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Thermalized solutions, statistical mechanics and turbulence: An overview of some recent results

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Abstract. In this study, we examine the intriguing connection between turbulence and equilibrium statistical mechanics. There are several recent works which emphasize this connection. Thus in the last few years, the first manifestations of the thermalization, predicted by T D Lee in 1952, was seen and a theoretical understanding of this was developed through detailed studies of finite-dimensional, Galerkin-truncated equations of hydrodynamics. Furthermore, the idea of the Galerkin truncation can be generalized for studying turbulence in non-integer (fractal) dimensions to yield a new, critical dimension with an equilibrium Gibbs state coinciding with a Kolmogorov spectrum. In this paper, we discuss these very exciting and recent developments in turbulence as well as open problems for the future.

Keywords. Turbulence; thermalization; Galerkin truncation; Fourier fractal decimation.

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1. Introduction

Turbulent flows are ubiquitous and abound. However, a detailed, microscopic understanding of turbulence still remains elusive and is widely regarded as one of the most challenging problems in classical physics [1–3]. Hence it is not surprising that scientists working in different areas ranging from fluid dynamics [4–7], astrophysics [8–11], geophysics [12,13], climate modelling [14], plasma physics [9–11,15,16], and statistical physics [17–26] have been deeply interested in problems related to turbulence [26a].

Although the fundamental equation to describe the motion of an ideal fluid (no viscosity) has been known for over two centuries due to the works of Leonhard Euler and Jean-Baptiste le Rond D'Alembert [27], and subsequently extended, independently, by Claude-Louis Navier and George Gabriel Stokes to include the effect of viscosity, we still do not have a complete mastery of the mathematical properties of these solutions. In particular, the question of finite-time singularities in the solutions to the Euler equation (in three dimensions) remains unanswered so far and its importance has been recognized by the Clay Institute by listing it as one of its millennium unsolved problems in mathematics [28].

Despite the mathematical complexity of the Euler or the Navier–Stokes equations, a major breakthrough in the last century gave an impetus for physicists to investigate turbulent flows seriously [28a]. This came in a set of remarkable papers by A N Kolmogorov in 1941 [29] and the theoretical and phenomelogical framework laid out in these papers came to be known, subsequently, as K41 in the literature. The Kolmogorov theory, or K41, invoking dimensional arguments, scaling laws, and universality to tackle fully developed turbulence, was crucial in providing a language accessible to theoretical, and in particular, statistical physicists to understand the complexity of turbulent flows.

Let us first briefly review the ideas contained in K41 [29]. We begin with the incompressible, viscous Navier–Stokes equations, at low Mach numbers, namely

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{f},$$

$$\nabla\cdot\mathbf{u} = 0, \tag{1}$$

where we have used units such that the density $\rho = 1$. The Eulerian velocity at the spatial position $\mathbf{r}$ and time t is denoted $\mathbf{u}(\mathbf{r}, t)$, ν is the kinematic viscosity, and $\mathbf{f}$ is an external force to maintain a non-equilibrium statistically stationary state. By using the incompressibility condition [1,4] it is often convenient to eliminate the pressure p via the Poisson equation $\nabla^2 p = -\partial_{ij}(u_i u_j)$. Frequently, flows are described by a non-dimensional parameter, the Reynolds number $\mathrm{Re} \equiv LV/\nu$, where L and V are characteristic length and velocity scales. Thus when Re is small, i.e., the typical velocity is small and the fluid is highly viscous, the flow is laminar. However, as Re increases, the flow starts becoming irregular, chaotic, and turbulent. Typically, this transition is accompanied by several symmetry breakings [1].

In three-dimensional flows, in the absence of force and viscosity, there are three conserved quantities, namely the momentum, the kinetic energy, and the helicity $H \equiv \int \mathrm{d}\mathbf{r}\omega\cdot\mathbf{u}/2$ ($\omega \equiv \nabla\times\mathbf{u}$ is known as the vorticity) [29a]. A phenomenological understanding of three-dimensional, homogeneous, isotropic turbulence rests on the notion of cascades: kinetic energy injected at a large scale L, often comparable to the system size and known as the integral length scale, is transferred to smaller and smaller scales till it reaches the scales η_d, where molecular viscosity becomes significant and energy is lost as heat. Furthermore, in the statistical steady state, the energy dissipation rate per unit volume ϵ, as $\mathrm{Re} \to \infty$, does not vanish but reaches a finite, positive value (dissipative anomaly). For scales r, known as the inertial range, such that $L \gg r \gg \eta_d$, statistical properties are determined uniquely by only r and ϵ when $\mathrm{Re} \to \infty$. By using such ideas of the cascade of energy from large to small scales and in the special case of fully developed homogeneous, isotropic turbulence, K41 yields, via dimensional analysis, the leading order behaviour of the energy spectrum $E(k)$, i.e., the distribution of the kinetic energy amongst the Fourier modes, as $E(k) \sim k^{-5/3}$. Subsequently, in a variety of numerical simulations and experiments, Kolmogorov's prediction for the leading order scaling behaviour of the energy spectra (in the intermediate wavenumbers) have been verified [29b].

