left[1\pm\coth\left(\pm\frac{D\beta^2}{k^3}\xi\right)\right]. \tag{39}$$

These solutions of eq. (2) are similar to the solutions obtained in [61] which are kink- and antikink-type soliton solutions. The solution $C(\xi)$ of eq. (37) can be specified in more simplified forms as

$$C(\xi)=\frac{\beta D}{k^2}\left[1\pm\tanh\left(\pm\frac{D\beta^2}{k^3}\xi+\xi_0\right)\right], \tag{40}$$

where $\xi_0=\tanh^{-1}(A_1/A_2)$, $|A_1/A_2|>1$ and

$$C(\xi)=\frac{\beta D}{k^2}\left[1\pm\coth\left(\pm\frac{D\beta^2}{k^3}\xi+\xi_0\right)\right], \tag{41}$$

where $\xi_0=\coth^{-1}(A_1/A_2)$, $|A_1/A_2|<1$.

Case (ii): When $(\lambda^2-4\mu)<0$, i.e. for $\alpha=-(8D^3\beta^4/k^6)$, we have a trigonometric function solution as

$$C(\xi)=\frac{\beta D}{k^2}\left[1\pm\sqrt{3}\times\left\{\frac{-A_1\sin\left(\pm\frac{\sqrt{3}D\beta^2}{k^3}\xi\right)+A_2\cos\left(\pm\frac{\sqrt{3}D\beta^2}{k^3}\xi\right)}{A_1\cos\left(\pm\frac{\sqrt{3}D\beta^2}{k^3}\xi\right)+A_2\sin\left(\pm\frac{\sqrt{3}D\beta^2}{k^3}\xi\right)}\right\}\right], \tag{42}$$

where $\xi=x+(2D^2\beta^2/k^3)t$. In a simplified form, eq. (42) can be written as

$$C(\xi)=\frac{\beta D}{k^2}\left[1\pm\sqrt{3}\,\cot\left(\pm\frac{\sqrt{3}D\beta^2}{k^3}\xi+\xi_0\right)\right], \tag{43}$$

where $\xi_0=\tan^{-1}(A_1/A_2)$.

Case (iii): When $(\lambda^2-4\mu)=0$, the rational solution can be written as

$$C(\xi)=\frac{\beta D}{k^2}+\frac{k}{\beta}\left(\frac{A_2}{A_1+A_2\xi}\right). \tag{44}$$

Equations (38) and (40) are diffusive kink- or antikink-shaped soliton solutions and wave speed of invasion of species ω depends on the parameter k, nonlinear diffusion coefficient D and parameter $\beta=r/K$, where r is the linear growth rate and K is the carrying capacity of the environment [35]. The travelling wave solutions (39), (41) and (43) are singular solutions and have no real physical meaning in explaining the dynamics of ecological or biological phenomenon.

4. Solutions by the generalised (G'/G)-expansion method

To explore the applications of the generalised (G'/G)-expansion method, in this section, we analyse eqs (1) and (2). As an outcome of the balancing procedure in eqs (20) and (21) we have $n=1$. This suggest the choice of $C(\xi)$ the same as in eq. (22). Here and now, on substituting eq. (22) into eq. (20) and with the help of eq. (11), we achieve a set of algebraic equations:

$$2Dq_2a_1-kq_2a_1^2=0, \tag{45a}$$
$$2Dq_0a_1+kq_0a_1^2=0, \tag{45b}$$
$$\omega q_2a_1-kq_2a_0a_1-\beta q_2a_1^2=0, \tag{45c}$$
$$-\omega q_0a_1+kq_0a_0a_1-\beta q_2a_1^2=0, \tag{45d}$$
$$\alpha a_0-\beta a_0^2-\beta q_1a_1^2=0, \tag{45e}$$
$$\alpha a_1-2\beta a_0a_1=0, \tag{45f}$$

which after solving provides the following results:

$$a_0=\frac{\alpha}{2\beta},\quad a_1=\frac{2D}{k},\quad \omega=\frac{(\alpha k^2+4D\beta^2)}{2\beta k},\quad q_0=0,\quad q_1=\frac{\alpha^2k^2}{16D^2\beta^2}. \tag{46}$$

Using eq. (46) into ansatz eq. (22), we obtain

$$C(\xi)=\frac{\alpha}{2\beta}+\frac{2D}{k}\left(\frac{G'}{G}\right), \tag{47}$$

where $\xi=x-((\alpha k^2+4D\beta^2)/2\beta k)t$.

Now, from Appendix A, we derive the following sets of exact solutions.

Set (i): If $q_0=0$, $q_1=1$, $q_2=-1$, then eq. (47) becomes

$$C(\xi)=\frac{\alpha}{2\beta}[1-\tanh(\xi)], \tag{48}$$

which is an antikink wave solution.

