Nonlinear structures for extended Korteweg–de Vries equation in multicomponent plasma

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Abstract. Using the fluid hydrodynamic equations of positive and negative ions, as well as q-nonextensive electron density distribution, an extended Korteweg–de Vries (EKdV) equation describing a small but finite amplitude dust ion-acoustic waves (DIAWs) is derived. Extended homogeneous balance method is used to obtain a new class of solutions of the EKdV equation. The effects of different physical parameters on the propagating nonlinear structures and their relevance to particle acceleration in space plasma are reported.

1. Introduction

Negative ion plasma is the plasma which contains both negative and positive ion species in addition to electrons. This type of plasma has great importance in various fields of plasma science and technology. The existence of a considerable number of negative ions in the Earth's ionosphere [1] and cometary comae [2] is well known. Positive–negative ion plasmas are found in plasma processing reactors [3,4], neutral beam sources [5], and low-temperature laboratory experiments [6,7]. Furthermore, negative ions are found in

the upper region of Titan atmosphere [8,9]. These particles may act as organic building blocks for even more complicated molecules. In this kind of plasma, the electron number density decreases according to the charge neutrality, i.e., $n_e = n_+ - n_-$, where n_e, n_+, and n_- are the electron, positive and negative ion densities, respectively. This results in a decrease in the shielding effect of the electrons. So, most of the phenomena are actually affected by the negative ions themselves, as well as by the lack of electrons [7]. Under specific laboratory conditions, the presence of nanodust clusters could change the plasma behaviour. These clusters could be considered as immobile or mobile charged nanodust grains. The presence of immobile nanodust grains changes the general properties of the propagated linear and nonlinear waves that are produced by the positive ions [10].

It is well known that different nonlinear equations are widely employed to describe many complex phenomena in science, e.g., fluid mechanics, plasma physics, optical fibres, solid-state physics, geophysics, etc. Various techniques such as inverse scattering method [11], bilinear transformation [12], tanh-function method [13], extended tanh method [14], sine–cosine method [15], F-expansion method [16], general expansion method [17], G'/G method [18,19], homogeneous balance (HB) [20], etc. were used to obtain the solutions of these nonlinear equations. The HB method is a direct and an effective algebraic method to determine the exact travelling wave solutions. Interestingly, the homogeneous balance (HB) method was extended to investigate other kinds of exact solutions [21,22] in addition to solitary solutions. Fan [23] described two new applications of the homogeneous balance method and explored for Backlund transformation and similarity reduction of nonlinear partial differential equations. Fan showed that there is a definite correlation among the HB, the Weiss–Tabor–Carnevale (WTC), and the Clarkson–Kruskal (CK) methods. The aim of this work is to use the HB method to solve the evolution equation describing the present model namely, the extended Korteweg–de Vries (EKdV) equation, and obtain a class of appropriate solutions to describe the possible nonlinear waves in negative ion plasma.

This paper is organized as follows: In §2, we present the governing equations for the positive–negative ion plasmas. In §3, the reductive perturbation method is employed to derive the EKdV equation describing the system. The HB method is applied to obtain possible solutions of the EKdV equation. Discussion and numerical results are presented in §4. Finally, the results are summarized in §5.

2. Basic equations and formulation of the problem

We consider a one-dimensional, collisionless, unmagnetized plasma consisting of positive ions, negative ions, electrons, and stationary (positive/negative) charged dust impurities. The description of such plasma is governed by the fluid equations

$$\frac{\partial n_+}{\partial t} + \frac{\partial}{\partial x}(n_+ u_+) = 0, \tag{1}$$

$$m_+ n_+ \left(\frac{\partial}{\partial t} + u_+ \frac{\partial}{\partial x}\right) u_+ + \frac{\partial P_+}{\partial x} + e Z_+ n_+ \frac{\partial \phi}{\partial x} = 0, \tag{2}$$

for positive ions,

$$\frac{\partial n_-}{\partial t} + \frac{\partial}{\partial x}(n_- u_-) = 0, \tag{3}$$

$$m_- n_- \left(\frac{\partial}{\partial t} + u_- \frac{\partial}{\partial x}\right) u_- + \frac{\partial P_-}{\partial x} - e Z_- n_- \frac{\partial \phi}{\partial x} = 0, \tag{4}$$

for negative ions, and

$$n_e = n_e^{(0)} \left[1 + (q-1)\frac{e\phi}{k_B T_e}\right]^{(q+1)/2(q-1)}, \tag{5}$$

for electrons.

Here $n_{+,-,e}$ is the positive ion/negative ion/electron number density, $u_{+,-}$ is the positive/negative ion fluid velocity, $m_{+,-}$ is the positive/negative ion mass, ϕ is the electrostatic wave potential and e is the magnitude of the electron charge. The ion pressure is assumed to be adiabatic and is expressed by $P_s = n_s^{(0)} k_B T_s n_s^3 (s = +, -)$, k_B is the Boltzmann constant, T_s and T_e are the positive (negative) ions, and electron temperatures, $n_{e,+,-}^{(0)}$ is the equilibrium density for the electrons, positive ions, and negative ions.

The system of equations is closed with the Poisson equation

$$\frac{\partial^2 \phi}{\partial x^2} = 4\pi e(Z_+ n_+ - Z_- n_- - n_e + \rho Z_d n_d), \tag{6}$$

where n_d is the dust number density, Z_d is the dust charge number, and the symbol $\rho = \pm$ is used for positively or negatively charged dust impurities. In equilibrium, the neutrality condition of the plasma is satisfied, viz., $Z_+ n_+^{(0)} - Z_- n_-^{(0)} - n_e^{(0)} + \rho Z_d n_d = 0$.

