PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 78, No. 3
March 2012
pp. 361–374

Robust chaos synchronization based on adaptive fuzzy delayed feedback $\mathcal{H}_\infty$ control

CHOON KI AHN
Faculty of the Department of Automotive Engineering, Seoul National University of Science & Technology, 172 Gongneung 2-dong, Nowon-gu, Seoul 139-743, Korea
E-mail: hironaka@snut.ac.kr

MS received 5 October 2010; revised 1 September 2011; accepted 14 October 2011

Abstract. In this paper, we propose a new adaptive $\mathcal{H}_\infty$ synchronization strategy, called an adaptive fuzzy delayed feedback $\mathcal{H}_\infty$ synchronization (AFDFHS) strategy, for chaotic systems with uncertain parameters and external disturbances. Based on Lyapunov–Krasovskii theory, Takagi–Sugeno (T–S) fuzzy model and adaptive delayed feedback $\mathcal{H}_\infty$ control scheme, the AFDFHS controller is presented such that the synchronization error system is asymptotically stable with a guaranteed $\mathcal{H}_\infty$ performance. It is shown that the design of the AFDFHS controller with adaptive law can be achieved by solving a linear matrix inequality (LMI), which can be easily facilitated by using some standard numerical packages. An illustrative example is given to demonstrate the effectiveness of the proposed AFDFHS approach.

Keywords. Chaos synchronization; Takagi–Sugeno (T–S) fuzzy model; adaptive $\mathcal{H}_\infty$ control; linear matrix inequality (LMI); delayed feedback control.

PACS Nos 05.45.Gg; 05.45.–a; 07.05.Mh

1. Introduction

Chaos is a very interesting nonlinear phenomenon and has extensive applications in many areas. Since the first work of Pecora and Carrol in 1990 [1], chaos synchronization has received increasing attention due to its theoretical challenge and its great potential applications in secure communication, economics, signal generator design, chemical reaction, biological systems, and so on [2]. In the literature, various synchronization schemes, such as variable structure control [3], OGY method [4], parameters adaptive control [5,6], observer-based control [7], active control [8,9], time-delay feedback approach [10], backstepping design technique [11,12], passivity-based control [13,14], and so on, have been successfully applied to the chaos synchronization. Over the past several years, the delayed feedback approach [15] has received considerable attention. The use of time delay in the feedback loop eliminates the need for explicitly determining any information

DOI: 10.1007/s12043-011-0239-4; ***e*Publication: 13 January 2012**

about the underlying dynamics other than the period of the desired orbit. The authors in [16–18] have used the linear and nonlinear parts of Lur'e chaotic systems to achieve synchronization. Despite some advances in the delayed feedback control, in general, it is difficult to get the corresponding linearized models along the system trajectory during the drive-response procedure.

In recent years, fuzzy logic has received much attention as a powerful tool for the nonlinear control. Among various kinds of fuzzy methods, Takagi–Sugeno (T–S) fuzzy model provides a successful method to describe certain complex nonlinear systems using some local linear subsystems [19,20]. These linear subsystems are smoothly blended together through fuzzy membership functions. It is therefore intuitive to believe that the T–S fuzzy model can be used to develop synchronization methods via the delayed feedback control without the assumption employed in [16–18]. Recently, a T–S fuzzy model-based delayed feedback synchronization controller was proposed for chaotic systems in [21]. However, this work was restricted to chaotic systems without unknown parameters and external disturbances. In real situation, some disturbances and unknown parameters always exist that may cause instability and poor performance. Therefore, knowledge of the adaptive synchronization for chaotic systems with unknown parameters and external disturbances is of considerable practical importance. To the best of our knowledge, however, for the T–S fuzzy model-based adaptive $\mathcal{H}_\infty$ delayed feedback synchronization of chaotic systems with both uncertain parameters and external disturbances, there is no result in the literature so far, which still remains open and challenging.

In this paper, a new adaptive $\mathcal{H}_\infty$ synchronization method based on the T–S fuzzy model and the adaptive delayed feedback control is proposed for chaotic systems with uncertain parameters and external disturbances. This method is called an adaptive fuzzy delayed feedback $\mathcal{H}_\infty$ synchronization (AFDFHS) method. By the proposed scheme, the synchronization error system is asymptotically stable with a guaranteed $\mathcal{H}_\infty$ norm bound. Based on Lyapunov–Krasovskii stability theory, the design of the proposed controller can be realized by solving a linear matrix inequality (LMI), which can be facilitated readily via standard numerical algorithms [22].

This paper is organized as follows. In §2, we formulate the problem. In §3, an LMI problem for the AFDFHS of chaotic systems is proposed. In §4, an application example for Lorenz system is given, and finally, conclusions are presented in §5.