In this short study, we look at a slightly different approach to understand turbulent flows. We begin by asking if there is a way to understand out-of-equilibrium, dissipative turbulence by adapting tools from equilibrium statistical mechanics. *A priori* such an approach would seem to have an inherent contradiction because although microscopically it is possible to model a flow through a Hamiltonian formulation with stationary states associated to an invariant Gibbs measure, at the macroscopic level self-consistency would invariably result in dissipative hydrodynamics and irreversible energy loss. Curiously, soon after Kolmogorov's prediction, Hopf [31] and Lee [32] studied the incompressible Euler equation (setting $\nu = 0$ in the Navier–Stokes equation) as a finite-dimensional system (by retaining only a finite number of Fourier modes via Galerkin truncation; see below). For such a finite-dimensional system with no viscosity, it is possible to apply the standard tools of equilibrium statistical mechanics. The long time solutions of this finite-dimensional system show an equipartition of energy and the thermalized states display energy spectra $E(k) \sim k^2$, very different from that predicted by Kolmogorov and seen in experiments and simulations [31a].

Given this obvious contradiction, it is important to ask if thermalized states are meaningful in the study of hydrodynamic turbulence? In this study, we explore the intriguing interplay between equilibrium statistical mechanics and turbulence which have attracted a lot of attention recently [33–37]. This paper is organized as follows. In the following section we discuss thermalized states and the onset of thermalization in equations of hydrodynamics. In §3 we discuss the related problems of turbulence in fractal dimensions and the existence of a critical dimension where the Kolmogorov solution coincides with equilibrium solutions. Finally, we make some concluding remarks and give possible future directions in this field.

2. Thermalization

As we have seen before, a straightforward extension of ideas of equilibrium statistical mechanics to the Galerkin-truncated, three-dimensional (3D), incompressible Euler equation by Hopf [31] and Lee [32] leads to an equipartition energy spectrum $E(k) \sim k^2$ which is very different from the spectrum observed in nature, experiments, and in direct numerical simulations (DNSs) [1] and which follow closely the celebrated Kolmogorov spectrum $E(k) \sim k^{-5/3}$ [29]. This underlines the inherent difficulty in adapting methods of statistical mechanics to problems in turbulence. However, in 1967, Kraichnan [38–40] (and later extended by Frisch *et al* in [36]) achieved success by using tools from equilibrium statistical mechanics to predict an inverse energy cascade in two-dimensional (2D) turbulence. The first clear evidence of how the Galerkin-truncated Euler equations actually thermalize was obtained by Cichowlas *et al* in [33] when they performed extremely high-resolution direct numerical simulations of the 3D Euler equation, with only a finite number of Fourier modes, and obtained long-lasting transients with energy spectra $E(k) \sim k^2$ at the high wavenumber end. Interestingly, if $k_{\text{threshold}}(t)$ is the wavenumber beyond which the equipartition spectra are clearly visible, then in the early times this threshold wavenumber is close to the largest mode of the system and with time it becomes smaller and smaller. A second interesting result of this calculation was that at any given time, for wavenumbers smaller than $k_{\text{threshold}}(t)$, the energy spectra seemed to follow the K41 scaling, namely $E(k) \sim k^{-5/3}$ [40a].

These results from 2005 (and predicted before in 1989 by Kraichnan and Chen [39]) lead us to ask how do systems thermalize? In this study, we explain the way truncated systems thermalize by using the Galerkin-truncated, one-dimensional (ID) Burgers equation. The results and insights from the ID system can be extended to the incompressible Euler equations in two [35] and three [41] dimensions.

The untruncated, inviscid Burgers equation for the velocity field $u(x, t)$, with initial condition $u(x, 0) = u_0(x)$ is written as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{2}$$

where the explicit dependence of the velocity field on the space variable x and time t has been omitted for notational convenience. For any positive integer K_G (the Galerkin truncation wavenumber), it is possible to define the Galerkin projector P_{K_G} which sets to zero all Fourier components with wavenumbers $|k| > K_G$, i.e.,

$$P_{K_G} u(x) = \sum_{|k|\le K_G} e^{ikx} \hat{u}_k. \tag{3}$$

By making use of this projector, we can write the associated inviscid Galerkin-truncated Burgers equation [42] for the truncated velocity field $v(x, t)$ as

$$\frac{\partial v}{\partial t} + P_{K_G}\left[v\frac{\partial v}{\partial x}\right] = 0, \quad v_0 = P_{K_G} u_0. \tag{4}$$

The solution of the inviscid Burgers equation, with smooth initial conditions such as trigonometric polynomials, has a finite time ($t_\star$) blow-up of the velocity gradient when the solution develops a cubic-root singularity or the preshock [43,44]. For times larger than $t_\star$, the solution of the inviscid Burgers equation is obtained by adding a small dissipative term $\nu\partial_x^2 u$ and taking the limit $\nu \to 0$ to obtain the inviscid-limit solution with at least one shock and dissipative anomaly. The solution to the Galerkin-truncated equation (4), however, stays smooth and conserves energy at all times.