The normalized set of the above dynamic equations can be written as

$$\frac{\partial \bar{n}_+}{\partial \bar{t}} + \frac{\partial(\bar{n}_+ \bar{u}_+)}{\partial \bar{x}} = 0, \tag{7}$$

$$\frac{\partial \bar{u}_+}{\partial \bar{t}} + \bar{u}_+ \frac{\partial \bar{u}_+}{\partial \bar{x}} + \sigma_+ \bar{n}_+ \frac{\partial \bar{n}_+}{\partial \bar{x}} + \frac{\partial \bar{\phi}}{\partial \bar{x}} = 0, \tag{8}$$

for the positive ions,

$$\frac{\partial \bar{n}_-}{\partial \bar{t}} + \frac{\partial(\bar{n}_- \bar{u}_-)}{\partial \bar{x}} = 0, \tag{9}$$

$$\frac{\partial \bar{u}_-}{\partial \bar{t}} + \bar{u}_- \frac{\partial \bar{u}_-}{\partial \bar{x}} + \sigma_- Q_- \bar{n}_- \frac{\partial \bar{n}_-}{\partial \bar{x}} - Q_- \Delta_- \frac{\partial \bar{\phi}}{\partial \bar{x}} = 0, \tag{10}$$

for the negative ions, and

$$n_e = \left[1 + (q-1)\bar{\phi}\right]^{(q+1)/2(q-1)}, \tag{11}$$

for electrons, and finally the Poisson's equation

$$\frac{\partial^2 \bar{\phi}}{\partial \bar{x}^2} = \bar{n}_+ - \alpha \Delta_- \bar{n}_- - \gamma \Delta_+ \bar{n}_e + \rho \beta. \tag{12}$$

Here, $Q_- = m_+/m_-$ is the mass ratio, $\sigma_{+,-} = 3T_{+,-}/T_e$, $\Delta_- = Z_-/Z_+$, and $\Delta_+ = 1/Z_+$.

Now, the neutrality condition is given by

$$1 = \alpha \Delta_- + \gamma \Delta_+ - \rho \beta, \tag{13}$$

where $\alpha = n_-^{(0)}/n_+^{(0)}$, $\gamma = n_e^{(0)}/n_+^{(0)}$, and $\beta = Z_d n_d/Z_+ n_+^{(0)}$.

In eqs (7)–(12), the densities for the positive ions, negative ions, and electrons are normalized with their equilibrium densities. The velocities of the positive ions and negative ions are normalized by the ion-acoustic speed of the positive ions, $C_{si} = (k_B T_e/m_+)^{1/2}$ and the potential ϕ is normalized by $k_B T_e/e$. The space and time are normalized by the positive ion Debye length $\lambda_{Di} = (k_B T_e/4\pi n_+^{(0)} e^2 Z_+^2)^{1/2}$ and the positive ion plasma period $\omega_{pi}^{-1} = (4\pi n_+^{(0)} e^2 Z_+^2/m_+)^{-1/2}$, respectively. The upper bar in eqs (7)–(12) will be omitted henceforth.

3. Derivation of the evolution equation

Now, we derive a dynamical equation for the nonlinear propagation of the dust ion-acoustic waves (DIAWs) using eqs (7)–(12). We employ the reductive perturbation technique, and accordingly we introduce the stretching space-time coordinates

$$\xi = \epsilon(x - \lambda t) \quad \text{and} \quad \tau = \epsilon^3 t, \tag{14}$$

where ϵ is a smallness parameter ($0 < \epsilon \ll 1$) measuring the strength of nonlinearity and λ is the wave propagation speed. Furthermore, the dependent variables are expanded as a power series in ϵ around their corresponding equilibrium values as

$$\Psi = \Psi^{(0)} + \sum_{n=1}^{\infty} \epsilon^n \Psi^{(n)}, \tag{15}$$

where

$$\Psi = \{n_+, n_-, n_e, u_+, u_-, \phi\}^T \tag{16}$$

and

$$\Psi^{(0)} = \{1, 1, 1, 0, 0, 0\}^T. \tag{17}$$

Substituting eqs (14)–(17) in eqs (7)–(12) allows us to develop equations in various powers of ϵ. The lowest-order equations of ϵ read as

$$n_e^{(1)} = \frac{q+1}{2}\phi^{(1)}, \quad n_-^{(1)} = \frac{-Q_- \Delta_-}{(\lambda^2 - \sigma_- Q_-)}\phi^{(1)}, \quad u_-^{(1)} = \frac{-\lambda Q_- \Delta_-}{(\lambda^2 - \sigma_- Q_-)}\phi^{(1)} \tag{18}$$

and

$$n_+^{(1)} = \frac{1}{(\lambda^2 - \sigma_+)}\phi^{(1)}, \quad u_+^{(1)} = \frac{\lambda}{(\lambda^2 - \sigma_+)}\phi^{(1)}. \tag{19}$$

The Poisson equation then provides the compatibility condition

$$\frac{1}{(\lambda^2 - \sigma_+)} + \frac{\alpha Q_- \Delta_-^2}{(\lambda^2 - \sigma_- Q_-)} - \gamma \Delta_+ \frac{q+1}{2} = 0. \tag{20}$$

To the next order in ϵ, we obtain a set of equations in the second-order perturbed quantities which can be solved using eqs (18) and (19) to give the second-order perturbed quantities as follows:

$$n_+^{(2)} = \frac{1}{(\lambda^2 - \sigma_+)}\left[\phi^{(2)} + \frac{(3\lambda^2 + \sigma_+)}{2(\lambda^2 - \sigma_+)^2}\phi^{(1)2}\right], \tag{21}$$

$$u_+^{(2)} = \frac{\lambda}{(\lambda^2 - \sigma_+)}\left[\phi^{(2)} + \frac{1}{2}\left(\frac{(3\lambda^2 + \sigma_+)}{(\lambda^2 - \sigma_+)^2} - \frac{2}{(\lambda^2 - \sigma_+)}\right)\phi^{(1)2}\right], \tag{22}$$

$$n_-^{(2)} = \frac{Q_-\Delta_-}{(\lambda^2 - \sigma_- Q_-)}\left[-\phi^{(2)} + \frac{Q_-\Delta_-(3\lambda^2 + \sigma_- Q_-)}{2(\lambda^2 - \sigma_- Q_-)^2}\phi^{(1)2}\right], \tag{23}$$

$$u_-^{(2)} = \frac{\lambda Q_-\Delta_-}{(\lambda^2 - \sigma_- Q_-)}\left[-\phi^{(2)} + \frac{Q_-\Delta_-}{2}\left(\frac{(3\lambda^2 + \sigma_- Q_-)}{(\lambda^2 - \sigma_- Q_-)^2} - \frac{2}{(\lambda^2 - \sigma_- Q_-)}\right)\phi^{(1)2}\right], \tag{24}$$

$$n_e^{(2)} = \frac{q+1}{2}\phi^{(2)} + \frac{(3 + 2q - q^2)}{8}\phi^{(1)2}, \tag{25}$$

while Poisson equation gives

$$\left[\frac{1}{(\lambda^2 - \sigma_+)} + \frac{\alpha Q_-\Delta_-^2}{(\lambda^2 - \sigma_- Q_-)} - \gamma\Delta_+\frac{q+1}{2}\right]\phi^{(2)} + B\phi^{(1)2} = 0, \tag{26}$$

where

$$B = \frac{1}{2}\left[\frac{(3\lambda^2 + \sigma_+)}{(\lambda^2 - \sigma_+)^3} - \frac{\alpha Q_-^2\Delta_-^3(3\lambda^2 + \sigma_- Q_-)}{(\lambda^2 - \sigma_- Q_-)^3} - \frac{\gamma\Delta_+(3 + 2q - q^2)}{8}\right]. \tag{27}$$

