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New exact travelling wave solutions of bidirectional wave equations

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Abstract. The surface water waves in a water tunnel can be described by systems of the form [Bona and Chen, *Physica* **D116**, 191 (1998)]

$$\begin{cases} v_t + u_x + (uv)_x + au_{xxx} - bv_{xxt} = 0, \\ u_t + v_x + uu_x + cv_{xxx} - du_{xxt} = 0, \end{cases} \tag{1}$$

where a, b, c and d are real constants. In general, the exact travelling wave solutions will be helpful in the theoretical and numerical study of the nonlinear evolution systems. In this paper, we obtain exact travelling wave solutions of system (1) using the modified tanh–coth function method with computerized symbolic computation.

Keywords. Travelling wave solutions; tanh–coth function method; Riccati equations; symbolic computation.

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1. Introduction

The investigation of exact travelling wave solutions to nonlinear evolution equations plays an important role in the study of nonlinear physical phenomena. In this paper, we consider the following system which was derived by Bona and Chen [1] having the form

$$\begin{cases} v_t + u_x + (uv)_x + au_{xxx} - bv_{xxt} = 0, \\ u_t + v_x + uu_x + cv_{xxx} - du_{xxt} = 0, \end{cases} \tag{2}$$

where a, b, c and d are real constants. Here x represents the distance along the channel, t is the elapsed time, the variable $v(x, t)$ is the dimensionless deviation of the water surface from its undisturbed position and $u(x, t)$ is the dimensionless horizontal velocity. This set of equations is used as a model equation for the propagation of long waves on the surface of water with a small amplitude [1,2].

In the past two decades, several methods such as Hirota's method [3], Jacobi elliptic function method [4], variational iteration method [5–10], exp-function method [11–17], homotopy perturbation method [18–21] and so on have been developed and extended for finding travelling wave solutions to nonlinear evolution equations. However, practically there is no unified method that can be used to handle all types of nonlinearity.

The tanh-function method is an effective and direct algebraic method for finding the exact solutions of nonlinear evolution problems [22,23]. The concept of tanh-function method was first proposed in [22] and subsequently some generalizations of this method such as extended tanh function method [24,25], the modified extended tanh-function method [26] and the modified tanh–coth method [27] have been proposed using different auxiliary ordinary differential equations and applied to many nonlinear problems [28,29]. The modified tanh–coth expansion method for finding solitary travelling wave solutions to nonlinear evolution equations has been used extensively in the literature. It is a natural extension to the basic tanh-function expansion method. Wazzan [27] used modified tanh–coth method and obtained new exact solutions for some important nonlinear problems. More recently, Lee and Sakthivel [30] implemented modified tanh–coth method to obtain single soliton solutions for the higher-dimensional integrable equations whereas the extended Jacobi elliptic function method is applied to derive doubly periodic wave solutions.

In this paper, we concentrate on finding travelling wave solutions of system (2) with the help of modified tanh–coth method. The travelling wave solutions may be useful in the theoretical and numerical studies of the model systems. The computer symbolic systems such as Maple and Mathematica allow us to perform complicated and tedious calculations.

2. Exact travelling wave solutions

The standard tanh method was developed by Malfliet [22], where the tanh was introduced as a new variable, because all derivatives of a tanh are represented by a tanh itself. In this section, we describe briefly the modified tanh–coth function method in its systematized form [26,28]. Suppose we are given a nonlinear evolution equation in the form of a partial differential equation (PDE) for a function $u(x,t)$. First, we seek travelling wave solutions by taking $u(x,t) = u(\eta)$, $\eta = kx - \omega t$, where k and ω represent the wave number and velocity of the travelling wave respectively. Substitution into the PDE yields an ordinary differential equation (ODE) for $u(\eta)$. The ordinary differential equation is then integrated as long as all terms contain derivatives, where the integration constants are considered as zero. The resulting ODE is then solved by the tanh–coth method which admits the use of a finite series of functions of the form