Set (ii): If $q_0=0$, $q_1=1$, $q_2=1$, then the solution acquires the following form:

$$C(\xi)=\frac{\alpha}{2\beta}[1-\coth(\xi)], \tag{49}$$

which represents the singular solution.

Set (iii): If $q_0 = 0, q_1 = -1, q_2 = 1$, then we have

$$C(\xi) = \frac{\alpha}{2\beta} + \frac{2D}{k} \tan(\xi), \tag{50}$$

which is a periodic wave solution with period π and range $(-\infty, \infty)$.

Set (iv): If $q_0 = 0, q_1 = 0, q_2 = 1$, then solution (47) becomes

$$C(\xi) = -\frac{(2D/k)}{\xi}, \tag{51}$$

which is the rational solution of nonlinear DR equation.

Set (v): If $q_0 = 0, q_1 = -(1+m^2), q_2 = m^2$, then we obtain

$$C(\xi) = \frac{\alpha}{2\beta} + \frac{2D}{k}\left\{\frac{\text{cn}(\xi)\text{dn}(\xi)}{\text{sn}(\xi)}\right\}, \tag{52}$$

which is a Jacobi elliptic function solution.

Now, using Appendices B and C, solution (52) changes into hyperbolic and trigonometric function forms according to the choice of modulus parameter m as follows.

Special cases

(a) When $m \to 1$, the combined hyperbolic solution of eq. (52) is written as

$$C(\xi) = \frac{\alpha}{2\beta} + \frac{2D}{k}\{\coth(\xi) - \tanh(\xi)\}. \tag{53}$$

(b) When $m \to 0$, solution (52) is written in the form of trigonometric function as

$$C(\xi) = \frac{\alpha}{2\beta} + \frac{2D}{k}\cot(\xi), \tag{54}$$

which is a periodic wave solution with period π and range $(-\infty, \infty)$.

Next, to find the exact solutions of eq. (21), we utilise similar procedure as above and obtain a set of simultaneous algebraic equations:

$$2Dq_2a_1 - 4\beta q_2a_0a_1^3 - 2kq_2a_0a_1^2 = 0, \tag{55a}$$

$$2Dq_0a_1 - 4\beta q_0a_0a_1^3 + 2kq_0a_0a_1^2 = 0, \tag{55b}$$

$$\omega q_2a_1 - kq_2a_0^2a_1 - kq_1q_2a_1^3 - 6\beta q_2a_0^2a_-^2 2\beta q_1q_2a_1^4 = 0, \tag{55c}$$

$$-\omega q_0a_1 + kq_0a_0^2a_1 + kq_1q_0a_1^3 - 6\beta q_0a_0^2a_-^2 2\beta q_1q_0a_1^4 = 0, \tag{55d}$$

$$-kq_2^2a_1^3 - \beta q_2^2a_1^4 = 0, \tag{55e}$$

$$-kq_0^2a_1^3 - \beta q_0^2a_1^4 = 0, \tag{55f}$$

$$\alpha a_0 - \beta a_0^4 - \beta q_1^2a_1^4 - 6\beta q_1a_0^2a_1^2 - 2\beta q_0q_2a_1^4 = 0, \tag{55g}$$

$$\alpha a_1 - 4\beta a_0^3a_1 - 4\beta a_0a_1^3 = 0, \tag{55h}$$

which, on solving provides the following results:

$$a_0 = \frac{\beta D}{k^2},\quad a_1 = -\frac{k}{\beta},\quad \omega = -\frac{5\beta^4D^2 + k^6q_1}{k^3\beta^2},$$
$$q_0 = 0,\quad \alpha = \pm\frac{8\beta^4D^3}{k^6},\quad q_1 = \frac{\alpha k^6 - 4\beta^4D^3}{4Dk^6}. \tag{56}$$

Substitution of eq. (56) into eq. (22) yields the following solution:

$$C(\xi) = \frac{\beta D}{k^2} - \frac{k}{\beta}\left(\frac{G'}{G}\right), \tag{57}$$

where

$$\xi = x + \left(\frac{5\beta^4D^2 + k^6q_1}{k^3\beta^2}\right)t.$$

Now, using Appendix A, we obtain the following sets of travelling wave solutions.