The coefficient of $\phi^{(2)}$ is zero due to the condition (20) and $\phi^{(1)} \neq 0$, and therefore, B should be at least of the order of ϵ and now $B\phi^{(1)2}$ becomes of the order of ϵ^3; so it should be included in the next order of Poisson equation. If we consider the next order in ϵ, we obtain a set of equations in the third-order perturbed quantities, which can be solved with the help of eqs (18)–(25) to give

$$\begin{aligned}
\frac{\partial n_+^{(3)}}{\partial \xi} = &\frac{2\lambda}{(\lambda^2 - \sigma_+)^2}\frac{\partial \phi^{(1)}}{\partial \tau} + \frac{(3\lambda^2 + \sigma_+)}{(\lambda^2 - \sigma_+)^3}\frac{\partial(\phi^{(1)}\phi^{(2)})}{\partial \xi} \\
&+ \frac{3}{(\lambda^2 - \sigma_+)^5}\left[\lambda^2(\lambda^2 + 3\sigma_+) + \frac{1}{2}(\lambda^2 + \sigma_+)(3\lambda^2 + \sigma_+)\right]\phi^{(1)2}\frac{\partial \phi^{(1)}}{\partial \xi} \\
&+ \frac{1}{(\lambda^2 - \sigma_+)}\frac{\partial \phi^{(3)}}{\partial \xi},
\end{aligned} \tag{28}$$

$$\frac{\partial n_-^{(3)}}{\partial \xi} = \frac{-2\lambda Q_- \Delta_-}{(\lambda^2 - \sigma_- Q_-)^2} \frac{\partial \phi^{(1)}}{\partial \tau} + Q_-^2 \Delta_-^2 \frac{(3\lambda^2 + \sigma_- Q_-)}{(\lambda^2 - \sigma_- Q_-)^3} \frac{\partial(\phi^{(1)} \phi^{(2)})}{\partial \xi}$$
$$- \frac{3 Q_-^3 \Delta_-^3}{(\lambda^2 - \sigma_- Q_-)^5} \left[\lambda^2(\lambda^2 + 3\sigma_- Q_-) + \frac{1}{2}(\lambda^2 + \sigma_- Q_-)(3\lambda^2 + \sigma_- Q_-) \right]$$
$$\times \phi^{(1)2} \frac{\partial \phi^{(1)}}{\partial \xi} - \frac{Q_- \Delta_-}{(\lambda^2 - \sigma_- Q_-)} \frac{\partial \phi^{(3)}}{\partial \xi}, \tag{29}$$

$$\frac{\partial n_e^{(3)}}{\partial \xi} = \frac{q+1}{2} \frac{\partial \phi^{(3)}}{\partial \xi} + \frac{(3 + 2q - q^2)}{4} \frac{\partial(\phi^{(1)} \phi^{(2)})}{\partial \xi}$$
$$+ \frac{1}{16}(q-3)(q+1)(3q-5)\phi^{(1)2} \frac{\partial \phi^{(1)}}{\partial \xi}. \tag{30}$$

The Poisson equation of this order yields

$$\frac{\partial^2 \phi^{(1)}}{\partial \xi^2} = \left(n_+^{(3)} - n_+^{(2)} \right) - \alpha \Delta_- \left(n_-^{(3)} - n_-^{(2)} \right) - \gamma \Delta_+ \left(n_e^{(3)} - n_e^{(2)} \right). \tag{31}$$

Differentiating eq. (31) and using eqs (21), (23), (25), (28)–(30), we obtain the EKdV equation

$$\frac{\partial u}{\partial \tau} + ABu \frac{\partial u}{\partial \xi} + ACu^2 \frac{\partial u}{\partial \xi} + \frac{1}{2} A \frac{\partial^3 u}{\partial \xi^3} = 0, \tag{32}$$

where $\phi^{(1)}$ is replaced by u for simplicity. The coefficients A and C are given as

$$A = -\frac{1}{\lambda} \left[\frac{1}{(\lambda^2 - \sigma_+)^2} + \frac{\alpha Q_- \Delta_-^2}{(\lambda^2 - \sigma_- Q_-)^2} \right]^{-1}, \tag{33}$$

$$C = -\frac{3}{2(\lambda^2 - \sigma_+)^5} \left[\lambda^2(\lambda^2 + 3\sigma_+) + \frac{1}{2}(\lambda^2 + \sigma_+)(3\lambda^2 + \sigma_+) \right]$$
$$- \frac{3\alpha Q_-^3 \Delta_-^4}{2(\lambda^2 - \sigma_- Q_-)^5} \left[\lambda^2(\lambda^2 + 3\sigma_- Q_-) + \frac{1}{2}(\lambda^2 + \sigma_- Q_-)(3\lambda^2 + \sigma_- Q_-) \right]$$
$$+ \frac{\gamma \Delta_+}{16}(q-3)(q+1)(3q-5). \tag{34}$$

It is known that if one solves the basic set of fluid eqs (7)–(12) exactly to obtain the energy equation including the Sagdeev potential, then the obtained evolution equation describes the large/finite amplitude wave. When the Sagdeev potential is expanded for small but finite amplitude limit, we obtain the same result as predicted by the reductive perturbation theory. The tricky point here is to make the expansion carefully to obtain the same coefficients. So, considering B has small value is just a bridge to maintain the large-amplitude limit with the small-amplitude limit that is covered by the perturbation theory. There are many papers to prove this point (e.g., [24] and [25]).