2. Problem formulation

Consider a class of uncertain chaotic systems represented by the following T–S fuzzy model:

Fuzzy Rule i:
IF ω_1 is μ_{i1} and $\ldots$ ω_s is μ_{is}, THEN

$$\dot{x}(t) = A_i x(t) + \eta_i(t) + \sum_{k=1}^{p} \Phi_k(x(t))\theta_k, \tag{1}$$

$$y(t) = Cx(t), \tag{2}$$

where $x(t) \in R^n$ is the state vector, $y(t) \in R^m$ is the output vector, $\Phi_k(x(t))$ ($k = 1, \ldots, p$): $R^n \to R^n$ is the smooth vector field satisfying the Lipschitz condition with the Lipschitz constant L_k, θ_k ($k = 1, \ldots, p$) represents the unknown constant parameter vector, $A_i \in R^{n\times n}$ and $C \in R^{m\times n}$ are known constant matrices, $\eta_i(t) \in R^n$ denotes a bias term which is generated by the fuzzy modelling procedure, ω_j ($j = 1, \ldots, s$) is the premise variable, μ_{ij} ($i = 1, \ldots, r$, $j = 1, \ldots, s$) is the fuzzy set that is characterized by membership function, p is the number of unknown parameters, r is the number of IF–THEN rules and s is the number of premise variables. Suppose the uncertain parameter vector θ_k is norm bounded by the known positive constant ξ_k. Note that the fuzzy model (1)–(2) can represent a general class of uncertain nonlinear system and we employ it for fuzzy modelling of uncertain chaotic systems.

Using a standard fuzzy inference method (using a singleton fuzzifier, product fuzzy inference and weighted average defuzzifier), the system (1)–(2) is inferred as follows:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(\omega)\left[A_i x(t) + \eta_i(t) + \sum_{k=1}^{p} \Phi_k(x(t))\theta_k\right], \tag{3}$$

$$y(t) = Cx(t), \tag{4}$$

where $\omega = [\omega_1, \ldots, \omega_s]$, $h_i(\omega) = \varpi_i(\omega)/\sum_{j=1}^{r}\varpi_j(\omega)$, ϖ_i: $R^s \to [0, 1]$ ($i = 1, \ldots, r$) is the membership function of the system with respect to the fuzzy rule i. h_i can be regarded as the normalized weight of each IF–THEN rule and it satisfies

$$h_i(\omega) \geq 0, \quad \sum_{i=1}^{r} h_i(\omega) = 1. \tag{5}$$

The system (3)–(4) is considered as a drive system.

The synchronization problem of system (3)–(4) is considered by using the drive-response configuration. According to the drive-response concept, the controlled fuzzy response system is described by the following rules:

Fuzzy Rule i:

IF ω_1 is μ_{i1} and $\ldots$ ω_s is μ_{is}, THEN

$$\dot{\hat{x}}(t) = A_i\hat{x}(t) + \eta_i(t) + \sum_{k=1}^{p} \Phi_k(\hat{x}(t))\hat{\theta}_k(t) + u(t) + G_i d(t), \tag{6}$$

$$\hat{y}(t) = C\hat{x}(t), \tag{7}$$

where $\hat{x}(t) \in R^n$ is the state vector of the response system, $\hat{y}(t) \in R^m$ is the output vector of the response system, $u(t) \in R^n$ is the control input, $d(t) \in R^q$ is the external disturbance, $\hat{\theta}_k(t)$ ($k = 1, \ldots, p$) is the estimate of θ_k and $G_i \in R^{n\times q}$ is a known constant matrix. The fuzzy response system can be inferred as

$$\begin{aligned}\dot{\hat{x}}(t) =& \sum_{i=1}^{r} h_i(\omega)\\ &\times\left[A_i\hat{x}(t) + \eta_i(t) + \sum_{k=1}^{p} \Phi_k(\hat{x}(t))\hat{\theta}_k(t) + u(t) + G_i d(t)\right],\end{aligned} \tag{8}$$

$$\hat{y}(t) = C\hat{x}(t). \tag{9}$$

Define the synchronization error $e(t) = \hat{x}(t) - x(t)$. Then we obtain the synchronization error system

$$\dot{e}(t) = \sum_{i=1}^{r} h_i(\omega)\Big[A_i e(t) + \sum_{k=1}^{p}(\Phi_k(\hat{x}(t))\tilde{\theta}_k(t) + \tilde{\Phi}_k(\hat{x}(t), x(t))\theta_k) + u(t) + G_i d(t)\Big], \tag{10}$$

where $\tilde{\Phi}_k(\hat{x}(t), x(t)) = \Phi_k(\hat{x}(t)) - \Phi_k(x(t))$ and $\tilde{\theta}_k(t) = \hat{\theta}_k(t) - \theta_k$.

The objective of this study is to design the AFDFHS controller $u(t)$ for the chaotic system (3)–(4) with a guaranteed performance in the $\mathcal{H}_\infty$ sense. Specifically, given a prescribed level of disturbance attenuation $\gamma > 0$, find the AFDFHS controller $u(t)$ such that the synchronization error system (10) with $d(t) = 0$ is asymptotically stable ($\lim_{t\to\infty} e(t) = 0$) and

$$\int_0^\infty e^T(t)Se(t)\mathrm{d}t < \gamma^2 \int_0^\infty d^T(t)d(t)\mathrm{d}t, \tag{11}$$

under zero-initial conditions for all nonzero $d(t) \in L_2[0,\infty)$, where $L_2[0,\infty)$ is the space of square integrable vector functions over $[0,\infty)$. In this case, the synchronization error system (10) is said to be asymptotically stable with γ as the $\mathcal{H}_\infty$ performance.