How does the system of the Galerkin truncated Burgers equation thermalize? In order to examine this, we begin by performing a direct numerical simulation of (4) by using standard pseudospectral methods and a fourth-order Runge–Kutta scheme for time integration. We use the total number of collocation points $N = 2^{14} = 16384$ points, an integration time step $\delta t = 10^{-4}$, and the Galerkin-projection wavenumber $K_G = 1000$. We choose, without any loss of generality, an initial condition $v_0 = \sin(x + 0.2) + 1.3 \sin(2x - 2.2)$ which has a $t_\star \approx 0.28$ [44a]. In figure 1 we show the time evolution of the solution, shown in blue, of (4). It is easy to observe that although the solution to the truncated equation coincides with the smooth solution of the inviscid limit for early times $t < t_\star$ (see figure 1a). At times close to $t_\star$, a tiny, symmetric, localized monochromatic bulge appears which, as time evolves, collapses (due to, e.g., Reynolds stresses) and eventually leads to a complete thermalization of the solution. Thus this is a visual demonstration of how thermalization sets in, in truncated, inviscid systems. The triggers for thermalization are localized, monochromatic (with the same wavenumber as the truncation wavenumber) oscillations – called ‘tygers’ by Ray *et al* in [35] – which are due to the resonant wave interactions between the fluid particles and truncation waves generated via the convolution between the projection operator and the

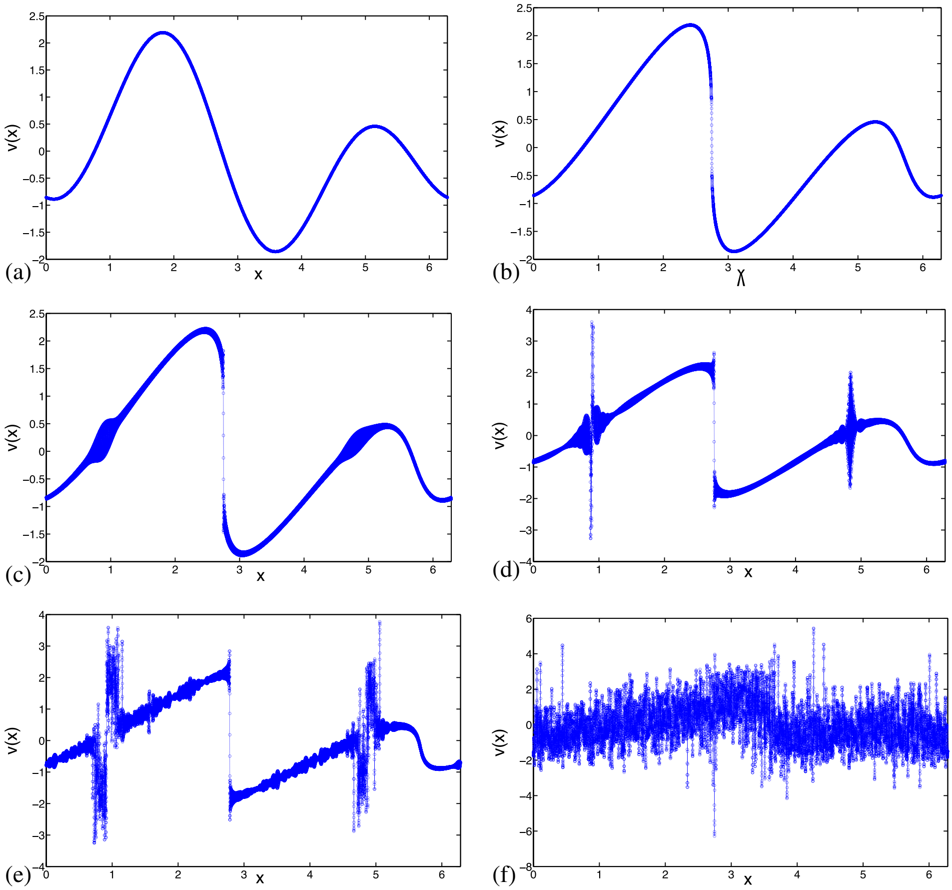

Figure 1. Plots of the solution of the Galerkin-truncated Burgers equation $v(x)$ vs. x with initial conditions ($t_\star \approx 0.28$) shown in **(a)** and at **(b)** $t = 0.28$, **(c)** $t = 0.30$, **(d)** $t = 0.35$, **(e)** $t = 0.50$, and **(f)** $t = 4.0$ showing the onset of, via localized symmetric oscillations in **(b)** and **(c)** and eventual thermalization **(f)**. Note the location of the localized oscillations around the spatial points whose velocity coincides, or is in resonance, with that of the shock.

nonlinearity. Typically, these waves are generated by the small structures, such as shocks. The first appearance of these bulges coincide with the time when singularities in the complex space reach distance $\lambda_G = 2\pi/K_G$ from the real domain. Furthermore 'tygers' are born at spatial points which have the same velocity as that of the generating small-scale structure, in this case the shock, and where the local velocity gradient is positive [44b]. As time evolves, $t \geq t_\star$, these bulges eventually collapse, grow and spread through the entire domain. This, then, is the first demonstration of the thermalization predicted by Lee [32] and whose signatures in spectral space were obtained in [33].

