



# Multiswitching compound antisynchronization of four chaotic systems

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**Abstract.** Based on three drive–one response system, in this article, the authors investigate a novel synchronization scheme for a class of chaotic systems. The new scheme, multiswitching compound antisynchronization (MSCoAS), is a notable extension of the earlier multiswitching schemes concerning only one drive–one response system model. The concept of multiswitching synchronization is extended to compound synchronization scheme such that the state variables of three drive systems antisynchronize with different state variables of the response system, simultaneously. The study involving multiswitching of three drive systems and one response system is first of its kind. Various switched modified function projective antisynchronization schemes are obtained as special cases of MSCoAS, for a suitable choice of scaling factors. Using suitable controllers and Lyapunov stability theory, sufficient condition is obtained to achieve MSCoAS between four chaotic systems and the corresponding theoretical proof is given. Numerical simulations are performed using Lorenz system in MATLAB to demonstrate the validity of the presented method.

**Keywords.** Chaos synchronization; multiswitching synchronization; compound synchronization; Lyapunov stability theory.

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## 1. Introduction

Synchronization of chaotic systems has been intensively investigated over the past two decades and lots of theoretical results have been obtained [1–5]. The potential interdisciplinary applications in physics, biological systems, electrical engineering, information processing, communication theory and many other fields have been extensively explored in the literature on chaos synchronization [6–10]. Due to the diverse nature of the chaotic systems, many types of synchronization methods have been proposed and investigated in the past years such as complete synchronization [11–13], antisynchronization [14,15], projective synchronization [16–18], lag synchronization [19–21], phase synchronization [22,23], reduced order synchronization [24,25], increased order synchronization [26,27], etc. Moreover, to achieve chaos synchronization, various methods have been developed and widely studied including active control method [28,29], adaptive control method

[30,31], sliding mode control [32,33], active backstepping method [34,35].

Most of the work in chaos synchronization, upto now, has been restricted to the synchronization studies between one drive system–one response system model. The synchronization problem among three or more chaotic systems is still a relatively unexplored area of research and deserves investigation. In the recent years, new ideas have been initiated in the study of chaos synchronization wherein three or more chaotic systems are involved. Synchronization schemes such as combination synchronization [36–39], combination–combination synchronization [40–43], compound synchronization [44,45], double compound synchronization [46], compound–combination synchronization [47, 48] etc. have recently been presented. In addition to their own intrinsic interest, these schemes are significant in enhancing the security of information transmitted via chaotic signals because of the complexity which they bring in transmitted signal.

Recently, Sun *et al* introduced the scheme of compound synchronization among four chaotic systems in [44]. In this method, the drive system is divided into two categories: Scaling drive system and base drive system. The scaling drive system scales the signals of two base drive systems, generating resultant signals. Then the response system is synchronized with the resultant signals. This scheme of compound synchronization is an extension and improvement of the existing synchronization schemes in the literature. In the existing literature on compound synchronization, the corresponding state variables of the drive systems have been combined to form a resultant signal which is in turn synchronized with the corresponding state variable of the response system.

Multiswitching synchronization was first proposed by Ucar *et al* in [49]. In the multiswitching synchronization scheme, different states of the drive system are synchronized with different state of the response system. Thus, a wide range of possible synchronization directions exist for multiswitching synchronization schemes. The relevance of this kind of synchronization schemes to information security makes them a very interesting topic to be explored. A few studies of this kind have been reported in the literature [50–52]. Almost the entire reported work in multiswitching synchronization relates to single drive and single response system. Only recently, multiswitching combination synchronization scheme involving multiple chaotic drive and response systems has been reported [53–56]. Nevertheless, the diverse possibilities of multiswitching synchronization have not been investigated yet with regards to compound synchronization involving three drive systems.

Motivated by the above discussions, in this paper, we present a new multiswitching compound antisynchronization (MSCoAS) scheme, wherein three drive chaotic systems are multiswitched in various manner to form a resultant signal which is then antisynchronized with some state variable of a single response chaotic system. To the best of our knowledge, multiswitching synchronization study involving four chaotic systems has not been reported before. Using Lyapunov stability theory and nonlinear controllers we propose sufficient condition for achieving MSCoAS. Numerical simulations have been performed to illustrate and verify the effectiveness of the proposed method. The main contributions of this study are: (a) The multiswitching scheme is extended to three drive and one response system and generalized for a class of chaotic systems. (b) The transmitted resultant compound signal is very complex and will thus provide improved performances for secure communication and information processing. (c) Suitable controllers are constructed which, in special cases,

adjust themselves accordingly to achieve novel modified function projective antisynchronization where the scaling factor is a chaotic system.

The paper is organized as follows. In §2 the formulation of MSCoAS is stated. In §3, three Lorenz multidrive chaotic systems are compound antisynchronized with Lorenz response chaotic system in multiswitched compound manner. Numerical simulations are performed to validate the scheme in §4. Finally, the conclusions are given in §5.

## 2. Formulation of multiswitching compound antisynchronization problem

In this section, we formulate the scheme of MSCoAS of chaotic systems. We need three chaotic drive systems and one response system. Let the scaling drive system be described by

$$\dot{x} = f_1(x) \quad (1)$$

and the base drive systems be given by

$$\dot{y} = f_2(y), \quad (2)$$

$$\dot{z} = f_3(z). \quad (3)$$

The response system is given by

$$\dot{w} = g(w) + u. \quad (4)$$

Here  $x = (x_1, x_2, x_3, \dots, x_n)^T$ ,  $y = (y_1, y_2, y_3, \dots, y_n)^T$ ,  $z = (z_1, z_2, z_3, \dots, z_n)^T$ , and  $w = (w_1, w_2, w_3, \dots, w_n)^T$  are state vectors of systems (1)–(4) respectively;  $f_1 = (f_{11}, f_{12}, f_{13}, \dots, f_{1n})^T$ ,  $f_2 = (f_{21}, f_{22}, f_{23}, \dots, f_{2n})^T$ ,  $f_3 = (f_{31}, f_{32}, f_{33}, \dots, f_{3n})^T$ , and  $g = (g_1, g_2, g_3, \dots, g_n)^T$  are four continuous vector functions,  $u = (u_1, u_2, \dots, u_n)^T : R^n \times R^n \times R^n \times R^n \rightarrow R^n$  are controllers to be designed for the response system (4). To solve the compound antisynchronization problem the error is defined as  $e = AX(BY + CZ) + DW$ .