Remark 1. The $\mathcal{H}_\infty$ norm [23] is defined as

$$\|T_{ed}\|_\infty = \frac{\sqrt{\int_0^\infty e^T(t)Se(t)\mathrm{d}t}}{\sqrt{\int_0^\infty d^T(t)d(t)\mathrm{d}t}},$$

where T_{ed} is a transfer function matrix from $d(t)$ to $e(t)$. For a given level $\gamma > 0$, $\|T_{ed}\|_\infty < \gamma$ can be restated in the equivalent form (11). If we define

$$H(t) = \frac{\int_0^t e^T(\sigma)Se(\sigma)\mathrm{d}\sigma}{\int_0^t d^T(\sigma)d(\sigma)\mathrm{d}\sigma}, \tag{12}$$

the relation (11) can be represented by

$$H(\infty) < \gamma^2. \tag{13}$$

In §4, through the plot of $H(t)$ vs. time, the relation (13) is verified.

3. Main result

In this section, we design the AFDFHS controller for the chaotic system (3)–(4). The following theorem presents an LMI-based criterion to obtain the AFDFHS controller.

Theorem 1. *For given $\gamma > 0$, $1 > \zeta > 0$ and $S = S^T > 0$, if there exist $P = P^T > 0$, $Q = Q^T > 0$, $R = R^T > 0$, $W = W^T > 0$, and M_j such that*

$$\begin{bmatrix} [1,1] & -\zeta M_j C & W & PG_i & P & I & I \\ -\zeta C^T M_j^T & -R & -W & 0 & 0 & 0 & 0 \\ W & -W & -\frac{1}{\tau}Q & 0 & 0 & 0 & 0 \\ G_i^T P & 0 & 0 & -\gamma^2 I & 0 & 0 & 0 \\ P & 0 & 0 & 0 & -I & 0 & 0 \\ I & 0 & 0 & 0 & 0 & -\frac{1}{\sum_{k=1}^{p}\xi_k^2 L_k^2} I & 0 \\ I & 0 & 0 & 0 & 0 & 0 & -S^{-1} \end{bmatrix} < 0, \quad (14)$$

where

$$[1,1] = A_i^T P + PA_i + M_j C + C^T M_j^T + \tau Q + R,$$

for $i, j = 1, 2, \ldots, r$, the synchronization error system (10) *is asymptotically stable with $\mathcal{H}_\infty$ performance γ. Then the controller and the adaptive law for the AFDFHS are given by*

$$u(t) = \sum_{j=1}^{r} h_j(\omega) P^{-1} M_j \left[(\hat{y}(t) - y(t)) - \zeta(\hat{y}(t-\tau) - y(t-\tau)) \right], \quad (15)$$

$$\dot{\hat{\theta}}_k(t) = -\Phi_k^T(\hat{x}(t)) Pe(t), \quad k = 1, \ldots, p, \quad (16)$$

where $\tau > 0$ is the chosen time delay or the propagation as it is in [16,17]. In addition, $\tilde{\theta}_k(t)$ is bounded for any bounded disturbance.

Proof. The AFDFHS controller can be constructed via the parallel distributed compensation. The controller is described by the following rules:

Fuzzy Rule j:

IF ω_1 is μ_{j1} and $\ldots$ ω_s is μ_{js}, THEN

$$u(t) = K_j \left[(\hat{y}(t) - y(t)) - \zeta(\hat{y}(t-\tau) - y(t-\tau)) \right], \quad (17)$$

where $K_j \in R^{n \times m}$ is the gain matrix of the controller for the fuzzy rule j. The fuzzy controller can be inferred as

$$u(t) = \sum_{j=1}^{r} h_j(\omega) K_j \left[(\hat{y}(t) - y(t)) - \zeta(\hat{y}(t-\tau) - y(t-\tau)) \right]$$

$$= \sum_{j=1}^{r} h_j(\omega) K_j C \left[e(t) - \zeta e(t-\tau) \right]. \quad (18)$$

The closed-loop synchronization error system with the control input (18) can be written as

$$\dot{e}(t)=\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(\omega)h_j(\omega)\Bigg[(A_i+K_jC)e(t)-\zeta K_jCe(t-\tau) + \sum_{k=1}^{p}(\Phi_k(\hat{x}(t))\tilde{\theta}_k(t)+\tilde{\Phi}_k(\hat{x}(t),x(t))\theta_k)+G_id(t)\Bigg]. \tag{19}$$

First, to establish the $\mathcal{H}_\infty$ performance for the synchronization error system (19), consider the following Lyapunov–Krasovskii functional:

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t), \tag{20}$$

where

$$V_1(t) = e^T(t)Pe(t) + \sum_{k=1}^{p}\tilde{\theta}_k^T(t)\tilde{\theta}_k(t), \tag{21}$$

$$V_2(t) = \int_{-\tau}^{0}\int_{t+\beta}^{t} e^T(\alpha)Qe(\alpha)\mathrm{d}\alpha\mathrm{d}\beta, \tag{22}$$

$$V_3(t) = \int_{t-\tau}^{t} e^T(\sigma)Re(\sigma)\mathrm{d}\sigma, \tag{23}$$

$$V_4(t) = \left[\int_{t-\tau}^{t} e(\sigma)\mathrm{d}\sigma\right]^T W \left[\int_{t-\tau}^{t} e(\sigma)\mathrm{d}\sigma\right]. \tag{24}$$