$$\begin{cases} u(\eta) = a_0 + \sum_{m=1}^{M} a_m Y^m(\eta) + \sum_{m=1}^{M} b_m Y^{-m}(\eta), \\ v(\eta) = c_0 + \sum_{n=1}^{N} c_n Y^n(\eta) + \sum_{n=1}^{N} d_n Y^{-n}(\eta) \end{cases} \tag{3}$$

and the Riccati equation

$$Y' = A + BY + CY^2, \tag{4}$$

where A, B and C are constants to be prescribed later. Here M and N are positive integers that will be determined. The parameters M and N are usually obtained by balancing the linear terms of highest order in the resulting equation with the highest order nonlinear terms. Substituting (3) in the ODE and using (4) results in an algebraic system of equations in powers of Y that will lead to the determination of the parameters a_m, b_m, c_n, d_n, k and ω. Having determined these parameters we obtain an analytic solution $u(x,t)$ in a closed form.

Note 2.1. In this paper, we shall consider the following special solutions of the Riccati equation (4):

(i) $A=\frac{1}{2}, B=0, C=-\frac{1}{2}$, eq. (4) has solutions $Y=\tanh\eta \pm i \operatorname{sech}\eta$ and $Y=\coth\eta \pm \operatorname{csch}\eta$.

(ii) $A=1,\ B=0,\ C=-4$, eq. (4) has solutions $Y=\frac{1}{2}\tanh 2\eta$ and $Y=\frac{1}{4}(\tanh\eta + \coth\eta)$.

(iii) $A=1,\ B=0,\ C=4$, eq. (4) has solutions $Y=\frac{1}{2}\tan 2\eta$ and $Y=\frac{1}{4}(\tan\eta - \cot\eta)$.

To look for the travelling wave solutions of eq. (2), we make the transformations $u(x,t)=u(\eta)$, $v(x,t)=v(\eta)$, $\eta=kx-\omega t$. Now eq. (2) can be written as

$$\begin{cases} -\omega v' + ku' + ku'v + kuv' + ak^3u''' + bk^2\omega v''' = 0, \\ -\omega u' + kv' + kuu' + ck^3v''' + dk^2\omega u''' = 0, \end{cases} \tag{5}$$

where the prime denotes derivative with respect to η.

To determine parameters M and N, we balance the linear terms of highest order in eq. (5) with the highest order nonlinear terms. This in turn gives $M=2$ and $N=2$. As a result, the modified tanh–coth method (3) admits the use of the finite expansion

$$\begin{cases} u(\eta) = a_0 + a_1Y + a_2Y^2 + \dfrac{b_1}{Y} + \dfrac{b_2}{Y^2}, \\ v(\eta) = c_0 + c_1Y + c_2Y^2 + \dfrac{d_1}{Y} + \dfrac{d_2}{Y^2}. \end{cases} \tag{6}$$

Substituting eq. (6) in the reduced ODE (5) and using eq. (4) collecting the coefficients of Y, yields a system of algebraic equations for $a_0, a_1, a_2, b_1, b_2, c_0, c_1, c_2, d_1, d_2, k$ and ω.

If we set $A=\frac{1}{2}, B=0, C=-\frac{1}{2}$ in eq. (4), and solving the system of algebraic equations using Maple, we obtain the following three sets of nontrivial solutions:

$$\left\{\begin{aligned} & a_0 = \mp(21b+8c+14d)\sqrt{E_1},\, a_1=\pm 6\sqrt{E_2},\, a_2=\pm 9(3b+2d)\sqrt{E_1}, \\ & b_1=0,\, b_2=0, \\ & c_0 = -1-\frac{a(21b-46d)(3b+2d)}{2c(b-6d)(3b-2d)},\, c_1=\mp 12(3b+2d)\sqrt{E_1}\sqrt{E_2}, \\ & c_2=\frac{9a(3b+2d)}{2c(3b-2d)} \\ & d_1=0,\, d_2=0,\, k=\pm\sqrt{\frac{3(3b+2d)}{c(b+2d)}},\, \omega=\mp 8c\sqrt{\frac{3(3b+2d)}{c(b+2d)}}\sqrt{E_1} \end{aligned}\right\}, \tag{7}$$