Set (i): If $q_0 = 0, q_1 = 1, q_2 = -1$, then the solution of eq. (2) becomes

$$C(\xi) = \frac{\beta D}{k^2}\{1 \pm \tanh(\xi)\}, \tag{58}$$

where a constraining relation between constant coefficients is given by $\alpha = 8\beta^4D^3/k^6$. However, when $\alpha = -8\beta^4D^3/k^6$, then $C(\xi)$ takes the form

$$C(\xi) = \frac{\beta D}{k^2}\{1 \pm \sqrt{3}R\ \tanh(\xi)\}. \tag{59}$$

Set (ii): If $q_0 = 0, q_1 = 1, q_2 = 1$ and $\alpha = \pm(8\beta^4D^3/k^6)$, the following solutions are obtained:

$$C(\xi) = \frac{\beta D}{k^2}\{1 \pm \coth(\xi)\}, \tag{60}$$

$$C(\xi) = \frac{\beta D}{k^2}\{1 \pm \sqrt{3}\mathrm{i}\coth(\xi)\}. \tag{61}$$

Set (iii): If $q_0 = 0, q_1 = -1, q_2 = 1$ and $\alpha = \pm(8\beta^4D^3/k^6)$, then eq. (57) becomes

$$C(\xi) = \frac{\beta D}{k^2}\{1 \mp \mathrm{i}\ \tan(\xi)\}, \tag{62}$$

$$C(\xi) = \frac{\beta D}{k^2}\{1 \pm \sqrt{3}\ \tan(\xi)\}, \tag{63}$$

which is a periodic wave solution with period π and range $(-\infty, \infty)$.

Set (iv): If $q_0 = 0$, $q_1 = -(1+m^2)$, $q_2 = m^2$ and $\alpha = \pm(8\beta^4 D^3/k^6)$, then we obtain

$$C(\xi) = \frac{\beta D}{k^2}\left\{1 \pm \frac{\mathrm{i}}{\sqrt{1+m^2}}\frac{\mathrm{cn}(\xi)\mathrm{dn}(\xi)}{\mathrm{sn}(\xi)}\right\}, \tag{64}$$

$$C(\xi) = \frac{\beta D}{k^2}\left\{1 \mp \sqrt{\frac{3}{1+m^2}}\frac{\mathrm{cn}(\xi)\mathrm{dn}(\xi)}{\mathrm{sn}(\xi)}\right\}. \tag{65}$$

Here, the parameter m varies between 0 and 1. When $m \to 0$, Jacobi elliptic functions will be converted to trigonometric functions and when $m \to 1$, Jacobi elliptic functions degenerate into hyperbolic functions.

Special cases

(a) In the limiting case when $m \to 1$, according to Appendices B and C, eqs (64) and (65) can be written as follows:

$$C(\xi) = \frac{\beta D}{k^2}\{1 \pm \sqrt{2}\mathrm{i}\ \mathrm{csch}(2\xi)\}, \tag{66}$$

$$C(\xi) = \frac{\beta D}{k^2}\{1 \mp \sqrt{6}\ \mathrm{csch}(2\xi)\}. \tag{67}$$

(b) When $m \to 0$, according to Appendices B and C, eqs (64) and (65) can be written as

$$C(\xi) = \frac{\beta D}{k^2}\{1 \pm \mathrm{i}\cot(\xi)\}, \tag{68}$$

$$C(\xi) = \frac{\beta D}{k^2}\{1 \mp \sqrt{3}\cot(\xi)\}. \tag{69}$$

Hence, we have found diverse solutions of DR equations in the form of Jacobi elliptic functions, hyperbolic and trigonometric functions. Using Appendix B, the solutions, which are in the forms of Jacobi elliptic function solutions, are reduced to hyperbolic, trigonometric function solutions. Many of the achieved solutions are kink–antikink solitons and some are travelling wave solutions. Also, we observed that some solutions are complex. These travelling wave solutions are singular as well as non-singular solutions. Only non-singular travelling wavefronts have physical meaning in order to explain natural biological and ecological processes.

5. Solutions by the Kudryashov method

To explore the applications of the Kudryashov method, here we analyse eqs (1) and (2). As a result of balancing procedure in eqs (20) and (21), we get $n = 1$. Consequently, we reach

$$C(\xi) = a_0 + a_1\Upsilon(\xi). \tag{70}$$

Using eqs (17) and (18), we have

$$\frac{\partial C(\xi)}{\partial \xi} = a_1\Upsilon(\xi)[\Upsilon(\xi) - 1], \tag{71}$$

$$\frac{\partial^2 C(\xi)}{\partial \xi^2} = a_1\Upsilon(\xi)[\Upsilon(\xi) - 1][2\Upsilon(\xi) - 1], \tag{72}$$

where a_0, a_1 are constants to be evaluated later, such that $a_1 \neq 0$. Substituting eq. (70) along with eqs (71) and (72) into eq. (20) and comparing all the coefficients of power of $\Upsilon(\zeta)$ to be zero, we achieve

$$-ka_1^2 + 2Da_1 = 0, \tag{73a}$$

$$-3Da_1 + \omega a_1 - \beta a_1^2 - ka_1a_0 + ka_1^2 = 0, \tag{73b}$$

$$-\omega a_1 + Da_1 - 2\beta a_0 a_1 + ka_1a_0 + \alpha a_0 = 0, \tag{73c}$$

$$-\beta a_0^2 + \alpha a_0 = 0. \tag{73d}$$

On evaluating the above system of algebraic eqs (73a)–(73d) with one of the packages for computer algebra, we get values of parameters as the following cases:

Set (i)

$$a_0 = 0, \quad a_1 = \frac{\alpha}{\beta}, \quad \omega = \alpha + D, \quad k = \frac{2\beta D}{\alpha}. \tag{74}$$

From eqs (74), (70) and (15), we have the particular travelling wave solutions of eq. (20) as follows:

$$C(\xi) = \frac{\alpha}{\beta}\frac{1}{\exp[x - (\alpha + D)t + \xi_0]}, \quad \alpha\beta > 0. \tag{75}$$

The above exact travelling wave solution can be further simplified as

$$C(\xi) = \frac{\alpha}{2\beta}\left[1 - \tanh\left(\frac{x - (\alpha + D)t}{2} + \frac{\xi_0}{2}\right)\right], \quad \alpha\beta > 0, \tag{76}$$

$$C(\xi) = \frac{\alpha}{2\beta}\left[1 - \coth\left(\frac{x - (\alpha + D)t}{2} + \frac{\xi_1}{2}\right)\right], \quad \alpha\beta > 0, \tag{77}$$

where ξ_0 is an arbitrary constant and ξ_1 also is a constant such that $\xi_0 = (\mathrm{i}\pi/2) + \xi_1$.

Set (ii)

$$a_0 = \frac{\alpha}{\beta}, \quad a_1 = -\frac{\alpha}{\beta}, \quad \omega = -\alpha - D, \quad k = -\frac{2\beta D}{\alpha}. \tag{78}$$

From eqs (78), (70) and (15), we get the exact hyperbolic function solutions of eq. (20) as follows:

$$C(\xi) = \frac{\alpha}{\beta}\left[1 - \frac{1}{\exp[x + (\alpha - D)t + \xi_0]}\right], \quad \alpha\beta < 0. \tag{79}$$

The exact travelling wave solution (79) can be written as

$$C(\xi) = \frac{\alpha}{\beta} - \frac{\alpha}{2\beta}\left[1 - \tanh\left(\frac{x + (\alpha - D)t}{2} + \frac{\xi_0}{2}\right)\right], \quad \alpha\beta < 0, \tag{80}$$

$$C(\xi) = \frac{\alpha}{\beta} - \frac{\alpha}{2\beta}\left[1 - \coth\left(\frac{x + (\alpha - D)t}{2} + \frac{\xi_1}{2}\right)\right], \quad \alpha\beta < 0, \tag{81}$$

where ξ_0 is an arbitrary constant and ξ_1 also is a constant such that $\xi_0 = (\mathrm{i}\pi/2) + \xi_1$.

Next, to find the exact solutions of eq. (21), we applied similar procedure as above and arrive at a set of algebraic relations:

$$-ka_1^3 - \beta a_1^4 = 0, \tag{82a}$$

$$-2ka_1^2a_0 + 2Da_1 + ka_1^3 - 4\beta a_0a_1^3 = 0, \tag{82b}$$

$$-ka_1a_0^2 + \omega a_1 - 3Da_1 - 6\beta a_0^2a_1^2 + 2ka_1^2a_0 = 0, \tag{82c}$$

$$-4\beta a_0^3a_1 + Da_1 - \omega a_1 + ka_1a_0^2 + \alpha a_1 = 0, \tag{82d}$$

$$-\beta a_0^4 + \alpha a_0 = 0. \tag{82e}$$

Solving (82a)–(82e), we obtain the values of parameters as follows:

Set (i)

$$a_0 = 0, \quad a_1 = \pm\sqrt{-\frac{2D}{k}}, \quad \omega = 3D, \quad \alpha = \pm\frac{k^3}{\beta^2}. \tag{83}$$

From eqs (83), (70) and (15), we obtain the exact travelling wave solutions of eq. (20) as follows:

$$C(\xi) = \pm\sqrt{-\frac{2D}{k}}\frac{1}{\exp[x - 3Dt + \xi_0]}, \quad kD < 0. \tag{84}$$

Moreover, the exact travelling wave solution (84) can be expressed as

$$C(\xi) = \pm\sqrt{-\frac{D}{2k}}\left[1 - \tanh\left(\frac{x - 3Dt}{2} + \frac{\xi_0}{2}\right)\right], \quad kD < 0, \tag{85}$$

$$C(\xi) = \pm\sqrt{-\frac{D}{2k}}\left[1 - \coth\left(\frac{x - 3Dt}{2} + \frac{\xi_1}{2}\right)\right], \quad kD < 0, \tag{86}$$

where ξ_0 is an arbitrary constant and ξ_1 is also a constant such that $\xi_0 = (\mathrm{i}\pi/2) + \xi_1$.