How easy is it to characterise the nature of this bulge which leads to eventual thermalization in truncated systems? In figure 2a we examine this bulge carefully. In order to do so, it is useful to subtract the solution to the inviscid, untruncated Burgers equation [44c]

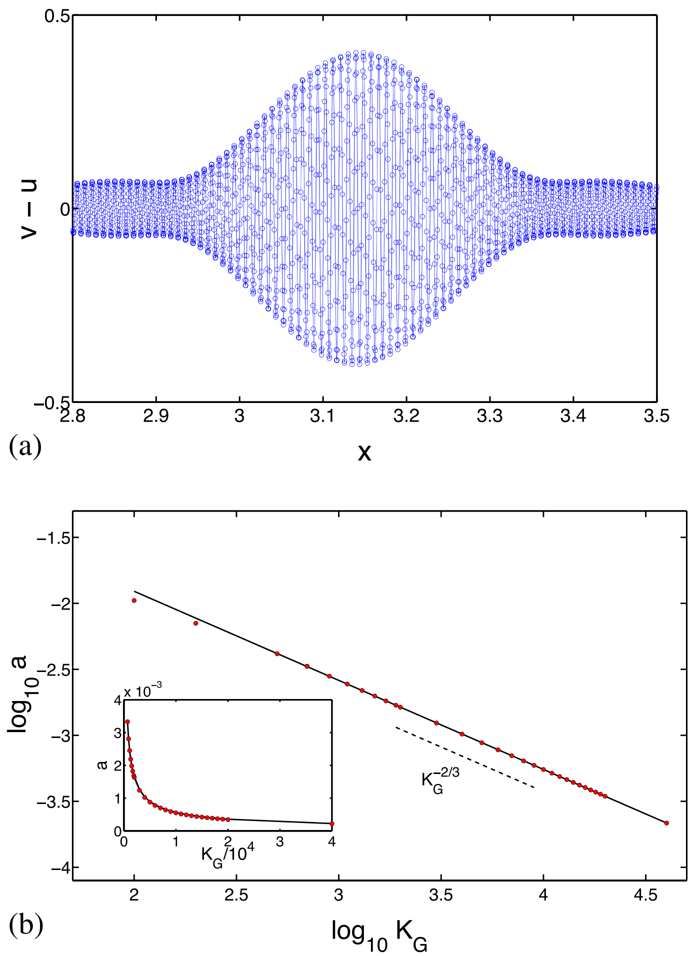

Figure 2. **(a)** The zoomed-in solution of the Galerkin-truncated Burgers equation, at $t = 1.05$, with the inviscid solution subtracted out, around the stagnation point. The initial condition for this simulation is $v_0 = \sin x$ and $K_{\mathrm{G}} = 700$. A clear symmetric, monochromatic bulge is seen. **(b)** Log–log plot of the amplitude a (solid red circles) of the 'tyger' at $t = t_\star$ as a function of K_{G}. The thick black line shows the theoretical prediction for the scaling $a \propto K_{\mathrm{G}}^{-2/3}$. Inset: The same but in a linear plot.

from the truncated solution as shown in figure 2a. The early symmetric tygers at $t \gtrsim t_\star$ are characterized by an amplitude a and a full-width-half-maximum w, which depends on the value of the truncation wavenumber K_{G}. At $t = t_\star$, it is possible to obtain scaling behaviour of both a and w in the following manner by using theoretical arguments [44d]. For the inviscid Burgers equation, the solution remains analytic for a finite time with at least one singularity in the complex domain within a distance $\delta(t)$. For times $t \ll t_\star$, $\delta(t) K_{\mathrm{G}} \gg 1$ ensuring that truncation effects are exponentially small. Truncation effects become important only at a time $O(K_{\mathrm{G}}^{-2/3})$ before $t_\star$ when $\delta(t)$ comes within a distance $2\pi/K_{\mathrm{G}}$ of the real domain. Therefore at $t_\star$, the effect of truncation has been important only for an interval of time $O(K_G^{-2/3})$. By using arguments of phase mixing, we can see that coherent structures can happen only at spatial points where the fluid velocity differs from the resonance velocity (the velocity of the shock) by an amount Δv such that

$$\Delta v \lesssim \frac{2\pi}{K_G^{-2/3} K_G} \propto K_G^{-1/3}. \tag{5}$$

As at $t_\star$, the numerical and theoretical arguments suggest that the velocity $v \approx u \propto x$, the width w of the bulge would scale as $w \propto K_G^{-1/3}$. This result has been rigorously obtained and confirmed by detailed numerical simulations in [35]. In figure 2b we show the scaling behaviour of the 'tyger' amplitude vs. K_G on a log–log scale (inset shows the same result in a linear plot) which gives clear evidence that $a \propto K_G^{-2/3}$. By using ideas of energy conservation and the fact that the Galerkin-truncated equation does conserve energy, it is possible to explain this scaling behaviour [35].