4. Use the HB method to solve the EKdV equation

Consider the EKdV eq. (32) in the form

$$\frac{\partial u}{\partial \tau} + \Gamma u \frac{\partial u}{\partial \xi} + \Lambda u^2 \frac{\partial u}{\partial \xi} + \Omega \frac{\partial^3 u}{\partial \xi^3} = 0, \tag{35}$$

where $\Gamma = AB$, $\Lambda = AC$, and $\Omega = A/2$. We seek for the special solution of eq. (35), the travelling wave solution, in the form

$$u(\xi, \tau) = u(\zeta), \quad \zeta = \xi - \vartheta\,\tau, \tag{36}$$

where ϑ is a constant to be determined later. Using the transformation (36) in eq. (35), eq. (35) reduces to a nonlinear ordinary differential equation (ODE) which will be solved later. The next crucial step is to express the solution of eq. (35) in the form

$$u(\zeta) = \sum_{i=0}^{n} a_i \omega^i + \sum_{i=1}^{n} b_i \left[1 + \omega\right]^{-i} \tag{37}$$

and

$$\omega' = k + M\omega + P\omega^2, \tag{38}$$

where a_i and b_i are constants, while k, M, and P are parameters to be determined later, $\omega = \omega(\zeta)$ and $\omega' = \mathrm{d}\omega/\mathrm{d}\zeta$. To determine the parameter n, it is necessary to create a balance between the highest-order linear term and the nonlinear terms. Substituting (37) and (38) in the relevant ODE form of eq. (35) yields a system of ODEs with respect to a_0, a_i, b_i, k, M, P, and ϑ (where $i = 1, ..., m$), because all the coefficients of ω^j (where $j = 0, 1, ...$) have to vanish. Using *Mathematica*, one can determine a_0, a_i, b_i, k, M, P, and ϑ.

It is noted that eq. (38) has a form of Riccati equation, which can be solved using the HB method as follows:

Case I. When $P = 1$ and $M = 0$, the Riccati eq. (38) has the following solutions:

$$\omega = \begin{cases} -\sqrt{-k}\,\tanh[\sqrt{-k}\zeta], & \text{with } k < 0, \\ -\sqrt{-k}\,\coth[\sqrt{-k}\zeta], & \text{with } k < 0, \end{cases} \tag{39}$$

$$\omega = -\frac{1}{\zeta}, \quad \text{with } k = 0, \tag{40}$$

and

$$\omega = \begin{cases} \sqrt{k}\,\tan[\sqrt{k}\zeta], & \text{with } k > 0, \\ -\sqrt{k}\,\cot[\sqrt{k}\zeta], & \text{with } k > 0. \end{cases} \tag{41}$$

As coth- and cot-type solutions appear in pairs with tanh- and tan-type solutions, respectively, they are omitted in this paper.

Case II. Let $\omega = \sum_{i=0}^{m} A_i \tanh^i (p_1\zeta)$. Balancing ω' with ω^2 leads to

$$\omega = A_0 + A_1 \tanh(p_1\zeta). \tag{42}$$

Substituting eq. (42) in (38), we have the following solution of eq. (38):

$$\omega = -\frac{p_1}{2P} \tanh\left(\frac{p_1}{2}\zeta\right) - \frac{M}{2P}, \quad \text{with} \quad Pk = \frac{M^2 - p_1^2}{4}. \tag{43}$$

Similarly, let $\omega = \sum_{i=0}^{m} A_i \coth^i (p_1 \zeta)$, then we obtain the following solution:

$$\omega = -\frac{p_1}{2P} \coth\left(\frac{p_1}{2}\zeta\right) - \frac{M}{2P},$$

with $Pk = (M^2 - p_1^2)/4$, where k, M, p_1, and P are constants.

Case III. We suppose that the Riccati eq. (38) has the following solutions of the form

$$\omega = A_0 + \sum_{i=0}^{m}(A_i f^i + B_i f^{i-1} g), \tag{44}$$

with

$$f = \frac{1}{\cosh\zeta + r} \quad \text{and} \quad g = \frac{\sinh\zeta}{\cosh\zeta + r}. \tag{45}$$

Substituting eqs (44) and (45) in (38), we have the following solution of eq. (38):

$$\omega = -\frac{1}{2P}\left(M + \frac{\sinh(\zeta) + \sqrt{r^2 - 1}}{\cosh(\zeta) + r}\right), \quad \text{with} \quad Pk = \frac{M^2 - 1}{4}, \tag{46}$$

where r is the arbitrary constant. It should be noticed that solution (46), as $r = 1$, degenerates to

$$\omega = -\frac{1}{2P}\left[M + \tanh\left(\frac{\zeta}{2}\right)\right]. \tag{47}$$

Case IV. We suppose that the Riccati eq. (38) has the following solutions of the form

$$\omega = A_0 + \sum_{i=0}^{m} \sinh^{i-1}(A_i \sinh\varrho + B_i \cosh\varrho), \tag{48}$$

where $d\varrho/d\zeta = \sinh\varrho$ or $d\varrho/d\zeta = \cosh\varrho$. Balancing ω' with ω^2 leads to $m = 1$

$$\omega = A_0 + A_1 \sinh\varrho + B_1 \cosh\varrho, \tag{49}$$

when $d\varrho/d\zeta = \sinh\varrho$, we substitute (49) and $d\varrho/d\zeta = \sinh\varrho$ into (38) and set the coefficient of $\sinh^i \varrho \cosh^j \varrho$, $i = 0, 1, 2$, $j = 0, 1$ to zero and on solving the obtained set of algebraic equations we get

$$A_0 = \frac{-M}{2P}, \quad A_1 = 0, \quad B_1 = \frac{1}{P}, \tag{50}$$

where $k = (M^2 - 4)/4P$ and

$$A_0 = \frac{-M}{2P}, \quad A_1 = \pm\sqrt{\frac{1}{2P}}, \quad B_1 = \frac{1}{P}, \tag{51}$$

where $k = (M^2 - 1)/4P$. When $\mathrm{d}\varrho/\mathrm{d}\zeta = \sinh\varrho$ we have

$$\sinh\varrho = -\operatorname{cosech}\zeta, \quad \cosh\varrho = -\coth\zeta. \tag{52}$$

From (50)–(52), we obtain

$$\omega = -\frac{M + 2\coth\zeta}{2P}, \tag{53}$$

for $k = (M^2 - 4)/4P$ and

$$\omega = -\frac{M \pm \operatorname{cosech}\zeta + \coth\zeta}{2P}, \tag{54}$$

for $k = (M^2 - 1)/4P$.