Set (ii)

$$a_0 = \mp\sqrt{\frac{2D}{k}}, \quad a_1 = \pm\sqrt{\frac{2D}{k}}, \quad \omega = -3D, \quad \alpha = \pm\frac{k^3}{\beta^2}, \tag{87}$$

From eqs (87), (70) and (15), we have the exact travelling wave solutions of eq. (20) as follows:

$$C(\xi) = \mp\sqrt{\frac{2D}{k}}\left[1 - \frac{1}{\exp[x + 3Dt + \xi_0]}\right], \quad kD > 0. \tag{88}$$

Solution (88) can be expressed in a more simplified form as

$$C(\xi) = \mp\sqrt{\frac{2D}{k}} \pm \sqrt{\frac{D}{2k}}\left[1 - \tanh\left(\frac{x + 3Dt}{2} + \frac{\xi_0}{2}\right)\right], \quad kD > 0, \tag{89}$$

$$C(\xi) = \mp\sqrt{\frac{2D}{k}} \pm \sqrt{\frac{D}{2k}}\left[1 - \coth\left(\frac{x + 3Dt}{2} + \frac{\xi_1}{2}\right)\right], \quad kD > 0, \tag{90}$$

where ξ_0 is an arbitrary constant and ξ_1 is also a constant such that $\xi_0 = (\mathrm{i}\pi/2) + \xi_1$.

Solutions (76), (80), (85) and (89) are diffusive kink or antikink wave solutions describing travelling population fronts depending on the parameter values involved in nonlinear DR equations. On the other hand, the obtained singular solutions (77), (81), (86) and (90) have no physical importance in the explanation of biological and ecological phenomena. It should be noted that the Kudryashov method offers all the results that can be found using the exp-function method and permits us to obtain all solitary wave solutions and all one-periodic solutions when we obtain the expansion of the general solution of the nonlinear differential equation in the Laurent series.

6. Graphical results and discussions

In order to understand the dynamics of nonlinear DR equations, we represent two-dimensional profiles of the obtained solutions. The profile of the obtained travelling wave solution describes the population density or concentration of species vs. space and time.

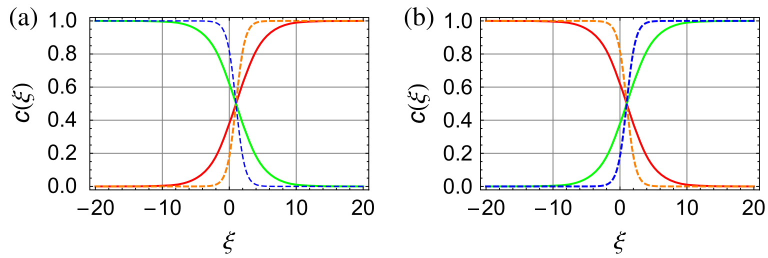

Figure 1. Profile of solution (27). (**a**) $C_{++}(\xi)$ and $C_{-+}(\xi)$ for $\alpha = 1.0$, $\beta = 1.0$, $D = 1.0$, $\omega = 1.0$, $t = 1.0$, for $k = 1.0$ (red curve), $k = 3.0$ (orange dotted curve), $k = 1.0$ (green curve) and $k = 3.0$ (blue dotted curve), respectively and (**b**) $C_{+-}(\xi)$ and $C_{--}(\xi)$ for $\alpha = 1.0$, $\beta = 1.0$, $D = 1.0$, $\omega = 1.0$, $t = 1.0$, for $k = 1.0$ (red curve), $k = 3.0$ (orange dotted curve), $k = 1.0$ (green curve) and $k = 3.0$ (blue dotted curve), respectively.