where

$$E_1 = -\frac{a}{2c(b-6d)(3b-2d)},$$

$$E_2 = -\frac{a(b-d)(3b+2d)}{c(b-6d)(3b-2d)}.$$

$$\left\{\begin{aligned} & a_0 = E_3, a_1 = 0, a_2 = b_2, b_1 = 0, b_2 = b_2, c_0 = E_4, c_1 = 0, \\ & c_2 = \frac{k^2 b_2\left((2d-b)b_2 \mp \sqrt{36abcdk^4 + (b-2d)^2 b_2^2}\right)}{6bc}, d_1 = 0, d_2 = c_2, \\ & k = -\sqrt{\frac{3(3b+2d)}{c(b+2d)}}, \\ & \omega = \frac{-(b+2d)b_2 \pm \sqrt{36abcdk^4 + (b-2d)^2 b_2^2}}{6bdk} \end{aligned}\right\}, \tag{8}$$

where

$$E_3 = -\frac{c\left\{\left(2d(c+d) + b(c-d+4cdk^2)\right)b_2 \mp (c+d)\sqrt{36abcdk^4 + (b-2d)^2 b_2^2}\right\}}{6bdk^2},$$

$$E_4 = \frac{c^2k^4\left\{18bc(ad-bc)k^4 + (b-2d+2bck^2)b_2\left((b-2d)b_2 \pm \sqrt{36abcdk^4 + (b-2d)^2 b_2^2}\right)\right\}}{18b^2}.$$

$$\left\{\begin{aligned} & a_0 = \mp 2(3b-4c+2d)\sqrt{E_1}, a_1 = \mp 3\sqrt{2}E_5, \\ & a_2 = \pm\frac{9(3b+2d)}{2}\sqrt{E_1}, b_1 = \pm 3\sqrt{2}E_5, \\ & b_2 = a_2, c_0 = -\frac{c(b-6d)(3b-2d) + 2a(3b+2d)^2}{c(b-6d)(3b-2d)}, \\ & c_1 = \mp 6\sqrt{2}(3b+2d)\sqrt{E_1}\sqrt{-E_2}, \\ & c_2 = -\frac{9a(3b+2d)}{4c(3b-2d)}, d_1 = -c_1, d_2 = c_2, k = \pm\sqrt{-\frac{3(3b+2d)}{2c(b+2d)}}, \\ & \omega = \pm 8c\sqrt{-\frac{3(3b+2d)}{2c(b+2d)}}\sqrt{E_1} \end{aligned}\right\}, \tag{9}$$

where

$$E_5 = \sqrt{-\frac{ad(b-d)}{c(b-6d)(3b-2d)}}. \tag{10}$$

Substituting $Y=\tanh\eta \pm i\,\mathrm{sech}\,\eta$ and $Y=\coth\eta\pm\mathrm{csch}\,\eta$ in eq. (6), the first two sets (7) and (8) gives the travelling wave solutions in the following form:

$$\begin{aligned} u_{1,1}(x,t) &= \mp(21b+8c+14d)\sqrt{E_1}\pm 6\sqrt{E_2}(\tanh\eta\pm i\,\mathrm{sech}\,\eta)\\ &\quad \pm 9(3b+2d)\sqrt{E_1}(\tanh\eta\pm i\,\mathrm{sech}\,\eta)^2, \end{aligned} \tag{11}$$

$$\begin{aligned} v_{1,1}(x,t) &= -1-\frac{a(21b-46d)(3b+2d)}{2c(b-6d)(3b-2d)}\\ &\quad \mp 12(3b+2d)\sqrt{E_1}\sqrt{E_2}(\tanh\eta\pm i\,\mathrm{sech}\,\eta)\\ &\quad +\frac{9a(3b+2d)}{2c(3b-2d)}(\tanh\eta\pm i\,\mathrm{sech}\,\eta)^2, \end{aligned} \tag{12}$$