On introducing different classes of solutions of the EKdV eq. (35), we determine the direct solutions of eq. (35) with clear details of the used methodology. We shall use the transformation $u(x, t) = U(\zeta), \zeta = x - \vartheta t$ in eq. (35). Therefore, eq. (35) reduces to the following ODE:

$$-\vartheta U' + \Gamma U U' + \Lambda U^2 U' + \Omega U''' = 0. \tag{55}$$

Integrating eq. (55) twice with respect to ζ, we get

$$-\lambda U + \frac{\Gamma}{2} U^2 + \frac{\Lambda}{3} U^3 + \Omega U'' = 0. \tag{56}$$

Using eq. (37) and balancing U'' with U^3 yields $n = 1$. Therefore, we are looking for the solution of the form

$$U = a_0 + b_0 + a_1\omega + b_1(1 + \omega)^{-1}. \tag{57}$$

Substituting eqs (57) and (38) in eq. (56), we get a polynomial equation ω. Hence, equating the coefficient of ω^j $(j = 0, 1, 2, ...)$ to zero and solving the obtained system of overde-termined algebraic equation using the symbolic manipulation package *Mathematica*, results in three sets of equations:

The first set is represented by

$$M = 0, \quad \Lambda \neq 0, \qquad a_0 = -\frac{\Gamma}{2\Lambda}, \quad a_1 = \frac{i\sqrt{6}P\sqrt{\Omega}}{\sqrt{\Lambda}}, \quad b_1 = 0,$$

$$\Gamma \neq 0, \quad \vartheta = \frac{\Gamma a_0}{3}, \quad P\Omega \neq 0, \qquad k = \frac{2\vartheta - \Gamma a_0}{4P\Omega}, \tag{58}$$

the second set is represented by

$$a_0 = \frac{\sqrt{\frac{3}{2}} M \sqrt{\Omega}}{\sqrt{\Lambda}}, \quad M \neq 0, \quad a_1 = \frac{2Pa_0}{M},$$

$$b_1 = 0, \quad P\Omega \neq 0, \quad k = \frac{\Omega M^2 + 2\vartheta}{4P\Omega}, \tag{59}$$

and the third set is represented by

$$M = 2P, \quad a_0 = \frac{\sqrt{6}P\sqrt{\Omega}}{\sqrt{\Lambda}}, \quad k \neq 0, \quad a_1 = a_0,$$

$$b_1 = \frac{\sqrt{6}\sqrt{-\Omega k^2 + 2P\Omega k - P^2\Omega}}{\sqrt{\Lambda}},$$

$$\vartheta = -2\Omega P^2 + 2k\Omega P + \Lambda a_0 b_1. \tag{60}$$

For the first set (58), when $P = 1$ we get the solutions satisfying Case I. Therefore, for $k > 0$ the solution of the EKdV eq. (35) will be

$$u_1(x, t) = \frac{i\sqrt{\Omega}\sqrt{\frac{\Gamma^2}{\Omega\Lambda}}\tan\left(\frac{\zeta\sqrt{\frac{\Gamma^2}{\Omega\Lambda}}}{2\sqrt{6}}\right)}{2\sqrt{\Lambda}} - \frac{\Gamma}{2\Lambda} \tag{61}$$

and

$$u_2(x, t) = \frac{i\sqrt{\Omega}\sqrt{\frac{\Gamma^2}{\Omega\Lambda}}\cot\left(\frac{\zeta\sqrt{\frac{\Gamma^2}{\Omega\Lambda}}}{2\sqrt{6}}\right)}{2\sqrt{\Lambda}} - \frac{\Gamma}{2\Lambda}, \tag{62}$$

while for $k < 0$

$$u_3(x, t) = -\frac{\Gamma}{2\Lambda} - \frac{i\sqrt{\Omega}\sqrt{-\frac{\Gamma^2}{\Omega\Lambda}}\tanh\left(\frac{\zeta\sqrt{-\frac{\Gamma^2}{\Omega\Lambda}}}{2\sqrt{6}}\right)}{2\sqrt{\Lambda}}, \tag{63}$$

$$u_4(x, t) = -\frac{\Gamma}{2\Lambda} - \frac{i\sqrt{\Omega}\sqrt{\frac{\Gamma^2}{\Omega\Lambda}}\coth\left(\frac{\zeta\sqrt{-\frac{\Gamma^2}{\Omega\Lambda}}}{2\sqrt{6}}\right)}{2\sqrt{\Lambda}}, \tag{64}$$

and for $k = 0$

$$u_5(x, t) = \frac{\Gamma\zeta + 2i\sqrt{6\Omega\Lambda}}{2\zeta\Lambda}. \tag{65}$$

For the second set (59), we apply the compatibility condition for the solutions satisfying Cases II, III, and IV as

$$Pk = \frac{M^2 - p_1^2}{4}. \tag{66}$$

Substituting P and k from (59), in eq. (66) and solving for p_1, we obtain

$$p_1 = \frac{i\sqrt{-\frac{\Gamma^2}{\Lambda}}}{\sqrt{3\Omega}} \quad \text{or} \quad p_1 = \frac{-i\sqrt{-\frac{\Gamma^2}{\Lambda}}}{\sqrt{3\Omega}}. \tag{67}$$

Therefore, the solution of eq. (35) will be

$$u_6(x, t) = -\frac{i\sqrt{6\Omega}p_1\tanh(\zeta p_1)}{\sqrt{\Lambda}} \tag{68}$$

and

$$u_7(x, t) = -\frac{i\sqrt{6\Omega}\, p_1 \coth(\zeta p_1)}{\sqrt{\Lambda}}. \tag{69}$$

In the same manner, Case III gives the solution

$$u_8(x, t) = -\frac{i\sqrt{\frac{3}{2}\Omega}\left(\sinh(\zeta) + \sqrt{r^2 - 1}\right)}{\sqrt{\Lambda}(r + \cosh(\zeta))}, \tag{70}$$

with the condition $p_1 = 1$.

For Case IV, the solution form is

$$u_9(x, t) = \frac{i\sqrt{\frac{3}{2}}\sqrt{\Omega}\,\text{cosech}\left(\frac{\zeta}{2}\right)\left(\cosh\left(\frac{\zeta}{2}\right) + 2M \sinh\left(\frac{\zeta}{2}\right)\right)}{\sqrt{\Lambda}}, \tag{71}$$

with the same condition $p_1 = 1$, and

$$u_{10}(x, t) = -\frac{i\sqrt{6}\sqrt{\Omega}\coth(\zeta)}{\sqrt{\Lambda}}, \tag{72}$$

with the condition $p_1 = 2$.