The time derivative of $V_1(t)$ along the trajectory of (19) is

$$\begin{aligned}
\dot{V}_1(t) &= \dot{e}(t)^TPe(t) + e^T(t)P\dot{e}(t) + 2\sum_{k=1}^{p}\tilde{\theta}_k^T(t)\dot{\hat{\theta}}_k(t)\\
&= \sum_{i=1}^{r}\sum_{j=1}^{r}h_i(\omega)h_j(\omega)\\
&\quad\times\Bigg\{e^T(t)[A_i^TP + PA_i + PK_jC + C^TK_j^TP]e(t)\\
&\quad - \zeta e^T(t)PK_jCe(t-\tau) - \zeta e^T(t-\tau)C^TK_j^TPe(t)\\
&\quad + e^T(t)PG_id(t) + d^T(t)G_i^TPe(t)\\
&\quad + 2e(t)^TP\sum_{k=1}^{p}(\Phi_k(\hat{x}(t))\tilde{\theta}_k(t) + \tilde{\Phi}_k(\hat{x}(t),x(t))\theta_k)\Bigg\}\\
&\quad + 2\sum_{k=1}^{p}\tilde{\theta}_k^T(t)\dot{\hat{\theta}}_k(t).
\end{aligned}$$

If we use the inequality $X^TY + Y^TX \leq X^T\Lambda X + Y^T\Lambda^{-1}Y$, which is valid for any matrices $X \in R^{n\times m}$, $Y \in R^{n\times m}$, $\Lambda = \Lambda^T > 0$, $\Lambda \in R^{n\times n}$, we have

$$e(t)^TPG_id(t) + d^T(t)G_i^TPe(t) \leq \gamma^2 d^T(t)d(t) + \frac{1}{\gamma^2}e(t)^TPG_iG_i^TPe(t), \tag{25}$$

$$\begin{aligned} 2e(t)^TP\tilde{\Phi}_k(\hat{x}(t), x(t))\theta_k &\leq e(t)^TPPe(t) \\ &\quad + \theta_k^T\tilde{\Phi}_k^T(\hat{x}(t), x(t))\tilde{\Phi}_k(\hat{x}(t), x(t))\theta_k \\ &= e(t)^TPPe(t) + \|\tilde{\Phi}_k(\hat{x}(t), x(t))\theta_k\|^2 \\ &\leq e(t)^TPPe(t) + \xi_k^2L_k^2\|e(t)\|^2 \\ &= e(t)^T[PP + \xi_k^2L_k^2I]e(t). \end{aligned} \tag{26}$$

Using (25) and (26), we obtain

$$\begin{aligned} \dot{V}_1(t) \leq \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\omega)h_j(\omega) \\ \times \Bigg\{ e^T(t)\Bigg[A_i^TP + PA_i + PK_jC + C^TK_j^TP + PP + \sum_{k=1}^{p}\xi_k^2L_k^2I \\ + \frac{1}{\gamma^2}PG_iG_i^TP\Bigg]e(t) - \zeta e^T(t)PK_jCe(t-\tau) - \zeta e^T(t-\tau)C^T \\ \times K_j^TPe(t) + 2e(t)^TP\sum_{k=1}^{p}\Phi_k(\hat{x}(t))\tilde{\theta}_k(t) + \gamma^2d^T(t)d(t)\Bigg\} \\ + 2\sum_{k=1}^{p}\tilde{\theta}_k^T(t)\dot{\hat{\theta}}_k(t). \end{aligned}$$

Since

$$2e(t)^TP\Phi_k(\hat{x}(t))\tilde{\theta}_k(t) = 2\tilde{\theta}_k^T(t)\Phi_k^T(\hat{x}(t))Pe(t), \tag{27}$$

we obtain

$$\begin{aligned} \dot{V}_1(t) \leq \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\omega)h_j(\omega) \\ \times \Bigg\{ e^T(t)\Bigg[A_i^TP + PA_i + PK_jC + C^TK_j^TP + PP + \sum_{k=1}^{p}\xi_k^2L_k^2I \\ + \frac{1}{\gamma^2}PG_iG_i^TP\Bigg]e(t) - \zeta e^T(t)PK_jCe(t-\tau) - \zeta e^T(t-\tau)C^T \\ \times K_j^TPe(t) + \gamma^2d^T(t)d(t)\Bigg\} \\ + 2\sum_{k=1}^{p}\tilde{\theta}_k^T(t)[\dot{\hat{\theta}}_k(t) + \Phi_k^T(\hat{x}(t))Pe(t)]. \end{aligned}$$

The time derivative of $V_2(t)$ is

$$\dot{V}_2(t) = \tau e^T(t)Qe(t) - \int_{t-\tau}^{t} e^T(\sigma)Qe(\sigma)\mathrm{d}\sigma. \tag{28}$$

Using the inequality [24]

$$\left[\int_{t-\tau}^{t} e(\sigma)\mathrm{d}\sigma\right]^T Q \left[\int_{t-\tau}^{t} e(\sigma)\mathrm{d}\sigma\right] \le \tau \int_{t-\tau}^{t} e(\sigma)^T Qe(\sigma)\mathrm{d}\sigma, \tag{29}$$

we have

$$\dot{V}_2(t) \le \tau e^T(t)Qe(t) - \frac{1}{\tau}\left[\int_{t-\tau}^{t} e(\sigma)\mathrm{d}\sigma\right]^T Q \left[\int_{t-\tau}^{t} e(\sigma)\mathrm{d}\sigma\right]. \tag{30}$$