Although it has been possible to have a complete and rigorous understanding of the onset of thermalization in the 1D truncated Burgers equation and insights into the higher-dimensional truncated Euler equations (see, e.g., [35] for details), the complete process of thermalization as seen in figure 1 is still far from obvious. In recent years, the issue of thermalization in finite-dimensional equations of hydrodynamics, obeying a Liouville's theorem, has also been studied by several other researchers for various other systems. These include truncated solutions of the Euler equations in two and three dimensions [33,35,36,47–49], the Gross-Pitaevskii equation [50] and the magnetohydrodynamic [51] equations. In a recent paper, it has been shown how weakly dissipative systems, under suitable conditions, can also thermalize [37]. Furthermore, ideas of partial thermalization have also been used to explain certain experimental and numerical results regarding the mild non-monotonicity of the energy spectrum in 3D turbulence [34].

Despite these exciting results and ongoing studies on finite-dimensional systems, the connection between thermalized states and real turbulence is moot. After all, experimental observations are at variance with the idea of equipartition. It is in this context, to make the connection between turbulence and statistical mechanics more precise, that the need for studying the Navier–Stokes equation in fractal dimensions becomes important. We discuss this in the following section.

3. Fractal turbulence

From the point of understanding real turbulent flows, it can be argued that thermalized states can be crucial if in some special dimension the equipartition spectrum k^{D-1} coincides with the physically relevant Kolmogorov spectrum $k^{-5/3}$. Unfortunately, equating the two spectra yields a special dimension $D = -2/3$. However, fortunately, it turns out that there is a way of obtaining a critical dimension D_c by using similar arguments of equipartition in the following way. Consider the 2D Navier–Stokes equation which has, as noted before, an additional conserved quantity in enstrophy $\Omega = \langle \frac{1}{2}\omega^2 \rangle$. The presence of the second conserved quantity leads to a dual cascade – an inverse cascade of energy from small to large scales and a forward cascade of enstrophy from large to small scales – in 2D flows and an energy spectrum which scales as $k^{-5/3}$ at wavenumbers smaller than those of the energy injection and $k^{-(3+\alpha)}$ for larger wavenumbers [38,51a,52]. Thus, at dimensions which have the additional conserved quantity of enstrophy, it is possible to obtain an enstrophy equipartition spectrum k^{D-3} (by using the fact that the vorticity is the curl of the velocity field) and then relate it to the Kolmogorov spectrum. This yields the critical dimension $D_c = 4/3$ [48], where the Kolmogorov spectrum coincides with the (enstrophy) equilibrium spectrum. But, is it possible to obtain such a critical dimension numerically? Frisch *et al* [36] demonstrate that it is possible to do this through the method of fractal Fourier decimation.

Consider the 2D, forced, incompressible Navier–Stokes equation which conserves the kinetic energy and enstrophy. As we consider 2π periodic solutions, it is possible to expand the velocity field in Fourier space. On this 2D Fourier (k_x, k_y) lattice the number of grid points within a circle of a large radius K would grow as $\sim K^2$. Now if we were to remove some of the modes in a way such that the number of remaining modes within a circle grows as K^D (with $D < 2$), then we would obtain an effective Fourier fractal dimension D. Mathematically, this can be done by defining the following decimation operator P_D on the Fourier series of the velocity field:

$$P_D \mathbf{u} = \sum_{\mathbf{k} \in \mathcal{Z}^2} \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} \theta_{\mathbf{k}} \hat{\mathbf{u}}_{\mathbf{k}}, \tag{6}$$

where $\theta_{\mathbf{k}}$ are numbers randomly and independently chosen with the constraints

$$\begin{aligned} \theta_{\mathbf{k}} &= \begin{cases} 1 & \text{with probability } h_k \\ 0 & \text{with probability } 1 - h_k\,, \quad k \equiv |\mathbf{k}| \end{cases}. \\ \theta_{\mathbf{k}} &= \theta_{-\mathbf{k}} \quad \text{Hermitian symmetry preserving.} \end{aligned} \tag{7}$$

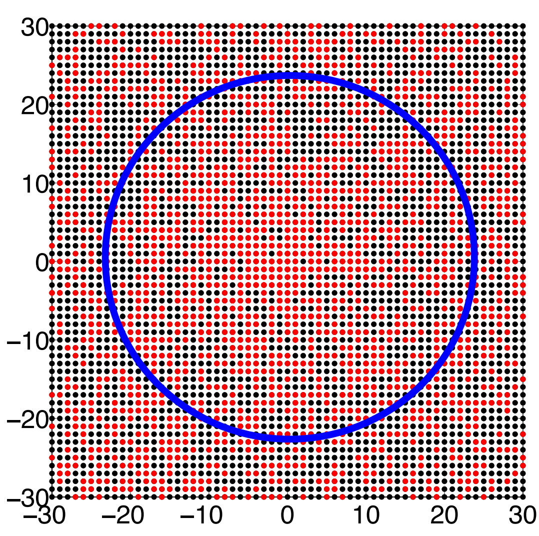

Figure 3. A cartoon representation of a fractal lattice for dimension $D = 1.55$ in the (k_x, k_y) plane. The blue circle indicates that the number of points within a circle of radius K would grow as K^D. The points in black are the modes which are decimated and the ones in red are the surviving modes (see text).