Hence, for the solutions satisfying Cases II–IV, we have the compatibility condition

$$Pk = \frac{M^2 - p_1^2}{4}.$$

Therefore, substituting for P and k, from (60) and solving for p_1, it is found that

$$p_1 = \sqrt{2}\sqrt{\frac{2P^2\Omega - 2kP\Omega}{\Omega}} \quad \text{or} \quad p_1 = -\sqrt{2}\sqrt{\frac{2P^2\Omega - 2kP\Omega}{\Omega}}. \tag{73}$$

The solution of eq. (55) will be

$$u_{11}(x, t) = \frac{-\sqrt{6}\sqrt{-(k - P)^2\Omega}P^2 - \sqrt{\Lambda}a_0\left(P - p_1\left(P + \tanh(\zeta p_1)\right)\right)^2}{P\sqrt{\Lambda}\left(p_1\left(P + \tanh(\zeta p_1)\right) - P\right)} \tag{74}$$

and

$$u_{12}(x, t) = \frac{-\sqrt{6}\sqrt{-(k - P)^2\Omega}P^2 - \sqrt{\Lambda}a_0\left(P - \left(P + \coth(\zeta p_1)\right)p_1\right)^2}{P\sqrt{\Lambda}\left(\left(P + \coth(\zeta p_1)\right)p_1 - P\right)}, \tag{75}$$

where a_0 is given by eq. (60), with the relative conditions. Similarly, Case III results in the solution

$$u_{13}(x, t)$$

$$= \frac{-8\sqrt{6}P^2\sqrt{-(k - P)^2\Omega}(r + \cosh(\zeta))^2 - \sqrt{\Lambda}\left(2r^2 + \cosh(2\zeta) + 4\sqrt{r^2 - 1}\sinh(\zeta) - 3\right)a_0}{4P\sqrt{\Lambda}(r + \cosh(\zeta))\left(\sinh(\zeta) + \sqrt{r^2 - 1}\right)}, \tag{76}$$

with the condition $p_1 = 1$.

For Case IV, the solution can be written in the form

$$u_{14}(x, t) = \frac{4\sqrt{6}\sqrt{-(k - P)^2\Omega}P^2 + a_0\sqrt{\Lambda}\left[4P + \coth\left(\frac{\zeta}{2}\right)\right]^2}{2P\sqrt{\Lambda}\left[4P + \coth(\zeta) + \text{cosech}(\zeta)\right]}, \tag{77}$$

with the same condition $p_1 = 1$, and

$$u_{15}(x, t) = \frac{-\sqrt{6}\sqrt{-(k - P)^2\Omega}\tanh(\zeta)P^2 - a_0\sqrt{\Lambda}\coth(\zeta)}{P\sqrt{\Lambda}}, \tag{78}$$

with the condition $p_1 = 2$.

4.1 *Numerical analysis and discussion*

We have considered a collisionless, unmagnetized plasma consisting of q-nonextensive electrons, positive ions, negative ions, as well as charged immobile dust grains. To investigate the nonlinear dynamics of the DIAWs, the reductive perturbation technique is employed to obtain an EKdV equation. The latter is solved using an extended homogeneous balance method. The extended homogeneous balance method gives different classes of solutions of the EKdV equation. These solutions include many types like rational, periodical, shock solutions, etc. For example, solution (65) represents the rational-type solutions, which may be helpful to explain the creation of very high energy in the plasma system. Because the rational solution is a discrete joint union of manifolds, particle systems describe the motion of a pole of the evolution equation. Solutions (61) and (62) are examples exhibiting the sinusoidal-type periodical solutions, which develop
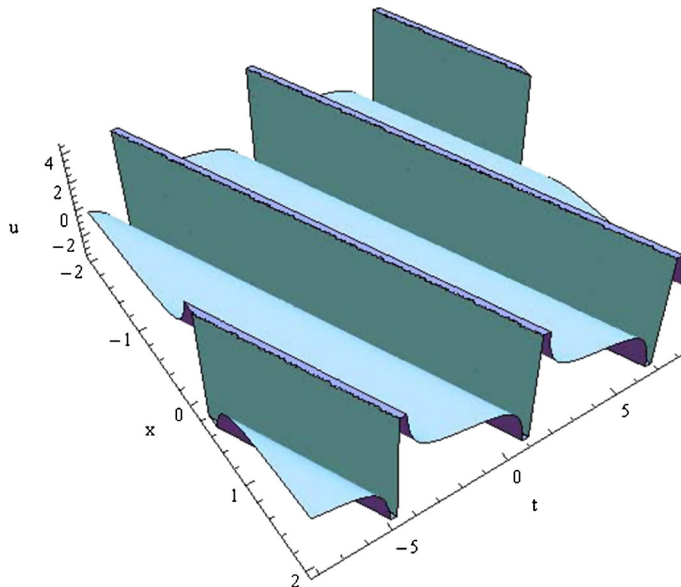

Figure 1. Three-dimensional profile of the periodic solution (eq. (61)) for $\alpha = 0.6$, $\beta = 0.1$, and $q = 0.7$.

a singularity at a finite point, i.e., for any fixed $t = t_0$ there exists a value of ζ_0 at which these solutions blow up (see figure 1). Note that these excitations never reach zero, except in a very specific combination of parameter values. The prediction for a potential excitation blow-up indicates that an instability in the system may occur due to the effect of nonlinearity. In simple terms, the balance between dispersion and nonlinearity may be disturbed by variations of plasma quantities (e.g., temperature, pressure, density, etc.). This might locally destroy the localized excitation stability leading to an amplitude increase to very high values; as this represents an increase in the electric potential, it might lead to an acceleration of the moving particles. It is important to note that eq. (69) is a form of explosive/blow-up solutions as depicted in figure 2.