$$\begin{aligned} u_{1,2}(x,t) &= \mp(21b+8c+14d)\sqrt{E_1}\pm 6\sqrt{E_2}(\coth\eta\pm \mathrm{csch}\,\eta)\\ &\quad \pm 9(3b+2d)\sqrt{E_1}(\coth\eta\pm \mathrm{csch}\,\eta)^2, \end{aligned} \tag{13}$$

$$\begin{aligned} v_{1,2}(x,t) &= -1-\frac{a(21b-46d)(3b+2d)}{2c(b-6d)(3b-2d)}\\ &\quad \mp 12(3b+2d)\sqrt{E_1}\sqrt{E_2}(\coth\eta\pm \mathrm{csch}\,\eta)\\ &\quad +\frac{9a(3b+2d)}{2c(3b-2d)}(\coth\eta\pm \mathrm{csch}\,\eta)^2, \end{aligned} \tag{14}$$

where

$$\eta=\pm\sqrt{\frac{3(3b+2d)}{c(b+2d)}}\left(x+8c\sqrt{-\frac{a}{2c(b-6d)(3b-2d)}t}\right).$$

$$\begin{aligned} u_{1,3}(x,t) &= -\frac{c\left\{\left(2d(c+d)+b(c-d+4cdk^2)\right)b_2\mp(c+d)\sqrt{36abcdk^4+(b-2d)^2b_2^2}\right\}}{6bdk^2}\\ &\quad +b_2\left\{(\tanh\eta\pm i\,\mathrm{sech}\,\eta)^2+\frac{1}{(\tanh\eta\pm i\,\mathrm{sech}\,\eta)^2}\right\}, \end{aligned} \tag{15}$$

$$\begin{aligned} v_{1,3}(x,t) &= \frac{c^2k^4\left\{18bc(ad-bc)k^4+(b-2d+2bck^2)b_2\left((b-2d)b_2\pm\sqrt{36abcdk^4+(b-2d)^2b_2^2}\right)\right\}}{18b^2}\\ &\quad +\frac{k^2b_2\left((2d-b)b_2\mp\sqrt{36abcdk^4+(b-2d)^2b_2^2}\right)}{6bc}\\ &\quad \times\left\{(\tanh\eta\pm i\,\mathrm{sech}\,\eta)^2+\frac{1}{(\tanh\eta\pm i\,\mathrm{sech}\,\eta)^2}\right\}, \end{aligned} \tag{16}$$

$$\begin{aligned} u_{1,4}(x,t) &= -\frac{c\left\{\left(2d(c+d)+b(c-d+4cdk^2)\right)b_2\mp(c+d)\sqrt{36abcdk^4+(b-2d)^2b_2^2}\right\}}{6bdk^2}\\ &\quad +b_2\left\{(\coth\eta\pm \mathrm{csch}\,\eta)^2+\frac{1}{(\coth\eta\pm \mathrm{csch}\,\eta)^2}\right\}, \end{aligned} \tag{17}$$

$$v_{1,4}(x,t) = \frac{c^2k^4\left\{18bc(ad-bc)k^4+(b-2d+2bck^2)b_2\left((b-2d)b_2\pm\sqrt{36abcdk^4+(b-2d)^2b_2^2}\right)\right\}}{18b^2} + \frac{k^2b_2\left((2d-b)b_2\mp\sqrt{36abcdk^4+(b-2d)^2b_2^2}\right)}{6bc} \times\left\{(\coth\eta\pm\operatorname{csch}\eta)^2+\frac{1}{(\coth\eta\pm\operatorname{csch}\eta)^2}\right\}, \tag{18}$$

where

$$\eta = -\sqrt{\frac{3(3b+2d)}{c(b+2d)}}x - \frac{-(b+2d)b_2\pm\sqrt{36abcdk^4+(b-2d)^2b_2^2}}{6bdk}t.$$