The time derivative of $V_3(t)$ is written as

$$\dot{V}_3(t) = e(t)^T Re(t) - e^T(t-\tau)Re(t-\tau). \tag{31}$$

Since $\dot{V}_4(t)$ yields the relation

$$\begin{aligned} \dot{V}_4(t) &= [e(t) - e(t-\tau)]^T W \left[\int_{t-\tau}^{t} e(\sigma)\mathrm{d}\sigma\right] \\ &\quad + \left[\int_{t-\tau}^{t} e(\sigma)\mathrm{d}\sigma\right]^T W[e(t) - e(t-\tau)], \end{aligned} \tag{32}$$

we have the derivative of $V(t)$ as

$$\begin{aligned} \dot{V}(t) &= \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) + \dot{V}_4(t) \\ &\le \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\omega)h_j(\omega) \\ &\quad \times \left\{ \begin{bmatrix} e(t) \\ e(t-\tau) \\ \int_{t-\tau}^{t} e(\sigma)d\sigma \end{bmatrix}^T \begin{bmatrix} (1,1) & -\zeta PK_jC & W \\ -\zeta C^T K_j^T P & -R & -W \\ W & -W & -\frac{1}{\tau}Q \end{bmatrix} \right. \\ &\quad \left. \times \begin{bmatrix} e(t) \\ e(t-\tau) \\ \int_{t-\tau}^{t} e(\sigma)d\sigma \end{bmatrix} - e^T(t)Se(t) + \gamma^2 d^T(t)d(t) \right\} \\ &\quad + 2\sum_{k=1}^{p} \tilde{\theta}_k^T(t)[\dot{\hat{\theta}}_k(t) + \Phi_k^T(\hat{x}(t))Pe(t)], \end{aligned}$$

where

$$\begin{aligned} (1,1) &= A_i^T P + PA_i + PK_jC + C^T K_j^T P + PP \\ &\quad + \sum_{k=1}^{p} \xi_k^2 L_k^2 I + \frac{1}{\gamma^2} PG_iG_i^T P + \tau Q + R + S. \end{aligned} \tag{33}$$

If the adaptive law (16) is used and the following matrix inequality is satisfied

$$\begin{bmatrix} (1,1) & -\zeta PK_jC & W \\ -\zeta C^T K_j^T P & -R & -W \\ W & -W & -\frac{1}{\tau}Q \end{bmatrix} < 0, \tag{34}$$

for $i, j = 1, 2, \ldots, r$, we have

$$\begin{aligned} \dot{V}(t) &< \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(\omega)h_j(\omega)\{-e^T(t)Se(t) + \gamma^2 d^T(t)d(t)\} \\ &= -e^T(t)Se(t) + \gamma^2 d^T(t)d(t). \end{aligned} \tag{35}$$

Since $V(t)$ is radially unbounded with regard to $\tilde{\theta}_k(t)$ $(k = 1, \ldots, p)$, the relation (35) guarantees that $\tilde{\theta}_k(t)$ is bounded for any bounded disturbance $d(t)$. Integrating both sides of (35) from 0 to ∞ gives

$$V(\infty) - V(0) < -\int_0^\infty e^T(t)Se(t)\mathrm{d}t + \gamma^2 \int_0^\infty d^T(t)d(t)\mathrm{d}t.$$

Since $V(\infty) \geq 0$ and $V(0) = 0$, we have the relation (11). From Schur complement, the matrix inequality (34) is equivalent to

$$\begin{bmatrix} \{1,1\} & -\zeta PK_jC & W & PG_i & P & I & I \\ -\zeta C^T K_j^T P & -R & -W & 0 & 0 & 0 & 0 \\ W & -W & -\frac{1}{\tau}Q & 0 & 0 & 0 & 0 \\ G_i^T P & 0 & 0 & -\gamma^2 I & 0 & 0 & 0 \\ P & 0 & 0 & 0 & -I & 0 & 0 \\ I & 0 & 0 & 0 & 0 & -\frac{1}{\sum_{k=1}^{p}\xi_k^2 L_k^2} I & 0 \\ I & 0 & 0 & 0 & 0 & 0 & -S^{-1} \end{bmatrix} < 0, \tag{36}$$

where

$$\{1,1\} = A_i^T P + PA_i + PK_jC + C^T K_j^T P + \tau Q + R.$$

If we let $M_j = PK_j$, eq. (36) is equivalently changed into the LMI (14). Then the gain matrix of the control input $u(t)$ is given by $K_j = P^{-1}M_j$.

Next, we show that, under the condition (14) of Theorem 1, the synchronization error system (19) with $d(t) = 0$ is asymptotically stable. When $d(t) = 0$, we obtain

$$\dot{V}(t) < -e^T(t)Se(t) \leq 0 \tag{37}$$

from (35). This guarantees

$$\lim_{t\to\infty} e(t) = 0 \tag{38}$$

from Lyapunov–Krasovskii stability theory. This completes the proof. □

Remark 2. The LMI problem given in Theorem 1 is to determine whether the solution exists or not. It is called the feasibility problem. The LMI problem can be solved efficiently by using the recently developed convex optimization algorithms [22]. In this paper, in order to solve the LMI problem, we utilize MATLAB LMI Control Toolbox [25], which implements state-of-the-art interior-point algorithms.

Remark 3. An especially powerful control scheme, which is called time-delayed feedback control, was introduced by Pyragas [15]. It constructs a control input from the difference of the present state of a given nondelayed system to its delayed value. For proper choices of the time delay, the control input vanishes if the state to be stabilized is reached. Thus, the method is noninvasive. This control scheme is easy to implement in an experimental set-up and numerical calculation. It can stabilize fixed points as well as periodic orbits even if the dynamics are very fast. The time-delayed feedback control can also be applied to the adaptive control problem for uncertain dynamical systems without time-delay [26]. In this paper, we extend the time-delayed feedback control to the adaptive time-delayed feedback $\mathcal{H}_\infty$ control problem for uncertain chaotic systems represented by the T–S fuzzy model without time-delay.