Another different nonlinear wave that could be of interest is represented by solution (63), which represent the shock waves. Equation (63) can be written as

$$u(\zeta) = \frac{1}{2}\phi_m \left[1 - \tanh\left(\frac{2\zeta}{W}\right) \right], \tag{79}$$

where ϕ_m and W are the amplitude and width of the shocks, respectively, and are given by

$$\phi_m = -\frac{B}{C} \quad \text{and} \quad W = 2\frac{\sqrt{-12/C}}{|\phi_m|}. \tag{80}$$

It is clear from eq. (80) that to have shock waves, C should acquire negative values; i.e., $C < 0$. The Earth's ionosphere plasma (H$^+$, H$^-$) will be used as an example to numerically investigate the nonlinear coefficient C and negative dust grains are considered. The numerical analysis in figure 3 defines the possible regions of negative C that is represented by green zone, while for white zone, C is greater than zero. Hence, our numerical analysis of the shock wave profile is limited within the blue region.
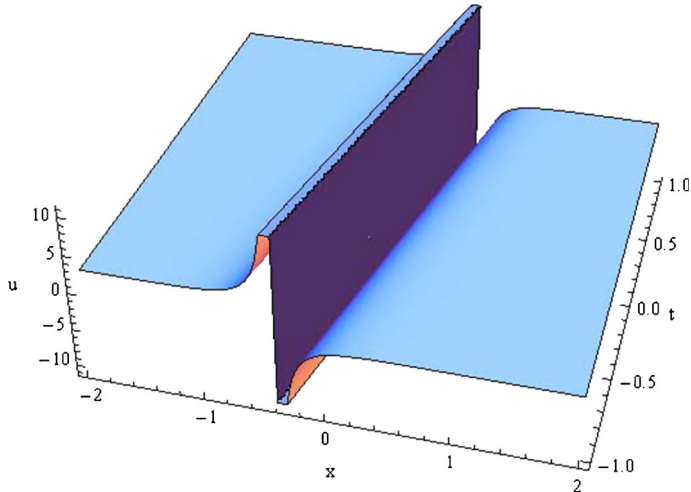

Figure 2. Three-dimensional profile of the explosive/blow-up pulse (eq. (69)) for the same parameters as in figure 1.
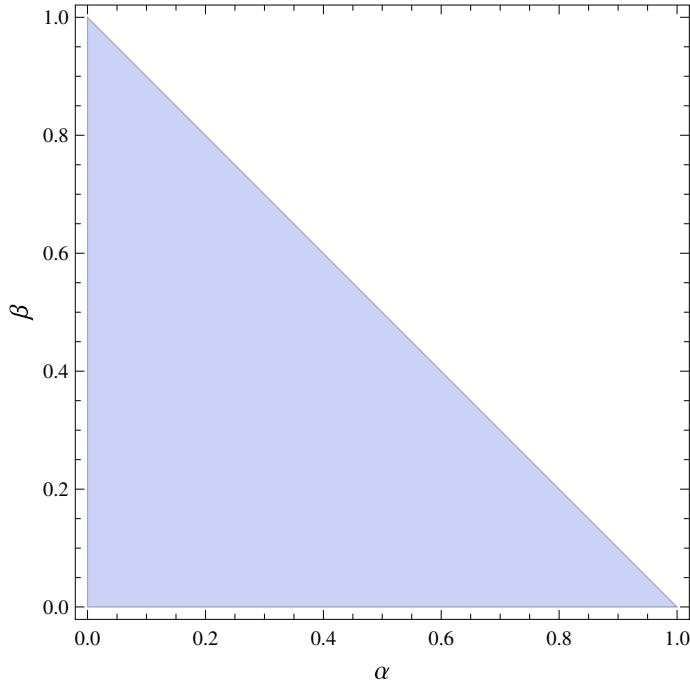
Figure 3. The contour plot of the coefficient C with α and β for negative dust particles, where $q = 0.6$, $\sigma_+ = \sigma_- = 0.5$, and $Q = 0.03$.

Now, we shall study the variation of the shock wave profile against q, σ_1, σ_2, α, and β as depicted in figures 4–6. Figure 4 shows that the increase of the nonextensive parameter q would lead to an enhancement in the shock amplitude. Actually, increasing the shock amplitude increases the potential difference and accelerates the particles to high
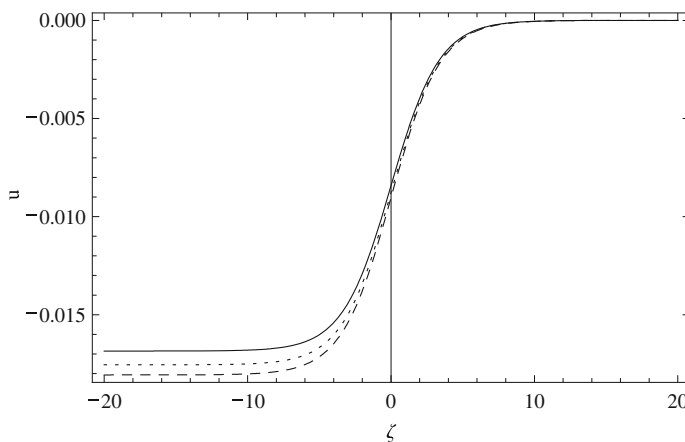
Figure 4. The shock wave profile for different values of q where $q = 0.6$ (——), 0.8 ($\cdots$), 0.97 (- - - -). Here, $\alpha = 0.5$, $\beta = 0.1$, $\sigma_+ = \sigma_- = 0.5$, and $Q = 0.03$.
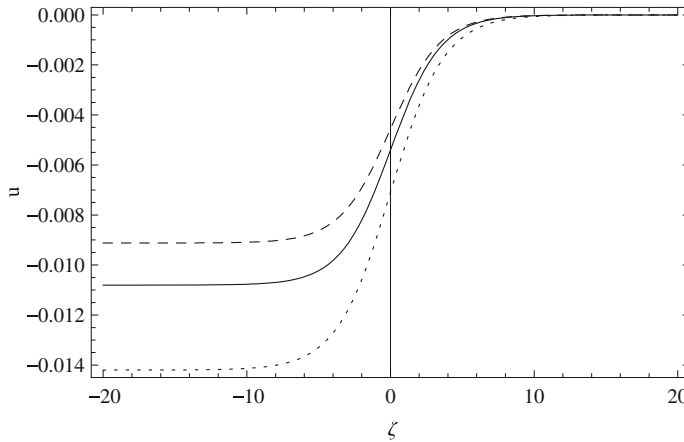
Figure 5. The shock wave profile for different values of σ_+ and σ_- where $\sigma_+ = \sigma_- = 0.5$ (——), $\sigma_+ = 0.6, = 0.5$ ($\cdots$), $\sigma_+ = 0.5, = 0.6$ (- - - - -). Here, $\alpha = 0.5$, $\beta = 0.1$, $q = 0.6$, and $Q = 0.03$.