### DEFINITION 1

If there exist four constant diagonal matrices  $A, B, C, D \in R^{n \times n}$  and  $D \neq 0$  such that

$$\lim_{t \rightarrow \infty} \|e\| = \lim_{t \rightarrow \infty} \|AX(BY + CZ) + DW\| = 0, \quad (5)$$

where  $\|\cdot\|$  is the matrix norm, then the drive systems (1)–(3) are said to be in compound antisynchronization with the response system (4). Here we assume  $X = \text{diag}(x_1, x_2, x_3, \dots, x_n)$ ,  $Y = \text{diag}(y_1, y_2, y_3, \dots, y_n)$ ,  $Z = \text{diag}(z_1, z_2, z_3, \dots, z_n)$ , and  $W = \text{diag}(w_1, w_2, w_3, \dots, w_n)$ . The constant matrices  $A, B, C, D$  are called the scaling matrices, drive system (1) is called the scaling drive system and drive systems (2) and (3) are called the base drive systems.

*Remark 1.* In Definition 1 the state vectors  $x = (x_1, x_2, x_3, \dots, x_n)^T$ ,  $y = (y_1, y_2, y_3, \dots, y_n)^T$ ,  $z = (z_1, z_2, z_3, \dots, z_n)^T$ , and  $w = (w_1, w_2, w_3, \dots, w_n)^T$  are designed as four diagonal matrices  $X = \text{diag}(x_1, x_2, x_3, \dots, x_n)$ ,  $Y = \text{diag}(y_1, y_2, y_3, \dots, y_n)$ ,  $Z = \text{diag}(z_1, z_2, z_3, \dots, z_n)$  and  $W = \text{diag}(w_1, w_2, w_3, \dots, w_n)$  respectively. Observe that by this definition error is obtained as a diagonal matrix

$$e = \text{diag}(e_1, e_2, e_3, \dots, e_n) = AX(BY + CZ) + DW. \tag{6}$$

*Remark 2.* If  $A = \text{diag}(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n)$ ,  $B = \text{diag}(\beta_1, \beta_2, \beta_3, \dots, \beta_n)$ ,  $C = \text{diag}(\gamma_1, \gamma_2, \gamma_3, \dots, \gamma_n)$ , and  $D = \text{diag}(\delta_1, \delta_2, \delta_3, \dots, \delta_n)$  then eq. (5) is equivalent to saying that compound antisynchronization is achieved between systems (1) and (4) when

$$\lim_{t \rightarrow \infty} e_l = \lim_{t \rightarrow \infty} [\alpha_i x_i (\beta_j y_j + \gamma_k z_k) + \delta_l w_l] = 0, \tag{7}$$

where  $e = \text{diag}(e_1, e_2, e_3, \dots, e_n)$  and  $l = 1, 2, \dots, n$ .

*Remark 3.* Let us rewrite the components of  $e$  as

$$e_{l(ijkl)} = \alpha_i x_i (\beta_j y_j + \gamma_k z_k) + \delta_l w_l, \tag{8}$$

where  $e = (e_{1(ijkl)}, e_{2(ijkl)}, e_{3(ijkl)}, \dots, e_{n(ijkl)})^T$ ,  $i, j, k, l = 1, 2, \dots, n$  and the subscript  $(ijkl)$  denotes the  $i$ th component of  $x$ ,  $j$ th component of  $y$ ,  $k$ th component of  $z$ , and  $l$ th component of  $w$ . In relation to Definition 1, the indices  $(ijkl)$  of the error states  $e_{l(ijkl)}$  are strictly chosen to satisfy  $i = j = k = l$  ( $i, j, k, l = 1, 2, \dots, n$ ).

**DEFINITION 2**

If the indices of the error states in (7) are redefined such that  $i = j = k \neq l$  or  $i = j = l \neq k$  or  $i = k = l \neq j$  or  $j = k = l \neq i$ ; or  $i = j \neq k = l$  or  $i = k \neq j = l$  or  $i = l \neq j = k$ ; or  $i = j \neq k \neq l$  or  $i = k \neq j \neq l$  or  $i = l \neq k \neq j$  or  $i \neq j = k \neq l$  or  $i \neq j \neq k = l$  or  $i \neq k \neq j = l$ ; or  $i \neq j \neq k \neq l$  and

$$\lim_{t \rightarrow \infty} e = \lim_{t \rightarrow \infty} \|AX(BY + CZ) + DW\| = 0, \tag{9}$$

where  $i, j, k, l = 1, 2, \dots, n$ , then the drive systems (1)–(3) are said to be in multiswitching compound antisynchronization with the response system (4).

*Remark 4.* If  $A \neq 0$ ,  $B = 0$  or  $C = 0$ , and  $D \neq 0$  then the MSCoAS will be turned into a novel type of switched modified function projective antisynchronization where the scaling factor is a chaotic system. Thus, the compound of two drive systems can synchronize a response system in multiswitching manner.

**3. Synchronization theory**

In this section, we achieve multiswitching compound antisynchronization among three chaotic drive systems (1)–(3) and one chaotic response system (4). Let the control functions be defined as

$$u_l = -\frac{1}{\delta_l} \phi_l - g_l - \frac{1}{\delta_l} K e, \quad i, j, k, l = 1, 2, \dots, n, \tag{10}$$

where

$$\phi_l = \alpha_1 f_{1i} (\beta_j y_j + \gamma_k z_k) + \alpha_i x_i (\beta_j f_{2j} + \gamma_k f_{3k}), \tag{11}$$

$$l, j, k, l = 1, 2, \dots, n$$

and  $K = (K_1, K_2, K_3, \dots, K_n)$  is the gain matrix controlling the rate of convergence and  $f_{1i}$ ,  $f_{2j}$ , and  $f_{3k}$ , are the  $i$ th,  $j$ th, and  $k$ th components of  $f_1$ ,  $f_2$ , and  $f_3$  respectively.

