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# Bulk viscous cosmology in early Universe

C P SINGH
Department of Applied Mathematics, Delhi College of Engineering, Bawana Road, Delhi 110 042, India
E-mail: cpsphd@rediffmail.com

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**Abstract.** The effect of bulk viscosity on the early evolution of Universe for a spatially homogeneous and isotropic Robertson–Walker model is considered. Einstein's field equations are solved by using 'gamma-law' equation of state $p = (\gamma - 1)\rho$, where the adiabatic parameter gamma ($\gamma$) depends on the scale factor of the model. The 'gamma' function is defined in such a way that it describes a unified solution of early evolution of the Universe for inflationary and radiation-dominated phases. The fluid has only bulk viscous term and the coefficient of bulk viscosity is taken to be proportional to some power function of the energy density. The complete general solutions have been given through three cases. For flat space, power-law as well as exponential solutions are found. The problem of how the introduction of viscosity affects the appearance of singularity, is briefly discussed in particular solutions. The deceleration parameter has a freedom to vary with the scale factor of the model, which describes the accelerating expansion of the Universe.

**Keywords.** Cosmology; viscous Universe; radiation phase; inflationary phase.

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## 1. Introduction

In the literature the cosmological fluid has so far been pictured as an ideal (non-viscous) fluid. Based upon the common experience in fluid mechanisms we would however expect that the viscosity concept may be important in cosmology also, especially in cases where turbulence effects occur. It is interesting to find out how the cosmological solutions of general relativity behave after introducing viscosity term. Will the initial singularity in cosmology remain when the Friedmann equation is enlarged by a viscosity term? The viscosity theory of relativistic fluids was first suggested by Eckart [1] and Landau and Lifshitz [2], who considered only first-order deviation from equilibrium. The relativistic second-order theory was founded by Israel [3] and developed by Israel and Stewart [4]. However, the character of the evolution equation is very complicated in the framework of the full causal theory. Therefore, the conventional theory [2] is still applied to phenomena which are quasi-stationary, i.e. slowly varying on space and time scales characterized by the mean free path and the mean collision time. In the case of isotropic and homogeneous cosmologies, the dissipative process can be modeled as a bulk viscosity within a thermodynamical approach. Different from shear viscosity and heat conductivity, it is compatible with the symmetry requirements of the homogeneous and isotropic Friedmann–Robertson–Walker (FRW) Universe. In the simplest cosmological models there is no way to study entropy producing processes except through bulk viscosity. Thus, bulk viscosity arises any time a fluid expands rapidly and ceases to be in thermodynamic equilibrium. The bulk viscosity, therefore, is a measure of the pressure required to restore equilibrium to a compressed form of expanding system.

Many general scenarios considering bulk viscosity have been studied in refs [5–13]. Santos *et al* [14] found exact solutions of the isotropic homogeneous model with bulk viscosity being a power function of energy density. Bulk viscosity associated with the grand unified theory (GUT) may lead to an inflationary cosmology. Grøn [15] studied inflationary Bianchi type models with bulk viscosity and shear. Later, viscous Universe models have been considered from various angles of view by Pavo′n *et al* [16,17], Burd and Coley [18], Maartens [19], Zimdahl [20,21], Chimento *et al* [22,23] and Fabris *et al* [24]. Bulk viscosity in Brans–Dicke theory, leading to an accelerated phase of the Universe today, has been studied in refs [25,26]. Applications of viscous fluid concerning the phantom models have been made by Cataldo *et al* [27]. Carlevaro and Montani [28] discussed the effects induced by bulk viscosity on the stability of very early Universe.