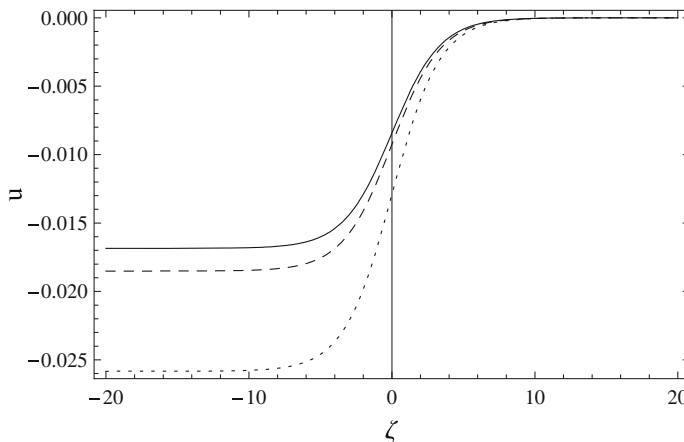**Figure 6.** The shock wave profile for different values of α and β where $\alpha = 0.5$, $\beta = 0.1$ (——), $\alpha = 0.6$, $\beta = 0.1$ ($\cdots$), $\alpha = 0.5$, $\beta = 0.2$ (- - - - -). Here, $\sigma_+ = \sigma_- = 0.5$, $q = 0.6$, and $Q = 0.03$.

velocity. However, for higher q values near to unity the system behaves like Maxwellian. Therefore, when the system nears either the Maxwellian state or the equilibrium state, the particles accelerates more.

Figure 5 clearly shows that the increase in positive ion-to-electron temperature ratio would make the shock amplitude taller but the negative ion-to-electron temperature ratio makes the shock amplitude shorter. On the other hand, the ion temperature accelerates the particles due to the generation of high potential shock waves.

It is obvious from figure 6 that the excess of negative-to-positive ion density ratio and the negative dust-to-positive ion density ratio would lead to an increase in the shock amplitude, but the former is more effective than the latter. In other words, the increase of

negative ions in the plasma system creates high potential difference due to their dynamics, while the negative stationary dust has less influence. Of course, the dynamics of the charged particles has a significant effect even if it has less mass than the stationary dust. The latter usually plays a role in neutralizing the background but does not play an effective role in wave dynamics.

5. Summary

In this paper, we have studied the nonlinear propagation of dust ion-acoustic waves in dusty plasmas, where a background of stationary dust was considered. We have derived the EKdV equation describing the system. Using homogeneous balance method we obtained a new class of solutions of the EKdV equation. These solutions include different rational solutions and shock wave solution. We have used the present model to investigate the behaviour of nonlinear structures in the Earth's ionosphere plasma environment. Numerical analysis of the solutions revealed that the profile of the nonlinear pulses suffer amplitude and width modifications due to the enhancement of the dust particle density, negative ion density, and nonextensive electron parameter. Furthermore, the necessary condition for the propagation of shock waves is examined.

References

[1] H Massey *Negative ions*, 3rd Edn (Cambridge University Press, Cambridge, 1976) p. 663
    W Swider, in: *Ionospheric modeling* edited by J N Korenkov (Birkhauser, Basel, 1988) p. 403
    Yu I Portnyagin *et al*, *Adv. Space Sci. Rev.* **12**, 121 (1992)
[2] P H Chaizy *et al*, *Nature (London)* **349**, 393 (1991)
[3] R A Gottscho and C E Gaebe, *IEEE Trans. Plasma Sci.* **14**, 92 (1986)
[4] S Ghosh *et al*, *Pramana–J. Phys.* **80**, 283 (2013)
    S S Duha, B Shikha and A A Mamun, *Pramana – J. Phys.* **77**, 357 (2011)
[5] M Bascal and G W Hamilton, *Phys. Rev. Lett.* **42**, 1538 (1979)
[6] J Jacquinot, B D McVey and J E Scharer, *Phys. Rev. Lett.* **39**, 88 (1977)
    Y Nakamura, T Odagiri and I Tsukabayashi, *Plasma Phys. Control. Fusion* **39**, 115004 (2001)
    A Weingarten, R Arad, Y Maron and A Fruchtman, *Phys. Rev. Lett.* **87**, 11504 (2001)
[7] R Ichiki, S Yoshimura, T Watanabe, Y Nakamura and Y Kawai, *Phys. Plasmas* **9**, 4481 (2002)
[8] A J Coates, F J Crary, G R Lewis, D T Young, J J H Waite Jr. and E C Sittler Jr., *Geophys. Res. Lett.* **34**, L22103 (2007)
[9] O Rahman and A A Mamun, *Pramana – J. Phys.* **80**, 1031 (2013)
    W M Moslem, U M Abdelsalam, R Sabry and P K Shukla, *New J. Phys.* **12**, 073010 (2010)
    U M Abdelsalam, *Physica B* **405**, 3914 (2010)

[10] P K Shukla and A A Mamun, *Introduction to dusty plasma physics* (Institute of Physics Publishing, Bristol, 2002)

[11] V O Vakhnenko, E J Parkes and A J Morrison, *Chaos, Solitons and Fractals* **17**, 683 (2003)

[12] R Hirota, *Direct method of finding exact solutions of nonlinear evolution equations* edited by R Bullough, P Caudrey, Backlund transform, Lecture Notes in Mathematics (Springer, Berlin, 1980) Vol. 515

[13] W Malfliet, *Am. J. Phys.* **60**, 650 (1992)

[14] E Fan, *Phys. Lett. A* **277**, 212 (2000)

[15] A M Wazwaz, *Appl. Math. Comp.* **159**, 559 (2004)

[16] G Cai, Q Wang and J Huang, *Int. J. Nonlinear Sci.* **2**, 122 (2006)

[17] R Sabry, M A Zahran and E Fan, *Phys. Lett. A* **326**, 326 (2004)

[18] M L Wang, X Li and J Zhang, *Phys. Lett. A* **372**, 417 (2008)

[19] U M Abdelsalam and M Selim, *J. Plasma Phys.* **79**, 163 (2013)

[20] M L Wang, *Phys. Lett. A* **199**, 169 (1995)

[21] E Fan and H Q Zhang, *Phys. Lett. A* **245**, 389 (1998)

[22] L Yang, Z Zhu and Y Wang, *Phys. Lett. A* **260**, 55 (1999)

[23] E Fan, *Phys. Lett. A* **265**, 353 (2000)

[24] S K El-Labany and A El-Sheikh, *Astrophys. Space Sci.* **197**, 289 (1992)

[25] W M Moslem, *J. Plasma Phys.* **61**, 177 (1999)