**Theorem 1.** *If the control functions are chosen as given in (10) then the drive systems (1)–(3) achieve multiswitching compound antisynchronization with the response system (4).*

*Proof.* Using (8) the error dynamical system can be written as

$$\begin{aligned} \dot{e}_{l(ijkl)} &= \alpha_i \dot{x}_i (\beta_j y_j + \gamma_k z_k) + \alpha_i x_i (\beta_j \dot{y}_j + \gamma_k \dot{z}_k) \\ &\quad + \delta_l \dot{w}_l, \quad i, j, k, l = 1, 2, \dots, n \\ &= \alpha_i f_{1i} (\beta_j y_j + \gamma_k z_k) + \alpha_i x_i (\beta_j f_{2j} + \gamma_k f_{3k}) \\ &\quad + \delta_l (g_l + u_l), \end{aligned} \tag{12}$$

where the indices  $(ijkl)$  satisfy one of the generic conditions given in Definition 2.

Let the Lyapunov function be defined as

$$\begin{aligned} V &= \frac{1}{2} e^T e \\ &= \frac{1}{2} \sum_{l=1}^n (e_{l(ijkl)})^2. \end{aligned}$$

The derivative  $\dot{V}$  is obtained as

$$\dot{V} = \sum_{l=1}^n e_{l(ijkl)} \dot{e}_{l(ijkl)}. \tag{13}$$

Using (12) in the above equation we get

$$\begin{aligned} \dot{V} &= \sum_{l=1}^n e_{l(ijkl)} [\alpha_i f_{1i} (\beta_j y_j + \gamma_k z_k) \\ &\quad + \alpha_i x_i (\beta_j f_{2j} + \gamma_k f_{3k}) + \delta_l (g_l + u_l)]. \end{aligned}$$

Using (10) in the above equation, we choose the gain matrix  $K$  in such a way that we get

$$\begin{aligned} \dot{V} &= \sum_{l=1}^n e_{l(ijkl)} (-K_l e_{l(ijkl)}) \\ &= \sum_{l=1}^n -K_l (e_{l(ijkl)})^2. \end{aligned}$$

Thus, we see that  $\dot{V}$  is negative definite. Using Lyapunov stability theory, we get  $\lim_{t \rightarrow \infty} \|e\| = 0$ . This means that the drive systems (1)–(3) achieve multiswitching compound antisynchronization with response system (4).  $\square$

The following corollaries are easily obtained from Theorem 1 and their proofs are omitted here.

**COROLLARY 1**

If  $A \neq 0, B = 0$ , and  $C \neq 0, i, j, k, l = 1, 2, \dots, n$ , and the control function is chosen as

$$\begin{aligned} u_l &= -\frac{\alpha_i \gamma_k}{\delta_l} [f_{1i} z_k + x_i f_{3k}] \\ &\quad -g_l - \frac{1}{\delta_l} K e, \quad i, j, k, l = 1, 2, \dots, n \end{aligned}$$

then the drive systems (1) and (3) achieve switched modified function projective antisynchronization with the response system (4).

**COROLLARY 2**

If  $A \neq 0, B \neq 0$ , and  $C = 0, i, j, k, l = 1, 2, \dots, n$ , and the control function is chosen as

$$\begin{aligned} u_l &= -\frac{\alpha_i \beta_j}{\delta_l} [f_{1i} y_j + x_i f_{2j}] - g_l - \frac{1}{\delta_l} K e, \\ i, j, k, l &= 1, 2, \dots, n \end{aligned}$$

then the drive systems (1) and (2) achieve switched modified function projective antisynchronization with the response system (4).

**COROLLARY 3**

If  $A = 0$ , and  $B = C = 0, i, j, k, l = 1, 2, \dots, n$ , and the control function is chosen as

$$u_l = -g_l - K e, \quad l = 1, 2, \dots, n$$

then the equilibrium point  $(0, 0, \dots, 0)$  of the response system (4) is asymptotically stable.

**4. Computational results and numerical simulations**

In this section, we achieve multiswitching compound antisynchronization among four identical Lorenz chaotic

systems. The scaling drive Lorenz system is represented by

$$\begin{cases} \dot{x}_1 = a_1(x_2 - x_1), \\ \dot{x}_2 = b_1 x_1 - x_1 x_3 - x_2, \\ \dot{x}_3 = x_1 x_2 - c_1 x_3, \end{cases} \tag{14}$$

and the two base drive systems are represented as follows:

$$\begin{cases} \dot{y}_1 = a_2(y_2 - y_1), \\ \dot{y}_2 = b_2 y_1 - y_1 y_3 - y_2, \\ \dot{y}_3 = y_1 y_2 - c_2 y_3, \end{cases} \tag{15}$$

$$\begin{cases} \dot{z}_1 = a_3(z_2 - z_1), \\ \dot{z}_2 = b_3 z_1 - z_1 z_3 - z_2, \\ \dot{z}_3 = z_1 z_2 - c_3 z_3. \end{cases} \tag{16}$$

The Lorenz response system is described as

$$\begin{cases} \dot{w}_1 = a_4(w_2 - w_1) + u_1, \\ \dot{w}_2 = b_4 w_1 - w_1 w_3 - w_2 + u_2, \\ \dot{w}_3 = w_1 w_2 - c_4 w_3 + u_3, \end{cases} \tag{17}$$

where  $u_1, u_2$ , and  $u_3$  are the controllers to be designed. In our synchronization scheme, we assume  $A = \text{diag}(\alpha_1, \alpha_2, \alpha_3)$ ,  $B = \text{diag}(\beta_1, \beta_2, \beta_3)$ ,  $C = \text{diag}(\gamma_1, \gamma_2, \gamma_3)$ , and  $D = \text{diag}(\delta_1, \delta_2, \delta_3)$ . The notations  $\alpha_i, \beta_j, \gamma_k, \delta_l$  ( $i, j, k, l = 1, 2, 3$ ) representing the scaling factors are set for the convenience of discussion and may assume different or same values in applications.