If we further choose

$$h_k = C(k/k_0)^{D-2}, \quad 0 < D \leq 2, \quad 0 < C \leq 1, \tag{8}$$

we obtain D-dimensional dynamics (k_0 is a reference wavenumber). We choose $C = k_0 = 1$. Fractal decimation thus produces a quenched fractal lattice which, on average, retains K^D active modes within a circle of radius K (see figure 3).

We can now use this projector to rewrite the Navier–Stokes equation in any fractal dimension $D < 2$ which preserves, because of the nature of the self-adjoint projector, the invariants of the 2D Navier–Stokes equation [36], as

$$\frac{\partial \mathbf{v}}{\partial t} + P_{\mathrm{D}}[\mathbf{v} \cdot \nabla \mathbf{v}] = P_{\mathrm{D}} \nabla^2 \mathbf{v} + P_{\mathrm{D}} \mathbf{f} \tag{9}$$

with the initial condition $\mathbf{v}_0 \equiv \mathbf{v}(t = 0) = P_{\mathrm{D}} \mathbf{u}_0$. Additionally, the above equation can be projected via Galerkin truncation to a finite-dimensional space thus allowing the use of ideas developed in the previous section of thermalization. In particular, it was found in [36], by using state-of-the-art direct numerical simulation that starting from the 2D Navier–Stokes equation, the (Fourier) dimension can be continuously reduced, still preserving the Kolmogorov spectrum $k^{-5/3}$ for low wavenumbers, till it approaches the critical dimension $D_{\mathrm{c}} = 4/3$. As we approach D_{c}, the flux of the solution vanishes (linearly) and eventually at the critical dimension the Kolmogorov spectrum coincides with the equilibrium Gibbs state with the same spectrum. This is indeed a remarkable result because it shows that there are special dimensions where conventional equilibrium statistical physics and turbulence theories are complimentary [52a]. We discuss the implication of this result in the last section [52b].

4. Open questions and conclusions

In theoretical physics there has been a long and successful history of extending the dimension from experimentally realizable integer dimensions to fractional ones, such as in $4 - \epsilon$ expansion in critical phenomena [54], to allow us to understand physical processes better. Similar attempts in turbulence had failed in the past because for $D < 2$, closure-type models lead to negative values of $E(k)$ for certain wavenumbers and hence become unphysical [55]. The fractal decimation method, outlined in the text, makes a probabilistic realization of turbulence in dimensions $D < 2$ possible and it also allows writing down closure equations of the eddy-damped-quasi-normal-Markovian (EDQNM) variety [55,56] by making velocity variances probabilistic. Such an approach is extremely exciting because as we have seen at the critical dimension, the equilibrium fluxless spectrum coincides with the observed spectrum. This allows standard tools of equilibrium statistical physics to be used to understand, e.g., the problem of turbulence in two dimension such as in soap films. Indeed, it has been noted [36] that the equilibrium Gibbsian distribution for the two-dimensional truncated, inviscid Navier–Stokes equation is very close to the energy spectrum in two-dimensional turbulence. It is still an open question, e.g., if this formal similarity can be exploited to understand the temporal multiscale nature of true turbulence via relaxation time-scales in the equilibrium problem obtained through velocity correlation functions. Another significant and unresolved question in turbulent flows is intermittency [1]. Indeed, operators such as the Galerkin projector or the decimation operator make the original equations of motion non-local and destroy the Lagrangian structure of the equations of hydrodynamics without changing the invariants or the symmetries of the equations. This fact allows the possibility of a better understanding of the source of intermittency in three-dimensional, isotropic and homogeneous turbulence. Thermalized, fluxless solutions, which are close to the Kolmogorov solutions, also suggest possibilities of perturbation techniques around the critical dimension which may be extendable to physically relevant dimensions of soap film experiments. There is another reason for studying truncated systems. Spectral and pseudospectral methods are widely used as being the most precise method for solving equations such as the Navier–Stokes and the Euler equations [57]. Thus, in order to obtain numerical evidence for or against finite-time blow up, i.e., whether $\delta(x) \to 0$ in a finite time, such techniques are used to calculate the velocity field and thence the width of the analyticity strip [58,59]. This approach has not yielded convincing arguments because the equations thermalize quickly at the high-k end of the spectrum leading to inaccurate and often impossible estimations of the temporal behaviour of $\delta(x)$ beyond very short times [59]. Understanding how thermalization sets in, in truncated system will lead to a better estimate of the temporal behaviour of $\delta(x)$, eventually resulting in a strong conjecture for the finite-time blow up problem [28].