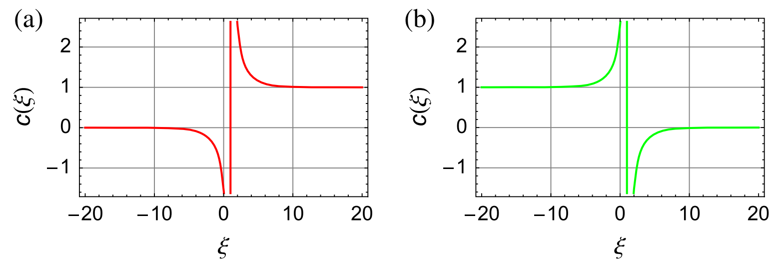

Figure 2. Profile of solution (28). (**a**) $C_{++}(\xi)$ for $\alpha = 1.0,\ \beta = 1.0$, $D = 1.0$, $\omega = 1.0$, $t = 1.0$, for $k = 1.0$ (red curve) and (**b**) $C_{+-}(\xi)$ for $\alpha = 1.0,\ \beta = 1.0$, $D = 1.0$, $\omega = 1.0$, $t = 1.0$, for $k = 1.0$ (green curve).

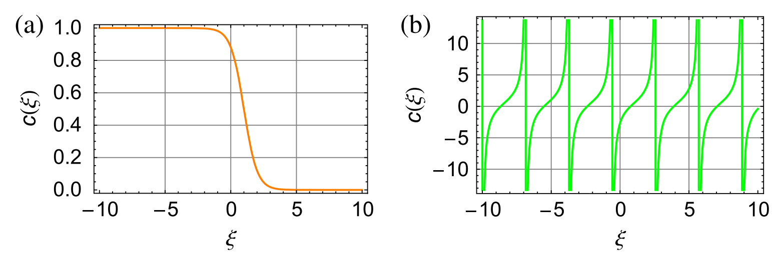

Figure 3. (**a**) Profile of solution (48) for $\alpha = 1.0$, $\beta = 1.0$, $\omega = 1.0$, $t = 1.0$ and (**b**) profile of solution (50) for $\alpha = 1.0$, $\beta = 1.0$, $D = 1.0$, $k = 1.0$, $\omega = 1.0$, $t = 1.0$.

The construction of the figures is carried out by taking suitable values of parameters to see the mechanism involved in DR equations. We have observed from the obtained results that eqs (27), (29), (38), (40), (48), (58), (76), (80), (85) and (89) are kink and antikink wave solutions. It is imperative to study the symmetry involved in the solutions which are expressed in eq. (27). We denote these solutions as C_{-+} and C_{+-} corresponding to upper and lower signs for positive values of k and as C_{--} and C_{++} for negative values of k. In this symmetry, it is found that positive values of k provide kink-shaped soliton solutions, and negative values of k provide antikink soliton solutions. Also, it should be illustrious that $C_{-+} = C_{+-}$ and $C_{--} = C_{++}$, as depicted in figures 1a and 1b. Note that the amplitude of $C(\xi)$ is independent of both the parameters D and k but decreases with the increase in β while wave speed ω depends on D and k. From solutions (38) and (39), we also found that the amplitude and velocity of wave ω depend on the nonlinear parameter β, diffusion coefficient D as well as density-dependent parameter k. Here, the amplitude and velocity of the wave increase with the increase in D and β but shows a decrease with an increase in k. The symmetry involved in the solutions is given in eqs (38) and (39), and it is found that the symmetry of the solutions C_{++}, C_{--}, C_{+-} and C_{-+} remains the same as in eq. (27). The solutions represented by eqs (28), (30), (34), (39), (41), (43), (44), (49), (51), (60), (77), (81), (86) and (90) are singular solutions and have no physical significance in the

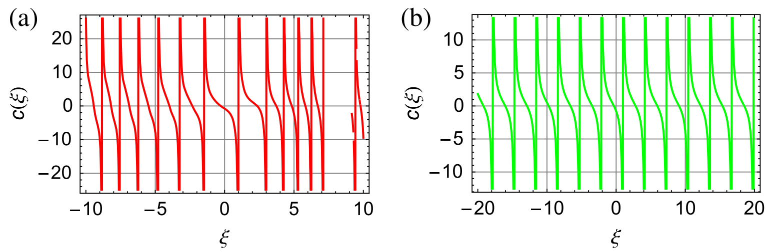

Figure 4. (**a**) Profile of solution (52) for $\alpha = 1.0,\ \beta = 1.0,\ D = 1.0, k = 1, \omega = 1.0, t = 1.0$ and (**b**) profile of solution (54) for $\alpha = 1.0,\ \beta = 1.0,\ D = 1.0, k = 1.0, \omega = 1.0, t = 1.0$.