Various switching combinations exist for defining the error states for the drive–response systems (14)–(17). In this paper, we present results for one randomly selected error space vector combination formed out of several switching possibilities. Let us define the error  $e = (e_{1(2231)}, e_{2(3132)}, e_{3(1323)})$  where

$$\begin{cases} e_{1(2231)} = \alpha_2 x_2 (\beta_2 y_2 + \gamma_3 z_3) + \delta_1 w_1, \\ e_{2(3132)} = \alpha_3 x_3 (\beta_1 y_1 + \gamma_3 z_3) + \delta_2 w_2, \\ e_{3(1323)} = \alpha_1 x_1 (\beta_3 y_3 + \gamma_2 z_2) + \delta_3 w_3. \end{cases} \tag{18}$$

We refer eq. (18) as Switch 1 for simplicity. For Switch 1, the error dynamical system is given as follows:

$$\begin{cases} \dot{e}_{1(2231)} = \alpha_2 \dot{x}_2 (\beta_2 y_2 + \gamma_3 z_3) + \alpha_2 x_2 (\beta_2 \dot{y}_2 + \gamma_3 \dot{z}_3) \\ \quad + \delta_1 \dot{w}_1, \\ \dot{e}_{2(3132)} = \alpha_3 \dot{x}_3 (\beta_1 y_1 + \gamma_3 z_3) + \alpha_3 x_3 (\beta_1 \dot{y}_1 + \gamma_3 \dot{z}_3) \\ \quad + \delta_2 \dot{w}_2, \\ \dot{e}_{3(1323)} = \alpha_1 \dot{x}_1 (\beta_3 y_3 + \gamma_2 z_2) + \alpha_1 x_1 (\beta_3 \dot{y}_3 + \gamma_2 \dot{z}_2) \\ \quad + \delta_3 \dot{w}_3. \end{cases} \tag{19}$$

Combining with eqs (14)–(17) the error system (19) is transformed into the following form:

$$\begin{cases} \dot{e}_{1(2231)} = \alpha_2(b_1x_1 - x_1x_3 - x_2)(\beta_2y_2 + \gamma_3z_3) \\ \quad + \alpha_2x_2(\beta_2(b_2y_1 - y_1y_3 - y_2) \\ \quad + \gamma_3(z_1z_2 - c_3z_3)) + \delta_1(a_4(w_2 - w_1)) \\ \quad + \delta_1u_1, \\ \dot{e}_{2(3132)} = \alpha_3(x_1x_2 - c_1x_3)(\beta_1y_1 + \gamma_3z_3) \\ \quad + \alpha_3x_3(\beta_1a_2(y_2 - y_1) + \gamma_3(z_1z_2 - c_3z_3)) \\ \quad + \delta_2(b_4w_1 - w_1w_3 - w_2) + \delta_2u_2, \\ \dot{e}_{3(1323)} = \alpha_1a_1(x_2 - x_1)(\beta_3y_3 + \gamma_2z_2) \\ \quad + \alpha_1x_1(\beta_3(y_1y_2 - c_2y_3) \\ \quad + \gamma_2(b_3z_1 - z_1z_3 - z_2)) \\ \quad + \delta_3(w_1w_2 - c_4w_3) + \delta_3u_3. \end{cases} \tag{20}$$

**Theorem 2.** *If the control functions  $u_1, u_2,$  and  $u_3$  are chosen such that*

$$\begin{cases} u_1 = -\frac{1}{\delta_1}\phi_1 - a_4(w_2 - w_1) \\ \quad - \frac{1}{\delta_1}[\alpha_2x_2(\beta_2y_2 + \gamma_3z_3) + \delta_1w_1] \\ \quad + \frac{a_1}{\delta_1}[\alpha_3x_3(\beta_1y_1 + \gamma_3z_3) + \delta_2w_2] \\ \quad - \frac{a_2}{\delta_1}[\alpha_1x_1(\beta_3y_3 + \gamma_2z_2) + \delta_3w_3], \\ u_2 = -\frac{1}{\delta_2}\phi_2 - (b_4w_1 - w_1w_3 - w_2) \\ \quad - \frac{1}{\delta_2}[\alpha_3x_3(\beta_1y_1 + \gamma_3z_3) + \delta_2w_2] \\ \quad - \frac{a_1}{\delta_2}[\alpha_2x_2(\beta_2y_2 + \gamma_3z_3) + \delta_1w_1] \\ \quad + \frac{a_3}{\delta_2}[\alpha_1x_1(\beta_3y_3 + \gamma_2z_2) + \delta_3w_3], \\ u_3 = -\frac{1}{\delta_3}\phi_3 - (w_1w_2 - c_4w_3) \\ \quad - \frac{1}{\delta_3}[\alpha_1x_1(\beta_3y_3 + \gamma_2z_2) + \delta_3w_3] \\ \quad + \frac{a_2}{\delta_3}[\alpha_2x_2(\beta_2y_2 + \gamma_3z_3) + \delta_1w_1] \\ \quad - \frac{a_3}{\delta_3}[\alpha_3x_3(\beta_1y_1 + \gamma_3z_3) + \delta_2w_2], \end{cases} \tag{21}$$

then the drive systems (14)–(16) will achieve multistitching compound antisynchronization with the response system (17) where

$$\begin{cases} \phi_1 = \alpha_2(b_1x_1 - x_1x_3 - x_2)(\beta_2y_2 + \gamma_3z_3) \\ \quad + \alpha_2x_2(\beta_2(b_2y_1 - y_1y_3 - y_2) \\ \quad + \gamma_3(z_1z_2 - c_3z_3)), \\ \phi_2 = \alpha_3(x_1x_2 - c_1x_3)(\beta_1y_1 + \gamma_3z_3) \\ \quad + \alpha_3x_3(\beta_1a_2(y_2 - y_1) + \gamma_3(z_1z_2 - c_3z_3)), \\ \phi_3 = \alpha_1a_1(x_2 - x_1)(\beta_3y_3 + \gamma_2z_2) \\ \quad + \alpha_1x_1(\beta_3(y_1y_2 - c_2y_3) \\ \quad + \gamma_2(b_3z_1 - z_1z_3 - z_2)). \end{cases} \tag{22}$$

*Proof.* For simplicity, we rewrite system (20) as

$$\begin{cases} \dot{E}_1 = \phi_1 + \delta_1(a_4(w_2 - w_1)) + \delta_1u_1, \\ \dot{E}_2 = \phi_2 + \delta_2(b_4w_1 - w_1w_3 - w_2) + \delta_2u_2, \\ \dot{E}_3 = \phi_3 + \delta_3(w_1w_2 - c_4w_3) + \delta_3u_3, \end{cases} \tag{23}$$

where  $E_1 = e_{1(2231)}, E_2 = e_{2(3132)},$  and  $E_3 = e_{3(1323)}$  and  $\phi_1, \phi_2,$  and  $\phi_3$  are as given in eq. (22). Consider the Lyapunov function in the form of