Finally, the third set gives the travelling wave solutions as

$$u_{1,5}(x,t) = \mp 2(3b-4c+2d)\sqrt{E_1} \mp 3\sqrt{2}E_5\left\{(\tanh\eta\pm i\operatorname{sech}\eta)-\frac{1}{(\tanh\eta\pm i\operatorname{sech}\eta)}\right\} \pm\frac{9(3b+2d)}{2}\sqrt{E_1}\left\{(\tanh\eta\pm i\operatorname{sech}\eta)^2+\frac{1}{(\tanh\eta\pm i\operatorname{sech}\eta)^2}\right\}, \tag{19}$$

$$v_{1,5}(x,t) = -\frac{c(b-6d)(3b-2d)+2a(3b+2d)^2}{c(b-6d)(3b-2d)} \mp 6\sqrt{2}(3b+2d)\sqrt{E_1}\sqrt{-E_2} \times\left\{(\tanh\eta\pm i\operatorname{sech}\eta)-\frac{1}{(\tanh\eta\pm i\operatorname{sech}\eta)}\right\}-\frac{9a(3b+2d)}{4c(3b-2d)} \times\left\{(\tanh\eta\pm i\operatorname{sech}\eta)^2+\frac{1}{(\tanh\eta\pm i\operatorname{sech}\eta)^2}\right\}, \tag{20}$$

$$u_{1,6}(x,t) = \mp 2(3b-4c+2d)\sqrt{E_1} \mp 3\sqrt{2}E_5\left\{(\coth\eta\pm\operatorname{csch}\eta)-\frac{1}{(\coth\eta\pm\operatorname{csch}\eta)}\right\} \pm\frac{9(3b+2d)}{2}\sqrt{E_1}\left\{(\coth\eta\pm\operatorname{csch}\eta)^2+\frac{1}{(\coth\eta\pm\operatorname{csch}\eta)^2}\right\}, \tag{21}$$

$$v_{1,6}(x,t) = -\frac{c(b-6d)(3b-2d)+2a(3b+2d)^2}{c(b-6d)(3b-2d)} \mp 6\sqrt{2}(3b+2d)\sqrt{E_1}\sqrt{-E_2} \times\left\{(\coth\eta\pm\operatorname{csch}\eta)-\frac{1}{(\coth\eta\pm\operatorname{csch}\eta)}\right\} -\frac{9a(3b+2d)}{4c(3b-2d)}\left\{(\coth\eta\pm\operatorname{csch}\eta)^2+\frac{1}{(\coth\eta\pm\operatorname{csch}\eta)^2}\right\}, \tag{22}$$

where

$$\eta = \pm\sqrt{-\frac{3(3b+2d)}{2c(b+2d)}}\left(x - 8c\sqrt{-\frac{a}{2c(b-6d)(3b-2d)}}t\right).$$

Note 2.2. The modified tanh–coth expansion method is a natural extension to the basic tanh-function expansion method. It gives three types of solutions, namely a tanh function expansion, a coth function expansion, and a tanh–coth expansion. For every tanh function expansion solution, there is a corresponding coth function expansion solution. It should be mentioned that by mistake in many papers, such tanh–coth solutions are claimed to be new. However, tanh–coth solutions may be delivered that are new in the sense that they would not be delivered via the basic tanh-function method.

Remark 2.3. If we set $A = 1,\ B = 0,\ C = -4$ in eq. (4) and by repeating the same calculation as above, we obtain the following travelling wave solutions for eq. (2):

$$\begin{aligned} u_{2,1}(x,t) &= \mp(21b+8c+14d)\sqrt{E_1} \pm 6\sqrt{E_2}\tanh 2\eta \\ &\quad \pm 9(3b+2d)\sqrt{E_1}\tanh^2 2\eta, \end{aligned} \tag{23}$$

$$\begin{aligned} v_{2,1}(x,t) &= -1 - \frac{a(21b-46d)(3b+2d)}{2c(b-6d)(3b-2d)} \mp 12(3b+2d)\sqrt{E_1}\sqrt{E_2}\tanh 2\eta \\ &\quad + \frac{9a(3b+2d)}{2c(3b-2d)}\tanh^2 2\eta, \end{aligned} \tag{24}$$