4. Numerical example

Consider the following Lorenz system:

$$\begin{aligned}\dot{x}_1(t) &= -10x_1(t) + 10x_2(t),\\ \dot{x}_2(t) &= 28x_1(t) - x_2(t) - x_1(t)x_3(t),\\ \dot{x}_3(t) &= x_1(t)x_2(t) - \kappa x_3(t). \end{aligned} \tag{39}$$

The parameter κ is assumed to be unknown. To apply the proposed scheme, we need the T–S fuzzy model representation of the Lorenz system. By defining two fuzzy sets, we can

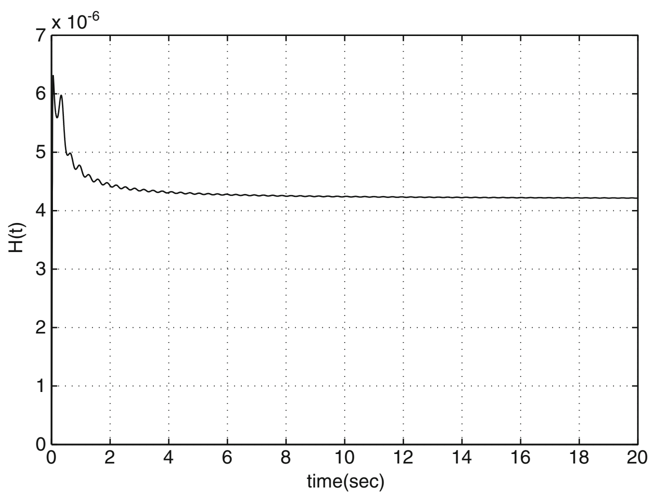

Figure 1. The plot of $H(t)$ vs. time.

obtain the following fuzzy drive system that exactly represents the nonlinear equation of the Lorenz system:

$$\dot{x}(t) = \sum_{i=1}^{2} h_i(\omega)[A_i x(t) + \eta_i(t) + \Phi(x(t))\theta], \tag{40}$$

where

$$A_1 = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & -d \\ 0 & d & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & d \\ 0 & -d & 0 \end{bmatrix},$$

$$\eta_1(t) = \eta_2(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \Phi(x(t)) = \begin{bmatrix} 0 \\ 0 \\ -x_3(t) \end{bmatrix}, \qquad \theta = \kappa. \tag{41}$$

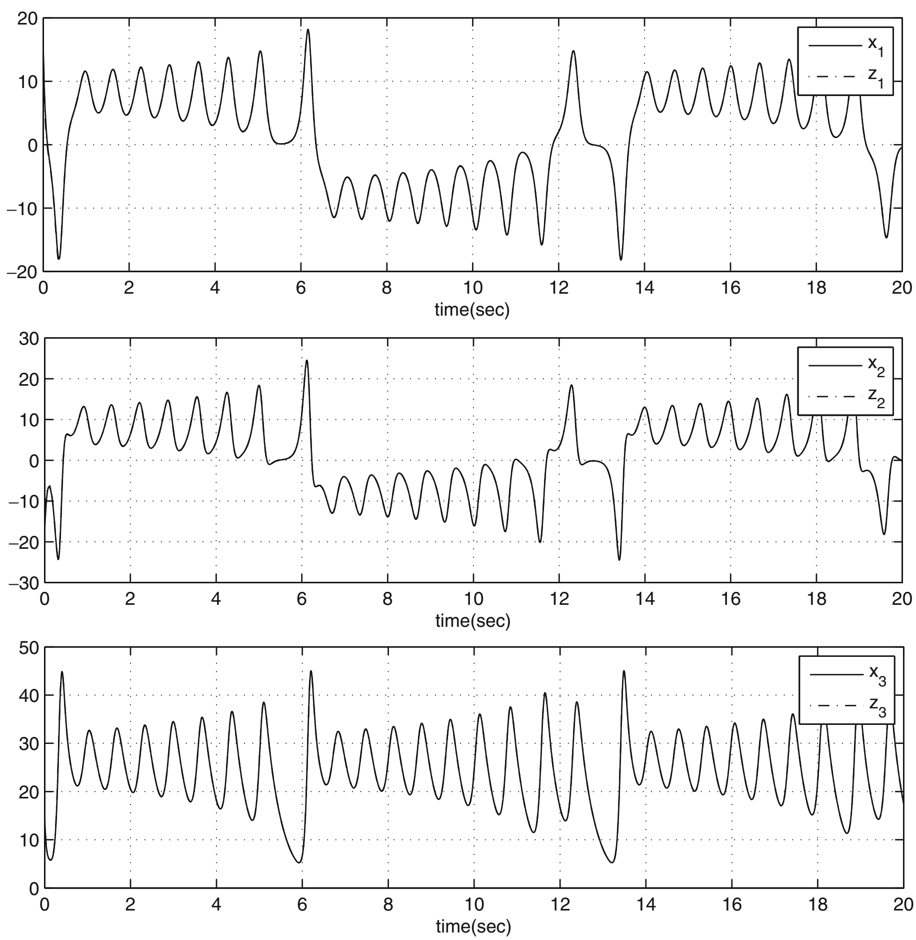

Figure 2. State trajectories.