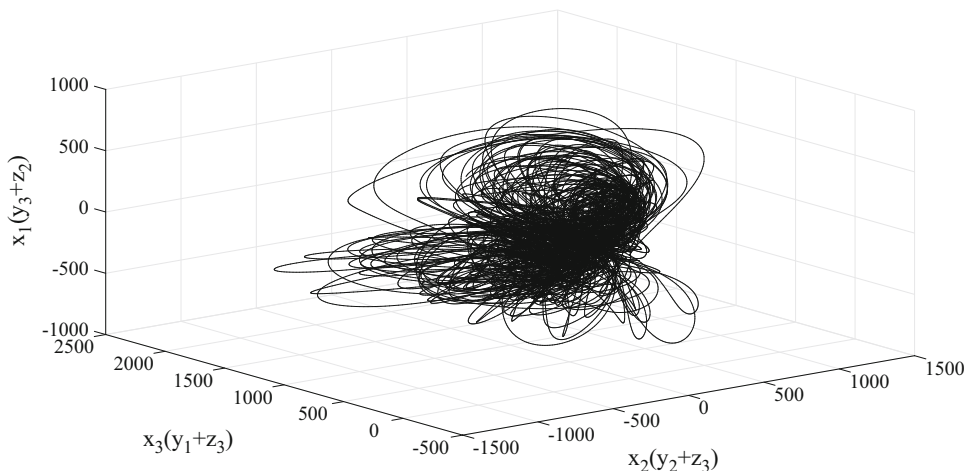
$$V(E(t)) = \frac{1}{2}(E_1^2 + E_2^2 + E_3^2). \tag{24}$$

The derivative of  $V$  along the trajectories of (23) is obtained as

$$\begin{aligned} \dot{V}(E(t)) &= E_1\dot{E}_1 + E_2\dot{E}_2 + E_3\dot{E}_3 \\ &= E_1(\phi_1 + \delta_1(a_4(w_2 - w_1)) + \delta_1u_1) \\ &\quad + E_2(\phi_2 + \delta_2(b_4w_1 - w_1w_3 \\ &\quad - w_2) + \delta_2u_2) \\ &\quad + E_3(\phi_3 + \delta_3(w_1w_2 - c_4w_3) + \delta_3u_3). \end{aligned} \tag{25}$$

Substituting the values of  $u_1, u_2,$  and  $u_3$  in (25), we get

$$\begin{aligned} \dot{V}(E(t)) &= E_1 \left\{ \phi_1 + \delta_1a_4(w_2 - w_1) \right. \\ &\quad + \delta_1 \left\{ -\frac{1}{\delta_1}\phi_1 - a_4(w_2 - w_1) \right. \\ &\quad - \frac{1}{\delta_1}[\alpha_2x_2(\beta_2y_2 + \gamma_3z_3) + \delta_1w_1] \\ &\quad + \frac{a_1}{\delta_1}[\alpha_3x_3(\beta_1y_1 + \gamma_3z_3) + \delta_2w_2] \\ &\quad \left. \left. - \frac{a_2}{\delta_1}[\alpha_1x_1(\beta_3y_3 + \gamma_2z_2) + \delta_3w_3] \right\} \right\} \\ &\quad + E_2 \left\{ \phi_2 + \delta_2(b_4w_1 - w_1w_3 - w_2) \right. \\ &\quad + \delta_2 \left\{ -\frac{1}{\delta_2}\phi_2 - (b_4w_1 - w_1w_3 - w_2) \right. \\ &\quad - \frac{1}{\delta_2}[\alpha_3x_3(\beta_1y_1 + \gamma_3z_3) + \delta_2w_2] \\ &\quad \left. \left. - \frac{a_1}{\delta_2}[\alpha_2x_2(\beta_2y_2 + \gamma_3z_3) + \delta_1w_1] \right\} \right\} \end{aligned}$$



**Figure 1.** 3D phase plot of compound of multidrive Lorenz system.

$$\begin{aligned}
 & + \frac{a_3}{\delta_2} [\alpha_1 x_1 (\beta_1 y_1 + \gamma_2 z_2) + \delta_3 w_3] \Big\} \\
 & + E_3 \left\{ \phi_3 + \delta_3 (w_1 w_2 - c_4 w_3) \right. \\
 & + \delta_3 \left\{ -\frac{1}{\delta_3} \phi_3 - (w_1 w_2 - c_4 w_3) \right. \\
 & - \frac{1}{\delta_3} [\alpha_1 x_1 (\beta_1 y_1 + \gamma_2 z_2) + \delta_3 w_3] \\
 & + \frac{a_2}{\delta_3} [\alpha_2 x_2 (\beta_2 y_2 + \gamma_3 z_3) + \delta_1 w_1] \\
 & \left. \left. - \frac{a_3}{\delta_3} [\alpha_3 x_3 (\beta_1 y_1 + \gamma_3 z_3) + \delta_2 w_2] \right\} \right\}.
 \end{aligned}$$

This can be written as

$$\begin{aligned}
 \dot{V}(E(t)) = & E_1 \{ -[\alpha_2 x_2 (\beta_2 y_2 + \gamma_3 z_3) + \delta_1 w_1] \\
 & + a_1 [\alpha_3 x_3 (\beta_1 y_1 + \gamma_3 z_3) + \delta_2 w_2] \\
 & - a_2 [\alpha_1 x_1 (\beta_1 y_1 + \gamma_2 z_2) + \delta_3 w_3] \} \\
 & + E_2 \{ -[\alpha_3 x_3 (\beta_1 y_1 + \gamma_3 z_3) + \delta_2 w_2] \\
 & - a_1 [\alpha_2 x_2 (\beta_2 y_2 + \gamma_3 z_3) + \delta_1 w_1] \\
 & + a_3 [\alpha_1 x_1 (\beta_1 y_1 + \gamma_2 z_2) + \delta_3 w_3] \} \\
 & + E_3 \{ -[\alpha_1 x_1 (\beta_1 y_1 + \gamma_2 z_2) + \delta_3 w_3] \\
 & + a_2 [\alpha_2 x_2 (\beta_2 y_2 + \gamma_3 z_3) + \delta_1 w_1] \\
 & - a_3 [\alpha_3 x_3 (\beta_1 y_1 + \gamma_3 z_3) + \delta_2 w_2] \}.
 \end{aligned}$$