$$\begin{aligned} u_{2,2}(x,t) &= \mp(21b+8c+14d)\sqrt{E_1} \pm 3\sqrt{E_2}(\tanh\eta + \coth\eta) \\ &\quad \pm \frac{9(3b+2d)}{4}\sqrt{E_1}(\tanh\eta + \coth\eta)^2, \end{aligned} \tag{25}$$

$$\begin{aligned} v_{2,2}(x,t) &= -1 - \frac{a(21b-46d)(3b+2d)}{2c(b-6d)(3b-2d)} \\ &\quad \mp 6(3b+2d)\sqrt{E_1}\sqrt{E_2}(\tanh\eta + \coth\eta) \\ &\quad + \frac{9a(3b+2d)}{8c(3b-2d)}(\tanh\eta + \coth\eta)^2, \end{aligned} \tag{26}$$

where

$$\eta = \pm\frac{1}{4}\sqrt{\frac{3(3b+2d)}{c(b+2d)}}\left(x - 8c\sqrt{-\frac{a}{2c(b-6d)(3b-2d)}}t\right).$$

$$\begin{aligned} u_{2,3}(x,t) &= \mp(21b+8c+14d)\sqrt{E_1} \pm 6\sqrt{E_2}\coth 2\eta \\ &\quad \pm 9(3b+2d)\sqrt{E_1}\coth^2 2\eta, \end{aligned} \tag{27}$$

$$\begin{aligned} v_{2,3}(x,t) &= -1 - \frac{a(21b-46d)(3b+2d)}{2c(b-6d)(3b-2d)} \mp 12(3b+2d)\sqrt{E_1}\sqrt{E_2}\coth 2\eta \\ &\quad + \frac{9a(3b+2d)}{2c(3b-2d)}\coth^2 2\eta, \end{aligned} \tag{28}$$

$$u_{2,4}(x,t) = \mp(21b+8c+14d)\sqrt{E_1} \pm \frac{12\sqrt{E_2}}{(\tanh\eta+\coth\eta)} \pm \frac{36(3b+2d)\sqrt{E_1}}{(\tanh\eta+\coth\eta)^2}, \tag{29}$$

$$v_{2,4}(x,t) = -1 - \frac{a(21b-46d)(3b+2d)}{2c(b-6d)(3b-2d)} \mp \frac{24(3b+2d)\sqrt{E_1}\sqrt{E_2}}{(\tanh\eta+\coth\eta)} + \frac{18a(3b+2d)}{c(3b-2d)}\frac{1}{(\tanh\eta+\coth\eta)^2}, \tag{30}$$

where

$$\eta = \pm\frac{1}{4}\sqrt{\frac{3(3b+2d)}{c(b+2d)}}\left(x+8c\sqrt{-\frac{a}{2c(b-6d)(3b-2d)}}t\right).$$

$$u_{2,5}(x,t) = \mp(3b-4c+2d)\sqrt{E_1} \pm 3\sqrt{2}\sqrt{-E_2}(\tanh 2\eta - \coth 2\eta) \pm \frac{9(3b+2d)}{2}\sqrt{E_1}(\tanh^2 2\eta + \coth^2 2\eta), \tag{31}$$

$$v_{2,5}(x,t) = -\frac{c(b-6d)(3b-2d)+2a(3b+2d)^2}{c(b-6d)(3b-2d)} \pm 6\sqrt{2}\sqrt{E_1}\sqrt{-E_2}(\tanh 2\eta - \coth 2\eta) - \frac{9a(3b+2d)}{4c(3b-2d)}(\tanh^2 2\eta + \coth^2 2\eta), \tag{32}$$

$$u_{2,6}(x,t) = \mp(3b-4c+2d)\sqrt{E_1} \pm 3\sqrt{2}\sqrt{-E_2}\left\{\frac{\tanh\eta+\coth\eta}{2} - \frac{2}{\tanh\eta+\coth\eta}\right\} \pm \frac{9(3b+2d)}{2} \times \sqrt{E_1}\left\{\left(\frac{\tanh\eta+\coth\eta}{2}\right)^2 + \left(\frac{2}{\tanh\eta+\coth\eta}\right)^2\right\}, \tag{33}$$