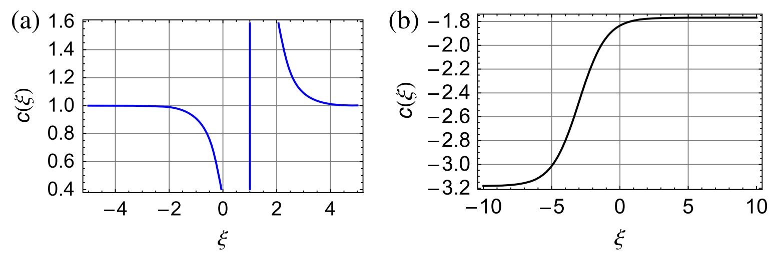

Figure 5. (**a**) Profile of solution (67) for $\beta = 1.0, D = 1,\ k = 1, \omega = 1.0, t = 1.0$ and (**b**) profile of solution (89), $C_{--}(\xi)$ for $\beta = 1.0, D = 1.0, k = 1.0, \omega = 1.0, t = 1.0$.

biological phenomenon. The two-dimensional profiles of a singular solution (28) are shown in figures 2a and 2b for different parameter values. The solutions represented by eqs (50), (52), (54), (63), (65) and (69) are periodic wave solutions. In figures 3a and 3b, we represent the two-dimensional profiles of a kink wave soliton (47) and a periodic wave solution (50), respectively. The profiles of the Jacobi elliptic solution (52) and trigonometric function solution (54) are shown in figures 4a and 4b, respectively. The two-dimensional profiles of singular soliton solution (67) and antikink soliton (89) are depicted in figures 5a and 5b, respectively.

7. Conclusion

We investigated the DR equations in the presence of quadratic and quartic nonlinearities with a nonlinear convective flux term arising in the biological systems. The travelling wave solutions are constructed in terms of hyperbolic, trigonometric and Jacobi elliptic functions. The general travelling wave solutions can provide periodic solutions and kink- or antikink-type soliton solutions under different parametric restrictions. The kink and antikink solutions can be used to describe the population dynamics of some insects and small rodents. The (G'/G)-expansion method, its generalised version and Kudryashov methods are direct, concise, powerful, standard and computerisable methods and can be applied for all integrable and non-integrable nonlinear evolution equations in diverse areas of nonlinear science. Performance of these methods is consistent, simple and offers different types of solutions. Moreover, these methods are capable of significantly reducing the size of computational work compared to other existing methods in the literature and one can easily recover solutions that are obtained from other methods.

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Appendix A

The general solutions to Jacobi elliptic equation and their derivative (see e.g. [53,55]) are listed in table 1.

In table 1, the elliptic modulus m of the Jacobi elliptic functions varies between $0 < m < 1$ and $\mathrm{i} = \sqrt{-1}$.

Table 1. Jacobi elliptic functions.

q_0	q_1	q_2	$G(\xi)$	$G'(\xi)$
1	$-(1+m^2)$	m^2	$\mathrm{sn}(\xi)$	$\mathrm{cn}(\xi)\mathrm{dn}(\xi)$
1	$-(1+m^2)$	m^2	$\mathrm{cd}(\xi)$	$-(1-m^2)\mathrm{sd}(\xi)\mathrm{nd}(\xi)$
$1-m^2$	$2m^2-1$	$-m^2$	$\mathrm{cn}(\xi)$	$-\mathrm{sn}(\xi)\mathrm{dn}(\xi)$
m^2-1	$2-m^2$	-1	$\mathrm{dn}(\xi)$	$-m^2\mathrm{sn}(\xi)\mathrm{cn}(\xi)$
m^2	$-(m^2+1)$	1	$\mathrm{ns}(\xi)$	$-\mathrm{ds}(\xi)\mathrm{cs}(\xi)$
m^2	$-(m^2+1)$	1	$\mathrm{dc}(\xi)$	$(1-m^2)\mathrm{nc}(\xi)\mathrm{sc}(\xi)$
$-m^2$	$2m^2-1$	$1-m^2$	$\mathrm{nc}(\xi)$	$\mathrm{sc}(\xi)\mathrm{dc}(\xi)$
-1	$2-m^2$	m^2-1	$\mathrm{nd}(\xi)$	$m^2\mathrm{sd}(\xi)\mathrm{cd}(\xi)$
$1-m^2$	$2-m^2$	1	$\mathrm{cs}(\xi)$	$-\mathrm{ns}(\xi)\mathrm{ds}(\xi)$
1	$2-m^2$	$1-m^2$	$\mathrm{sc}(\xi)$	$\mathrm{nc}(\xi)\mathrm{dc}(\xi)$
1	$2m^2-1$	$m^2(m^2-1)$	$\mathrm{sd}(\xi)$	$\mathrm{nd}(\xi)\mathrm{cd}(\xi)$
$m^2(m^2-1)$	$2m^2-1$	1	$\mathrm{ds}(\xi)$	$-\mathrm{cs}(\xi)\mathrm{ns}(\xi)$
$\frac{1}{4}$	$\frac{1}{2}(1-2m^2)$	$\frac{1}{4}$	$\mathrm{ns}(\xi)\pm\mathrm{cs}(\xi)$	$-\mathrm{ds}(\xi)\mathrm{cs}(\xi)\mp\mathrm{ns}(\xi)\mathrm{ds}(\xi)$
$\frac{1}{4}(1-m^2)$	$\frac{1}{2}(1+m^2)$	$\frac{1}{4}(1-m^2)$	$\mathrm{nc}(\xi)\pm\mathrm{sc}(\xi)$	$\mathrm{sc}(\xi)\mathrm{dc}(\xi)\pm\mathrm{nc}(\xi)\mathrm{dc}(\xi)$
$\frac{m^2}{4}$	$\frac{1}{2}(m^2-2)$	$\frac{1}{4}$	$\mathrm{ns}(\xi)\pm\mathrm{ds}(\xi)$	$-\mathrm{ds}(\xi)\mathrm{cs}(\xi)\mp\mathrm{cs}(\xi)\mathrm{ns}(\xi)$
$\frac{m^2}{4}$	$\frac{1}{2}(m^2-2)$	$\frac{m^2}{4}$	$\mathrm{sn}(\xi)\pm i\mathrm{cn}(\xi)$	$\mathrm{dn}(\xi)\mathrm{cn}(\xi)\mp i\mathrm{sn}(\xi)\mathrm{dn}(\xi)$
0	1	-1	$\mathrm{sech}(\xi)$	$-\mathrm{sech}(\xi)\tanh(\xi)$
0	1	1	$\mathrm{csch}(\xi)$	$-\mathrm{csch}(\xi)\coth(\xi)$
0	-1	1	$\sec(\xi)$	$\sec(\xi)\tan(\xi)$
0	0	1	$\frac{1}{\xi}$	$-\frac{1}{\xi^2}$
0	$-(1+m^2)$	m^2	$\mathrm{sn}(\xi)$	$\mathrm{cn}(\xi)\mathrm{dn}(\xi)$