This can be further rewritten as

$$\begin{aligned}
 \dot{V}(E(t)) = & E_1 (-E_1 + a_1 E_2 - a_2 E_3) \\
 & + E_2 (-E_2 - a_1 E_1 + a_3 E_3) \\
 & + E_3 (-E_3 + a_2 E_1 - a_3 E_2) \\
 = & -E_1^2 - E_2^2 - E_3^2 \\
 = & -E^T E,
 \end{aligned}$$

where  $E^T = (E_1, E_2, E_3)^T$ . Thus, we see that  $\dot{V}(E(t))$  is negative definite. According to Lyapunov stability theory, we know  $E_i \rightarrow 0 (i = 1, 2, 3)$ , that is,  $\lim_{t \rightarrow \infty} \|E\| = 0$ , which means that the drive systems (14)–(16) will achieve multiswitching compound anti-synchronization with the response system (17).  $\square$

*Remark 5.* In many previous studies, one common problem on the compound of multiple drive system is that the compound signal often is asymptotically stable or emanative. This is not desirable as the dynamic evolution of the signal obtained by compound of multidrive system is either too easy or completely useless for transmitting information signals. However, in our work, the resulting compound system is still chaotic and the dynamic evolution is more abundant and complex as can be seen in figure 1. This can be utilized to attain improved performances for secure communication and information processing in the future.

*Remark 6.* In Theorem 2, the designed control inputs  $u_1, u_2$ , and  $u_3$  are highly nonlinear in nature due to the high nonlinearity present in the structural design of the drive system signals where the resultant signal of the sum of two drive systems is being scaled by signals of the scaling drive system. To design a less complicated control input for achieving desired multiswitching compound antisynchronization will be the topic of our future research.

The following corollaries can be easily obtained from Theorem 2, but their proofs are omitted here for brevity. Suppose  $A \neq 0, B = 0$ , and  $C \neq 0$ , then we have the following corollary:

**COROLLARY 4**

If the control functions  $u_1, u_2,$  and  $u_3$  are chosen such that

$$\begin{aligned}
 u_1 &= -\frac{\alpha_2\gamma_3}{\delta_1}[(b_1x_1 - x_1x_3 - x_2)z_3 \\
 &\quad + x_2(z_1z_2 - c_3z_3)] \\
 &\quad - a_4(w_2 - w_1) - \frac{1}{\delta_1}[\alpha_2\gamma_3x_2z_3 + \delta_1w_1] \\
 &\quad + \frac{a_1}{\delta_1}[\alpha_3\gamma_3x_3z_3 + \delta_2w_2] \\
 &\quad - \frac{a_2}{\delta_1}[\alpha_1\gamma_2x_1z_2 + \delta_3w_3], \\
 u_2 &= -\frac{\alpha_3\gamma_3}{\delta_2}[(x_1x_2 - c_1x_3)z_3 + x_3(z_1z_2 - c_3z_3)] \\
 &\quad - (b_4w_1 - w_1w_3 - w_2) \\
 &\quad - \frac{1}{\delta_2}[\alpha_3\gamma_3x_3z_3 + \delta_2w_2] \\
 &\quad - \frac{a_1}{\delta_2}[\alpha_2\gamma_3x_2z_3 + \delta_1w_1] \\
 &\quad + \frac{a_3}{\delta_2}[\alpha_1\gamma_2x_1z_2 + \delta_3w_3], \\
 u_3 &= -\frac{\alpha_1\gamma_2}{\delta_3}[a_1(x_2 - x_1)z_2 + x_1(b_3z_1 - z_1z_3 - z_2)] \\
 &\quad - (w_1w_2 - c_4w_3) - \frac{1}{\delta_3}[\alpha_1\gamma_2x_1z_2 + \delta_3w_3] \\
 &\quad + \frac{a_2}{\delta_3}[\alpha_2\gamma_3x_2z_3 + \delta_1w_1] \\
 &\quad - \frac{a_3}{\delta_3}[\alpha_3\gamma_3x_3z_3 + \delta_2w_2],
 \end{aligned}$$

then the drive systems (14) and (16) will achieve a novel type of switched modified function projective antisynchronization with response system (17).

Suppose  $A \neq 0, B \neq 0,$  and  $C = 0,$  then we have the following corollary:

**COROLLARY 5**

If the control functions  $u_1, u_2,$  and  $u_3$  are chosen such that

$$\begin{aligned}
 u_1 &= -\frac{\alpha_2\beta_2}{\delta_1}[(b_1x_1 - x_1x_3 - x_2)y_2 \\
 &\quad + x_2(b_2y_1 - y_1y_3 - y_2)] \\
 &\quad - a_4(w_2 - w_1) - \frac{1}{\delta_1}[\alpha_2\beta_2x_2y_2 + \delta_1w_1] \\
 &\quad + \frac{a_1}{\delta_1}[\alpha_3\beta_1x_3y_1 + \delta_2w_2] \\
 &\quad - \frac{a_2}{\delta_1}[\alpha_1\beta_3x_1y_3 + \delta_3w_3],
 \end{aligned}$$

$$\begin{aligned}
 u_2 &= -\frac{\alpha_3\beta_1}{\delta_2}[(x_1x_2 - c_1x_3)y_1 + a_2x_3(y_2 - y_1)] \\
 &\quad - (b_4w_1 - w_1w_3 - w_2) \\
 &\quad - \frac{1}{\delta_2}[\alpha_3\beta_1x_3y_1 + \delta_2w_2] \\
 &\quad - \frac{a_1}{\delta_2}[\alpha_2\beta_2x_2y_2 + \delta_1w_1] \\
 &\quad + \frac{a_3}{\delta_2}[\alpha_1\beta_3x_1y_3 + \delta_3w_3], \\
 u_3 &= -\frac{\alpha_1\beta_3}{\delta_3}[a_1(x_2 - x_1)y_3 + x_1(y_1y_2 - c_2y_3)] \\
 &\quad - (w_1w_2 - c_4w_3) - \frac{1}{\delta_3}[\alpha_1\beta_3x_1y_3 + \delta_3w_3] \\
 &\quad + \frac{a_2}{\delta_3}[\alpha_2\beta_2x_2y_2 + \delta_1w_1] \\
 &\quad - \frac{a_3}{\delta_3}[\alpha_3\beta_1x_3y_1 + \delta_2w_2],
 \end{aligned}$$

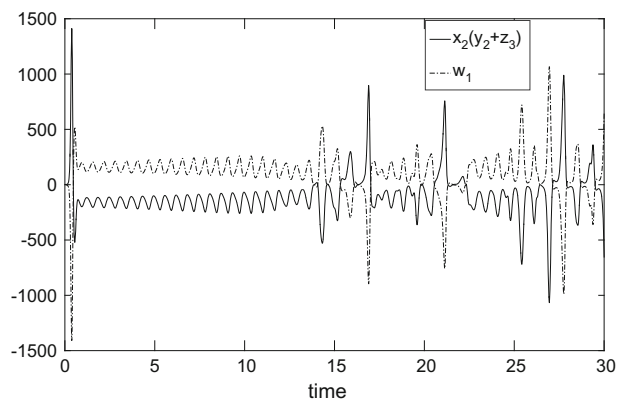
then the drive systems (14) and (15) will achieve a novel type of switched modified function projective antisynchronization with response system (17).