The membership functions are

$$h_1(\omega) = \frac{1}{2}\left(1 + \frac{x_1}{d}\right), \qquad h_2(\omega) = \frac{1}{2}\left(1 - \frac{x_1}{d}\right). \tag{42}$$

For the numerical simulation, we use the following parameters:

$$d = 30, \quad \tau = 0.2, \quad \zeta = 0.1, \quad \kappa = \frac{8}{3}, \tag{43}$$

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad G_1 = G_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad S = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}. \tag{44}$$

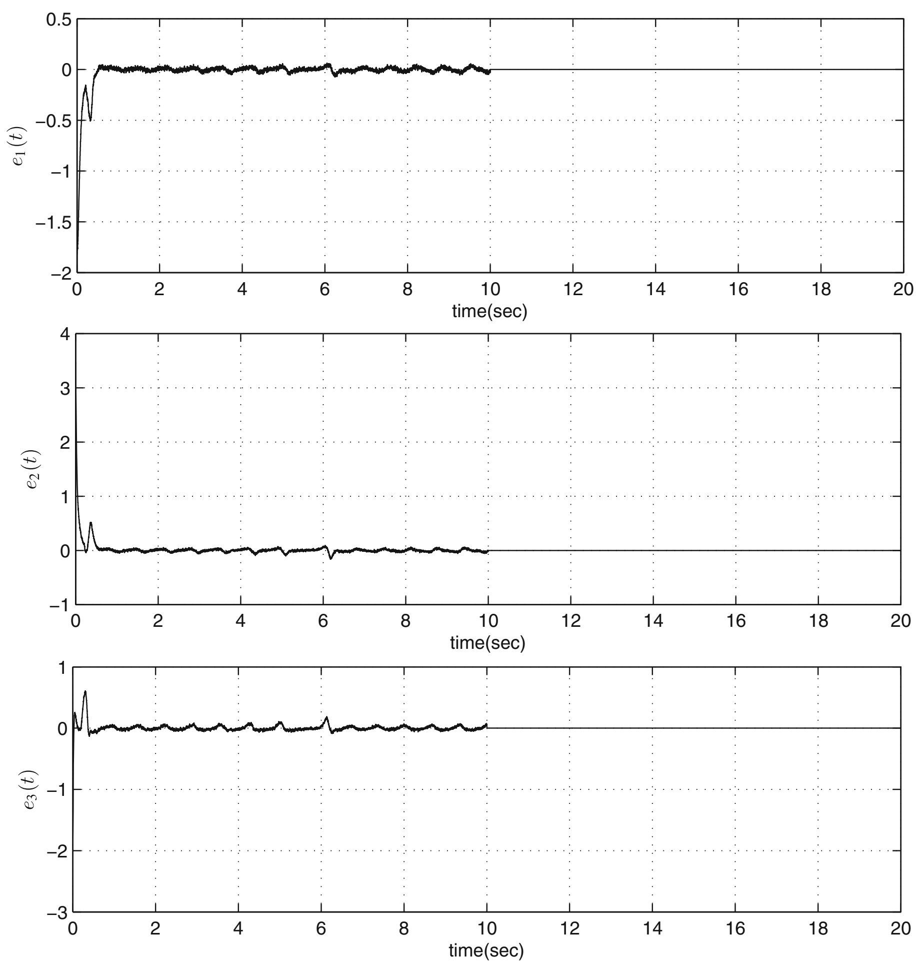

Figure 3. Synchronization errors.

Applying Theorem 1 to the fuzzy system (40) with the $\mathcal{H}_\infty$ performance $\gamma = 0.1$ yields

$$P = \begin{bmatrix} 2.9323 & -1.4547 & -0.9776 \\ -1.4547 & 2.2975 & -0.4132 \\ -0.9776 & -0.4132 & 2.0402 \end{bmatrix},$$

$$M_1 = M_2 = \begin{bmatrix} -48.7828 & -12.0774 \\ -93.8064 & 11.7248 \\ 0.2058 & -133.0279 \end{bmatrix}.$$

Figure 1 shows the plot of $H(t)$ vs. time when $d(t) = \sin(10t)$. Figure 1 verifies $H(\infty) < \gamma^2 = 0.01$. This means that the $\mathcal{H}_\infty$ norm from the external disturbance $d(t)$ to the synchronization error $e(t)$ is reduced within the $\mathcal{H}_\infty$ norm bound γ. Figure 2 shows state trajectories for drive and response systems when the initial conditions are given by

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 15.8 \\ -17.48 \\ 15.64 \end{bmatrix}, \quad \begin{bmatrix} \hat{x}_1(0) \\ \hat{x}_2(0) \\ \hat{x}_3(0) \end{bmatrix} = \begin{bmatrix} 13.8 \\ -14 \\ 13 \end{bmatrix}, \quad \hat{\theta}(0) = 0, \qquad (45)$$

and the external disturbance $d(t)$ is given by

$$d(t) = \begin{cases} w(t), & 0 \le t \le 10, \\ 0, & \text{otherwise,} \end{cases}$$

where $w(t)$ means a Gaussian noise with mean 0 and variance 100. Figure 3 shows that the proposed method reduces the effect of external disturbance $d(t)$ on the synchronization error $e(t)$. In addition, it is shown that the synchronization error $e(t)$ goes to zero after the external disturbance $d(t)$ disappears. The estimate $\hat{\theta}(t)$ of the unknown parameter θ is