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References

[1] U Frisch, *Turbulence: The legacy of A N Kolmogorov* (Cambridge University Press, Cambridge, UK, 1996)

[2] G Falkovich, K Gawedzki and M Vergassola, *Rev. Mod. Phys.* **73**, 913 (2001)

[3] R Pandit, P Perlekar and S S Ray, *Pramana – J. Phys.* **73**, 157 (2009)

[4] S B Pope, *Turbulent flows* (Cambridge University Press, Cambridge, UK, 2000)

[5] H Tennekes and J L Lumley, *A first course in turbulence* (MIT Press, Cambridge, Massachusetts, 1972)

[6] M Lesieur, *Turbulence in fluids* (Springer, The Netherlands, 2008)

[7] P A Davidson, *Turbulence: An introduction for scientists and engineers* (Oxford University Press, Oxford, 2007)

[8] V M Canuto and J Christensen-Dalsgaard, *Annu. Rev. Fluid Mech.* **30**, 167 (1998)
Turbulence and magnetic fields in astrophysics. edited by E Falgarone and T Passot (Springer, New York, 2003)
M S Miesch and J Toomre, *Annu. Rev. Fluid Mech.* **41**, 317 (2009)

[9] A Rai Choudhuri, *The physics of fluids and plasmas: An introduction for astrophysicists* (Cambridge University Press, Cambridge, UK, 1998)

[10] V Krishan, *Astrophysical plasmas and fluids* (Kluwer, Dordrecht, 1999)

[11] H Goedbloed and S Poedts, *Principles of magnetohydrodynamics with applications to laboratory and astrophysical plasmas* (Cambridge University Press, Cambridge, UK, 2004)

[12] A S Monin, *Russ. Math. Surv.* **38** 127 (1983)
P Bradshaw and J D Woods, in: *Turbulence* edited by P Bradshaw (Springer, New York, 1978)
A J Majda and X Wang, *Nonlinear dynamics and statistical theories of basic geophysical flows* (Cambridge University Press, Cambridge, UK, 2006)

[13] G Boer and T Shepherd, *J. Atmos. Sci.* **40**, 164 (1983)

[14] R A Shaw, *Annu. Rev. Fluid Mech.* **35**, 183 (2003)

[15] D Biskamp, *Magnetohydrodynamic turbulence* (Cambridge University Press, Cambridge, UK, 2003)

[16] M K Verma, *Phys. Rep.* **401**, 229 (2004)

[17] S Orszag, in: *Fluid dynamics, Les Houches* edited by R Balian and J Peube (Gordon and Breach, New York, 1977) pp. 237–374

[18] H Rose and P Sulem, *J. Phys. France* **39**, 441 (1978)

[19] C D Dominicis and P Martin, *Phys. Rev. A* **19**, 419 (1979)

[20] V Yakhot and S Orszag, *Phys. Rev. Lett.* **57**, 1722 (1986)

[21] W D McComb, *The physics of fluid turbulence* (Oxford University Press, Oxford, 1991)

[22] G L Eyink, *Phys. Fluids A* **6**, 3063 (1994)
G L Eyink and N Goldenfeld, *Phys. Rev. E* **50**, 4679 (1994)

[23] T Bohr, M H Jensen, G Paladin and A Vulpiani, *Dynamical systems approach to turbulence* (Cambridge University Press, Cambridge, UK, 1998)

[24] L Ts Adzhemyan, N V Antonov and A N Vasiliev, *The field theoretic renormalization group in fully developed turbulence* (Gordon and Breach, 1999)

[25] J K Bhattacharjee and S Bhattacharyya, *Nonlinear dynamics near and far from equilibrium* (Hindustan Book Agency, New Delhi, 2007) Ch. 8, pp. 231–274

[26] *Non-equilibrium statistical mechanics and turbulence* edited by S Nazarenko and O V Zaboronski (Cambridge University Press, Cambridge, UK, 2008)

[26a] See also the scholarpedia article by R Benzi and U Frisch (*Scholarpedia* **5(3)**, 3439 (2010)), which gives a nice introduction to turbulent flows and the phenomenology for non-experts.

[27] G Eyink, U Frisch, R Moreau and A Sobolevskiĭ, *Proceedings of Euler Equations: 250 Years On* (Aussois, June 18–23, 2007); *Physica D* **237** no. 14–17 (2008)

[28] C Fefferman, *Existence and smoothness of the Navier–Stokes equation. Clay Millenium Prize Problem Description* (2000); http://www.claymath.org/millennium/Navier-Stokes_Equations/Official_Problem_Description.pdf

[28a] A second breakthrough came from the advances in computers, as predicted in 1949 by von Neumann [30], which allowed solving numerically, the evolution equations of hydrodynamics with increasing precision.

[29] A N Kolmogorov, *Dokl. Akad. Nauk SSSR* **30**, 301 (1941); *Dokl. Akad. Nauk SSSR* **31**, 538 (1941)