Suppose  $A = 0$  or  $B = C = 0,$  then we have the following corollary:

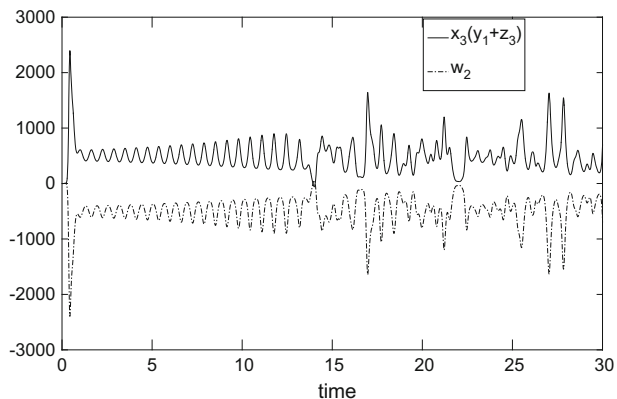
**COROLLARY 6**

If the control functions  $u_1, u_2,$  and  $u_3$  are chosen such that

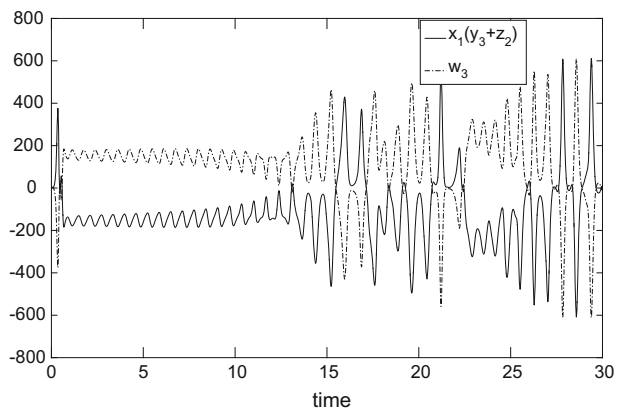
$$\begin{aligned}
 u_1 &= a_4(w_1 - w_2) - w_1 + \frac{a_1}{\delta_1}\delta_2w_2 - \frac{a_2}{\delta_1}\delta_3w_3, \\
 u_2 &= (w_1w_3 + w_2 - b_4w_1) - w_2 - \frac{a_1}{\delta_2}\delta_1w_1 \\
 &\quad + \frac{a_3}{\delta_2}\delta_3w_3, \\
 u_3 &= (c_4w_3 - w_1w_2) - w_3 + \frac{a_2}{\delta_3}\delta_1w_1 - \frac{a_3}{\delta_3}\delta_2w_2,
 \end{aligned}$$



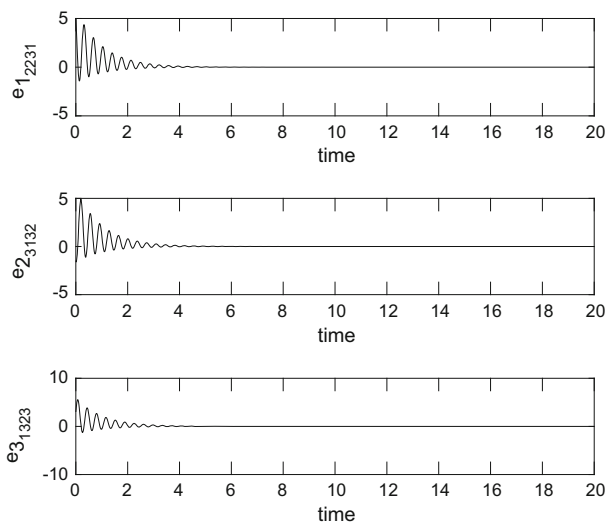
**Figure 2.** Response for states  $x_2(y_2 + z_3)$  and  $w_1$  for the drive systems (14)–(16) and the response system (17).



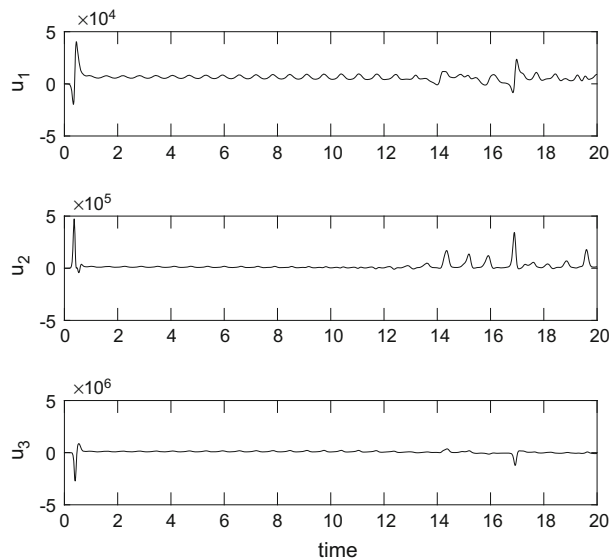
**Figure 3.** Response for states  $x_3(y_1 + z_3)$  and  $w_2$  for the drive systems (14)–(16) and the response system (17).