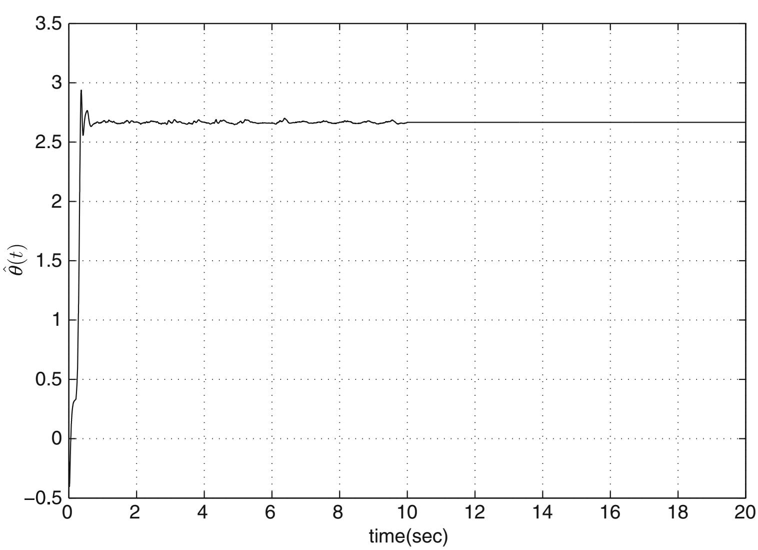

Figure 4. The estimate value $\hat{\theta}(t)$ of parameter θ.

illustrated in figure 4, which shows that the estimate $\hat{\theta}(t)$ is bounded around target value 8/3. However, the estimate $\hat{\theta}(t)$ approaches rapidly to target value 8/3 after the external disturbance $d(t)$ disappears.

5. Conclusion

In this paper, we have proposed the AFDFHS controller, which is a new adaptive $\mathcal{H}_\infty$ synchronization controller, for chaotic systems with uncertain parameter and external disturbance. Based on Lyapunov–Krasovskii theory and LMI approach, the synchronization error system was shown to be asymptotically stable with a guaranteed $\mathcal{H}_\infty$ performance. It was also shown that the AFDFHS controller can be determined by solving the delay-dependent LMI. The synchronization for the Lorenz system was given to illustrate the effectiveness of the proposed scheme. Finally, the proposed AFDFHS scheme has the advantage that it can be effectively used to adaptive $\mathcal{H}_\infty$ control and synchronization of other uncertain nonlinear systems described by a T–S fuzzy model.

References

[1] L M Pecora and T L Carroll, *Phys. Rev. Lett.* **64**, 821 (1996)
[2] G Chen and X Dong, *From chaos to order* (World Scientific, Singapore, 1998)
[3] C C Wang and J P Su, *Chaos, Solitons and Fractals* **20**, 967 (2004)
[4] E Ott, C Grebogi and J A Yorke, *Phys. Rev. Lett.* **64**, 1196 (1990)
[5] Y Wang, Z H Guan and H O Wang, *Phys. Lett.* **A312**, 34 (2003)
[6] J H Park, *Int. J. Nonlinear Sci. Numer. Simul.* **6**, 201 (2005)
[7] X S Yang and G Chen, *Chaos, Solitons and Fractals* **13**, 1303 (2002)
[8] E Bai and K Lonngen, *Phys. Rev.* **E8**, 51 (1997)
[9] E W Bai and K E Lonngren, *Chaos, Solitons and Fractals* **11**, 1041 (2000)
[10] O M Kwon and J H Park, *Chaos, Solitons and Fractals* **23**, 445 (2005)
[11] X Wu and J Lu, *Chaos, Solitons and Fractals* **18**, 721 (2003)
[12] J Hu, S Chen and L Chen, *Phys. Lett.* **339**, 455 (2005)
[13] C K Ahn, *Mod. Phys. Lett.* **B23**, 3531 (2009)
[14] C K Ahn, *Chin. Phys. Lett.* **27**, 010503 (2010)
[15] K Pyragas, *Phys. Lett.* **A170**, 421 (1992)
[16] M E Yalcin, J A K Suykens and J Vandewalle, *Int. J. Bifurcation and Chaos* **11**, 1707 (2001)
[17] X X Liao and G R Chen, *Int. J. Bifurcation and Chaos* **13**, 207 (2003)
[18] F L Zhu, *Phys. Lett.* **A372**, 223 (2008)
[19] T Takagi and M Sugeno, *IEEE Trans. Syst., Man. Cyb.* **15**, 116 (1985)
[20] K Tanaka and M Sugeno, *Fuzzy Sets Syst.* **45**, 135 (1992)
[21] H K Lam, W K Ling, H H Iu and S S H Ling, *IEEE Trans. Circuits Syst. I* **55**, 893 (2008)
[22] S Boyd, L E Ghaoui, E Feron and V Balakrishnan, *Linear matrix inequalities in systems and control theory* (SIAM, Philadelphia, PA, 1994)
[23] Anton Stoorvogel, *The $\mathcal{H}_\infty$ control problem: A state-space approach* (Prentice Hall, London, 1992)
[24] E Noldus, *Int. J. Control* **41**, 947 (1985)
[25] P Gahinet, A Nemirovski, A J Laub and M Chilali, *LMI control toolbox* (The Mathworks Inc., Natik, 1995)
[26] B Robert, H H C Iu and M Feki, *J. Circuit. Syst. Comp.* **13**, 519 (2004)