$$v_{2,6}(x,t) = -\frac{c(b-6d)(3b-2d)+2a(3b+2d)^2}{c(b-6d)(3b-2d)} \pm 6\sqrt{2}(3b+2d)\sqrt{E_1}\sqrt{-E_2} \times \left\{\frac{\tanh\eta+\coth\eta}{2} - \frac{2}{\tanh\eta+\coth\eta}\right\} - \frac{9a(3b+2d)}{4c(3b-2d)}\left\{\left(\frac{\tanh\eta+\coth\eta}{2}\right)^2 + \left(\frac{2}{\tanh\eta+\coth\eta}\right)^2\right\}, \tag{34}$$

where

$$\eta = \pm\frac{1}{4\sqrt{2}}\sqrt{-\frac{3(3b+2d)}{c(b+2d)}}\left(x+8c\sqrt{-\frac{a}{2c(b-6d)(3b-2d)}}t\right).$$

Remark 2.4. If we set $A = 1$, $B = 0$, $C = 4$ in eq. (4), and solving the system of algebraic equations using Maple by the same calculation as above, we obtain the following travelling wave solutions of eq. (2):

$$\begin{aligned} u_{3,1}(x,t) &= \pm(21b + 8c + 14d)\sqrt{E_1} \\ &\quad \pm 6\sqrt{-E_2}\tan 2\eta \pm 9(3b + 2d)\sqrt{E_1}\tan^2 2\eta, \end{aligned} \tag{35}$$

$$\begin{aligned} v_{3,1}(x,t) &= -1 - \frac{a(21b - 46d)(3b + 2d)}{2c(b - 6d)(3b - 2d)} \\ &\quad \pm 12(3b + 2d)\sqrt{E_1}\sqrt{-E_2}\tan 2\eta \\ &\quad - \frac{9a(3b + 2d)}{2c(3b - 2d)}\tan^2 2\eta, \end{aligned} \tag{36}$$

$$\begin{aligned} u_{3,2}(x,t) &= \pm(21b + 8c + 14d)\sqrt{E_1} \pm 3\sqrt{-E_2}(\tan\eta - \cot\eta) \\ &\quad \pm \frac{9(3b + 2d)}{4}\sqrt{E_1}(\tan\eta - \cot\eta)^2, \end{aligned} \tag{37}$$

$$\begin{aligned} v_{3,2}(x,t) &= -1 - \frac{a(21b - 46d)(3b + 2d)}{2c(b - 6d)(3b - 2d)} \\ &\quad \pm 6(3b + 2d)\sqrt{E_1}\sqrt{-E_2}(\tan\eta - \cot\eta) \\ &\quad - \frac{9a(3b + 2d)}{8c(3b - 2d)}(\tan\eta - \cot\eta)^2, \end{aligned} \tag{38}$$

where

$$\eta = \pm\frac{1}{4}\sqrt{-\frac{3(3b + 2d)}{c(b + 2d)}}\left(x - 8c\sqrt{-\frac{a}{2c(b - 6d)(3b - 2d)}}t\right).$$

$$\begin{aligned} u_{3,3}(x,t) &= \pm(21b + 8c + 14d)\sqrt{E_1} \pm 6\sqrt{-E_2}\cot 2\eta \\ &\quad \pm 9(3b + 2d)\sqrt{E_1}\cot^2 2\eta, \end{aligned} \tag{39}$$

$$\begin{aligned} v_{3,3}(x,t) &= -1 - \frac{a(21b - 46d)(3b + 2d)}{2c(b - 6d)(3b - 2d)} \\ &\quad \pm 12(3b + 2d)\sqrt{E_1}\sqrt{-E_2}\cot 2\eta \\ &\quad - \frac{9a(3b + 2d)}{2c(3b - 2d)}\cot^2 2\eta, \end{aligned} \tag{40}$$

$$\begin{aligned} u_{3,4}(x,t) &= \pm(21b + 8c + 14d)\sqrt{E_1} \pm \frac{12\sqrt{-E_2}}{(\tan\eta - \cot\eta)} \\ &\quad \pm \frac{36(3b + 2d)\sqrt{E_1}}{(\tan\eta - \cot\eta)^2}, \end{aligned} \tag{41}$$