Appendix B

When $m\to 1$, the Jacobi elliptic functions $\mathrm{sn}(\xi)$, $\mathrm{cn}(\xi)$, $\mathrm{dn}(\xi)$, $\mathrm{ns}(\xi)$, $\mathrm{cs}(\xi)$, $\mathrm{ds}(\xi)$, $\mathrm{sc}(\xi)$ and $\mathrm{sd}(\xi)$ degenerate into hyperbolic functions as follows:

$$\begin{aligned}&\mathrm{sn}(\xi)\to\tanh(\xi), && \mathrm{cn}(\xi)\to\mathrm{sech}(\xi),\\ &\mathrm{dn}(\xi)\to\mathrm{sech}(\xi), && \mathrm{cs}(\xi)\to\mathrm{cosech}(\xi),\\ &\mathrm{ds}(\xi)\to\mathrm{cosech}(\xi), && \mathrm{ns}(\xi)\to\coth(\xi),\\ &\mathrm{sc}(\xi)\to\sinh(\xi), && \mathrm{sd}(\xi)\to\sinh(\xi)\end{aligned}$$

and into trigonometric functions if $m\to 0$ as follows:

$$\begin{aligned}&\mathrm{sn}(\xi)\to\sin(\xi), && \mathrm{cn}(\xi)\to\cos(\xi),\\ &\mathrm{dn}(\xi)\to 1, && \mathrm{ns}(\xi)\to\mathrm{cosec}(\xi),\\ &\mathrm{cs}(\xi)\to\cot(\xi), && \mathrm{ds}(\xi)\to\mathrm{cosec}(\xi),\\ &\mathrm{sc}(\xi)\to\tan(\xi), && \mathrm{sd}(\xi)\to\sin(\xi).\end{aligned}$$

Appendix C

$$\mathrm{cd}(\xi)=\frac{\mathrm{cn}(\xi)}{\mathrm{dn}(\xi)},\quad \mathrm{dc}(\xi)=\frac{\mathrm{dn}(\xi)}{\mathrm{cn}(\xi)},$$

$$\mathrm{nc}(\xi)=\frac{1}{\mathrm{cn}(\xi)},\quad \mathrm{nd}(\xi)=\frac{1}{\mathrm{dn}(\xi)},$$

$$\mathrm{cs}(\xi)=\frac{\mathrm{cn}(\xi)}{\mathrm{sn}(\xi)},\quad \mathrm{sc}(\xi)=\frac{\mathrm{sn}(\xi)}{\mathrm{cn}(\xi)},$$

$$\mathrm{sd}(\xi)=\frac{\mathrm{sn}(\xi)}{\mathrm{dn}(\xi)},\quad \mathrm{ds}(\xi)=\frac{\mathrm{dn}(\xi)}{\mathrm{sn}(\xi)}.$$

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