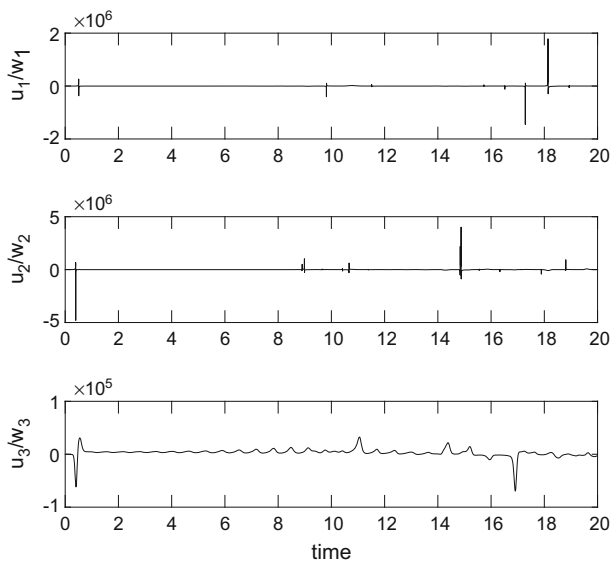
**Figure 4.** Response for states  $x_1(y_3 + z_2)$  and  $w_3$  for the drive systems (14)–(16) and the response system (17).



**Figure 5.** Synchronization errors  $e_{1(2231)}$ ,  $e_{2(3132)}$ , and  $e_{3(1323)}$  between the drive systems (14)–(16) and the response system (17).



**Figure 6.** Time response of the controllers used to achieve compound antisynchronization between the drive systems (14)–(16) and the response system (17).

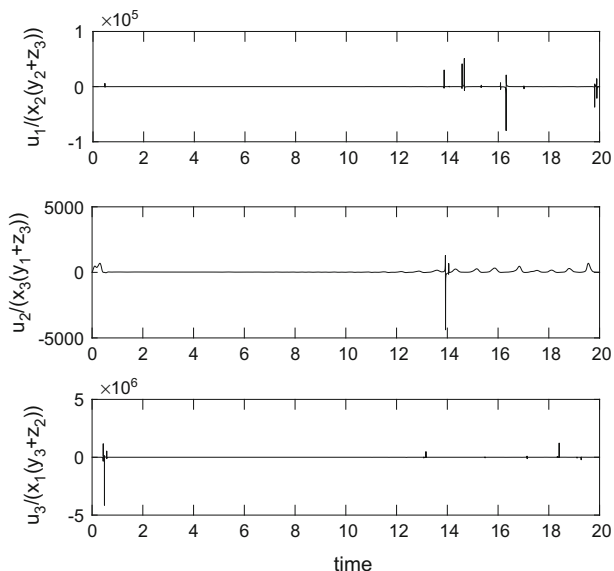


**Figure 7.** Ratios between the controllers  $u_1$ ,  $u_2$  and  $u_3$  and the uncontrolled response system signals  $w_1$ ,  $w_2$ , and  $w_3$ .

then the equilibrium point  $(0, 0, 0)$  of the response system (17) is asymptotically stable.

To demonstrate the effectiveness of our proposed method we perform numerical simulations in MATLAB using fourth-order Runge–Kutta method and give the results for Switch 1. In the simulation process we assume  $\alpha_1 = \alpha_2 = \alpha_3 = 1$ ,  $\beta_1 = \beta_2 = \beta_3 = 1$ ,  $\gamma_1 = \gamma_2 = \gamma_3 = 1$ , and  $\delta_1 = \delta_2 = \delta_3 = 1$ . Note that  $\delta_i$  is the scaling factor of the response system, and its value is set to





**Figure 8.** Ratios between the controllers  $u_1$ ,  $u_2$  and  $u_3$  and the drive system signals  $x_2(y_2 + z_3)$ ,  $x_3(y_1 + z_3)$ , and  $x_1(y_3 + z_2)$ .

unity to ensure that only the response system antisynchronizes with compound of multidrive system, while  $\alpha_i$ ,  $\beta_j$ , and  $\gamma_k$  may take any values. The system parameters of Lorenz system are taken as  $a_1 = a_2 = a_3 = a_4 = 10$ ,  $b_1 = b_2 = b_3 = b_4 = 28$ , and  $c_1 = c_2 = c_3 = c_4 = 8/3$  and initial states of the chaotic drive and response system are given by  $(x_1(0), x_2(0), x_3(0)) = (0, 1, 0.5)$ ,  $(y_1(0), y_2(0), y_3(0)) = (0.1, 0, 1.5)$ ,  $(z_1(0), z_2(0), z_3(0)) = (1, 0.5, 1)$  and  $(w_1(0), w_2(0), w_3(0)) = (4, -2, 3)$ . Figure 1 shows that the switched compound drive system remains chaotic. The time response of the synchronized states  $w_1$ ,  $w_2$ , and  $w_3$  of the response system with states  $x_2(y_2 + z_3)$ ,  $x_3(y_1 + z_3)$ , and  $x_1(y_3 + z_2)$  of the drive systems (14)–(16) respectively is illustrated in figures 2, 3, 4. Figure 5 displays the time response of synchronization errors  $e_{1(2231)}$ ,  $e_{2(3132)}$ , and  $e_{3(1323)}$ . Figures 2–5 show that the drive systems (14)–(16) achieve multiswitching compound antisynchronization successfully with the response system (17). Figure 6 shows the time response of the controllers used in Theorem 2 and figures 7 and 8 display the ratios of the controllers with the corresponding uncontrolled response system signal and compound drive system signal respectively.

### 5. Conclusions

In this paper, we have introduced a new type of synchronization involving four chaotic systems, namely MSCoAS. Using Lyapunov stability theory some sufficient conditions are obtained for achieving MSCoAS

of four chaotic systems. In this new synchronization scheme, the state variables involved in the compound of multidrive system are multiswitched in various ways to antisynchronize with different state variables of the response system. The main advantages of the proposed scheme can be summarized as:

(i) For synchronization achieved in this manner, the possible combinations for error space vectors in which synchronization may take place is very large due to multiswitching. In the context of secure communication applications [44,45], this scheme will provide better resistance and antiattack ability than normal synchronization schemes as it would be very difficult for the intruder to predetermine the combination in which synchronization would occur.

(ii) The proposed scheme theoretically guarantees good control performance.

(iii) The proposed scheme will be helpful in synchronizing multiple chaotic systems and producing complex resultant signals which will further strengthen the security of the transmitted signals.

(iv) A novel type of switched modified function projective synchronization is obtained as a special case of MSCoAS.

The main disadvantage of the obtained results lies in the highly nonlinear nature of the designed controller. Numerical simulations are performed using Lorenz system to demonstrate the validity and effectiveness of our proposed scheme. The presented scheme MSCoAS may form the basis of various other synchronization studies in future. Using fractional chaotic systems as the drive and the response systems, or utilizing function scaling factors or considering the chaotic system with parameter uncertainty are some interesting directions for future work.

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