[29a] In two-dimensional flows, there are more conserved quantities: $\langle \frac{1}{2}\omega^n \rangle$, for all n; a special case is $\Omega = \langle \frac{1}{2}\omega^2 \rangle$ known as the mean enstrophy.
[29b] For a detailed discussion, the reader is referred to [1].
[30] J Glimm, J Impagliazzo and I M Singer, *The legacy of John Von Neumann* (American Mathematical Society, New York, 1990)
[31] E Hopf, *Commun. Pure Appl. Math.* **3**, 201 (1950)
[31a] The equilibrium (equipartition spectrum) in any dimension d is given by $E(k) \sim k^{d-1}$.
[32] T D Lee, *Q. J. Appl. Math.* **10** (1952)
[33] C Cichowlas, P Bonaïti, F Debbash and M Brachet, *Phys. Rev. Lett.* **95**, 264502 (2005)
[34] U Frisch, S Kurien, R Pandit, W Pauls, S S Ray, A Wirth and J-Z Zhu, *Phys. Rev. Lett.* **101**, 144501 (2008)
[35] S S Ray, U Frisch, S Nazarenko and T Matsumoto, *Phys. Rev. E* **84**, 16301 (2011)
[36] U Frisch, A Pomyalov, I Procaccia and S S Ray, *Phys. Rev. Lett.* **108**, 074501 (2012)
[37] D Banerjee and S S Ray, *Phys. Rev. E* **90**, 041001(R) (2014)
[38] R H Kraichnan, *Phys. Fluids* **10**, 1417 (1967)
[39] R H Kraichnan and S Chen, *Physica D* **37**, 160 (1989)
[40] G L Eyink, U Frisch and Robert H Kraichnan, in: *A journey through turbulence in twelve chapters* edited by P Davidson, Y Kaneda, H K Moffatt and K S R Sreenivasan (Cambridge University Press, Cambridge, 2010)
[40a] See, e.g., [33,34,39] for an explanation.
[41] T Matsumoto, Private Communications
[42] A J Majda and I Timofeyev, *Proc. Natl. Acad. Sci. USA* **97**, 12413 (2000)
[43] J D Fournier and U Frisch, *J. Mec. Theor. Appl.* **2**, 699 (1983)
[44] U Frisch and J Bec, *Les Houches 2000: New trends in turbulence* edited by M Lesieur, A Yaglom and F David (Springer EDP-Sciences, 2001) pp. 341–383
J Bec and K Khanin, *Phys. Rep.* **447**, 1 (2007)
[44a] We refer the reader to [35] for a more detailed discussion on solutions to different initial conditions. We also refer the reader to [45] for a wavelet-based approach to truncated systems.
[44b] At points where the velocity matches those of the shock but have a negative gradient, 'tygers' cannot form as the negative gradients would result in the creation of wavelengths higher than K_G and thus be suppressed by the Galerkin projection.
[44c] It is possible to obtain, numerically, the true inviscid solution by using the method of fast Legendre transform [46].
[44d] A more rigorous derivation is found in [35].
[45] R Pereira, R Nguyen van Yen, M Farge and K Schneider, *Phys. Rev. E* **87**, 033017 (2013)
[46] A Noullez and M Vergassola, *J. Sci. Comput.* **9**, 259 (1994)
[47] A J Majda and I Timofeyev, *Proc. Natl. Acad. Sci. USA* **97**, 12413 (2000)
[48] V L'vov, A Pomyalov and I Procaccia, *Phys. Rev. Lett.* **89**, 064501 (2002)
[49] G Krstulovic, P D Mininni, M E Brachet and A Pouquet, *Phys. Rev. E* **79**, 056304 (2009)
[50] G Krstulovic and M Brachet, *Phys. Rev. Lett.* **106**, 115303 (2011); *Phys. Rev. E* **83**, 066311 (2011)
V Shukla, M Brachet and R Pandit, *New J. Phys.* **15**, 113025 (2013)
[51] G Krstulovic, M E Brachet and A Pouquet, *Phys. Rev. E* **84**, 016410 (2011)
[51a] The value of α is non-universal and depends on the precise parameters, such as the nature of the damping.
[52] C Leith, *Phys. Fluids* **11**, 671 (1968)
G Batchelor, *Phys. Fluids Suppl. II* **12**, 233 (1969)
P Perlekar, S S Ray, D Mitra and R Pandit, *Phys. Rev. Lett.* **106**, 054501 (2011)
S S Ray, D Mitra, P Perlekar and R Pandit, *Phys. Rev. Lett.* **107**, 184503 (2011)

[52a] The linear vanishing of the flux has severe implications on the behaviour of the Kolmogorov constant and its divergence as we approach the critical dimension (see [36]).
[52b] It is worth pointing out that similar ideas of fractal decimation have been used by Biferale and co-workers in a series of recent papers to understand both the regularity problem as well as the nature of cascades for the helically decimated version of the 3D Navier–Stokes equations [53].
[53] L Biferale and E S Titi, *J. Stat. Phys.* **151**, 1089 (2013)
L Biferale, S Musacchio and F Toschi, *J. Fluid Mech.* **730**, 309 (2013)
L Biferale, S Musacchio and F Toschi, *Phys. Rev. Lett.* **108**, 164501 (2012)
[54] K G Wilson and M E Fisher, *Phys. Rev. Lett.* **28**, 240 (1972)
[55] J-D Fournier and U Frisch, *Phys. Rev. A* **17**, 747 (1978)
[56] S A Orszag, Statistical theory of turbulence, in: *Fluid dynamics*, Les Houches 1973, 237, edited by R Balian and J L Peube (Gordon and Breach, 1977)
[57] D Gottlieb and S A Orszag, *CBMS-NSF Regional Conf. Ser. Appl. Math.* **26**, SIAM (1977)
[58] C Sulem, P-L Sulem and H Frisch, *J. Comp. Phys.* **50**, 138 (1983)
[59] M D Bustamante and M-E Brachet, *Phys. Rev. E* **86**, 066302 (2012)