$$\begin{aligned} v_{3,4}(x,t) &= -1 - \frac{a(21b - 46d)(3b + 2d)}{2c(b - 6d)(3b - 2d)} \pm \frac{24(3b + 2d)\sqrt{E_1}\sqrt{-E_2}}{(\tan\eta - \cot\eta)} \\ &\quad - \frac{18a(3b + 2d)}{c(3b - 2d)}\frac{1}{(\tan\eta - \coth\eta)^2}, \end{aligned} \tag{42}$$

where

$$\eta = \pm\frac{1}{4}\sqrt{-\frac{3(3b+2d)}{c(b+2d)}}\left(x - 8c\sqrt{-\frac{a}{2c(b-6d)(3b-2d)}}t\right).$$

$$\begin{aligned} u_{3,5}(x,t) &= \pm 2(3b-4c+2d)\sqrt{E_1} \pm 3\sqrt{2}\sqrt{E_2}(\tan 2\eta - \cot 2\eta) \\ &\pm \frac{9(3b+2d)}{2}\sqrt{E_1}(\tan^2 2\eta - \cot^2 2\eta), \end{aligned} \tag{43}$$

$$\begin{aligned} v_{3,5}(x,t) &= -\frac{c(b-6d)(3b-2d)+2a(3b+2d)^2}{c(b-6d)(3b-2d)} \\ &\mp 6\sqrt{2}(3b+2d)\sqrt{E_1}\sqrt{E_2}(\tan 2\eta + \cot 2\eta) \\ &+ \frac{9a(3b+2d)}{4c(3b-2d)}(\tan^2 2\eta + \cot^2 2\eta), \end{aligned} \tag{44}$$

$$\begin{aligned} u_{3,6}(x,t) &= \pm 2(3b-4c+2d)\sqrt{E_1} \\ &\pm 3\sqrt{2}\sqrt{E_2}\left\{\frac{\tan\eta - \cot\eta}{2} + \frac{2}{\tan\eta - \cot\eta}\right\} \\ &\pm \frac{9(3b+2d)}{2}\sqrt{E_1}\left\{\left(\frac{\tan\eta - \cot\eta}{2}\right)^2 + \left(\frac{2}{\tan\eta - \cot\eta}\right)^2\right\}, \end{aligned} \tag{45}$$

$$\begin{aligned} v_{3,6}(x,t) &= -\frac{c(b-6d)(3b-2d)+2a(3b+2d)^2}{c(b-6d)(3b-2d)} \\ &\mp 6\sqrt{2}(3b+2d)\sqrt{E_1}\sqrt{E_2}\left\{\frac{\tan\eta - \cot\eta}{2} + \frac{2}{\tan\eta - \cot\eta}\right\} \\ &+ \frac{9a(3b+2d)}{4c(3b-2d)}\left\{\left(\frac{\tan\eta - \cot\eta}{2}\right)^2 + \left(\frac{2}{\tan\eta - \cot\eta}\right)^2\right\}, \end{aligned} \tag{46}$$

where

$$\eta = \pm\frac{1}{4\sqrt{2}}\sqrt{\frac{3(3b+2d)}{c(b+2d)}}\left(x + 8c\sqrt{-\frac{a}{2c(b-6d)(3b-2d)}}t\right).$$

Remark 2.5. It should be mentioned that we have verified all the obtained solutions by putting them back into the original equation. To the authors' knowledge this is the first attempt to solve the bidirectional wave equations with tanh function method, all solutions are new and cannot be found in the literature.

3. Conclusion

In this paper, using the solution of the auxiliary equation (4) in the modified tanh–coth function method, we have found some new exact travelling wave solutions for bidirectional wave equations. This method also suggests that one can get different exact solutions by choosing different auxiliary equations in the tanh-function. It should be noted that the method used here can generate not only regular solutions but also singular ones involving csch and coth functions.

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