Usually, the field equations are solved and analysed separately for different epochs, although some authors have given unified solutions. For instance, Carvalho [29] studied Friedmann–Robertson–Walker (FRW) model for perfect fluid in general relativity and presented a unified solution using varying adiabatic parameter 'gamma' of 'gamma-law' equation of state. In the present paper we discuss the effects of bulk viscosity on the early evolution of Universe for flat FRW model. The matter filling the cosmological (isotropic and homogeneous) background is discussed by a viscous fluid having 'gamma-law' equation of state and whose viscosity coefficient $\zeta$ is related to the energy density via a power-law of the form $\zeta \propto \rho^n$, where $n$ is a positive numerical constant. We study the evolution of the Universe as it goes from an inflationary phase to a radiation-dominated era. We keep the 'gamma-law' form of equation of state, but let the parameter 'gamma' varies continuously with the cosmic time as the Universe expands. Our approach is based on Carvalho's work [29] that studied the FRW models using functional form of gamma, depends on scale factor, and presented a unified solution for two early phases of Universe. In the previous work Ram and Singh [30,31], Singh *et al* [32], Singh [33] extended Carvalho's work in general relativity and Brans–Dicke theory with the introduction of bulk viscous term. Barrow [13] and Santos *et al* [14] investigated FRW models with bulk viscous term. But both of them have given the solutions using equation of state with constant gamma. In this work we are studying whether a varying adiabatic parameter 'gamma' of 'gamma-law' equation of state can explain the early evolution of Universe with bulk viscosity when it transits from an inflationary to a radiation-dominated phase. We present a complete general solution via three cases, which describe the singular and non-singular nature of the model. This work is more general than the author's previous works [30–33]. The physical significances of the solutions are discussed in detail by taking some particular cases.

## 2. Model and field equations

Let us consider a spatially homogeneous and isotropic Friedmann–Robertson–Walker (FRW) line element (in the unit $c = 1$)

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right], \tag{1}$$

where $R(t)$ is the scale factor and $k = 0, -1$ or $+1$ is the curvature parameter for flat, open and closed Universe, respectively. In the following we use the units $8\pi G = 1$. We assume that the fluid has bulk viscosity, so that the energy–momentum tensor can be written as

$$T_{\mu\nu} = (\rho + \bar{p})u_\mu u_\nu - \bar{p} g_{\mu\nu} \tag{2}$$

with

$$\bar{p} = p - \zeta u^\mu_{;\mu} = p - 3\zeta H, \tag{3}$$

where $\rho$, $p$ and $u_\mu$ have the usual meanings; $\zeta$ stands for the coefficient of bulk viscosity and $H$ denotes Hubble parameter. Since the bulk viscous pressure represents only a small correction to the thermodynamical pressure, it is a reasonable assumption that the inclusion of viscous term in the energy–momentum tensor does not change fundamentally the dynamics of the cosmic evolution. The Einstein's field equations are given by

$$R_{\mu\upsilon} - \frac{1}{2} g_{\mu\upsilon} R = T_{\mu\upsilon}. \tag{4}$$

The field equations (4), with the metric (1) and energy–momentum tensor (2), are given by

$$\left(\frac{\dot{R}}{R}\right)^2 = H^2 = \frac{\rho}{3}, \tag{5}$$

$$\frac{\ddot{R}}{R} = \dot{H} + H^2 = -\frac{1}{6}(\rho + 3\bar{p}), \tag{6}$$

where an overdot represents derivative with respect to cosmic time $t$ and $H = \dot{R}/R$. The conservation equation $T^{\mu\nu}_{;\nu} = 0$ gives

$$\dot{\rho} + 3H(\rho + p) - 9\zeta H^2 = 0. \tag{7}$$

In order to solve the field equations (5)–(7), usually we assume that the perfect fluid contribution is related through the 'gamma-law' equation of state

$$p = (\gamma - 1)\rho, \tag{8}$$

where $\gamma$ is the adiabatic parameter. In general, the value of $\gamma$ is taken to be constant satisfying the condition $0 \leq \gamma \leq 2$ and many authors have solved the field equations for different epochs by taking this constraint of $\gamma$. Our aim in this paper is to let the parameter $\gamma$ varies continuously with the cosmic time as the Universe expands and later on to present the unified solutions of two early phases of Universe. Carvalho [29] assumed a scale-dependent $\gamma$ of the form

$$\gamma(R) = \frac{4}{3}\frac{A(R/R_*)^2 + (a/2)(R/R_*)^a}{A(R/R_*)^2 + (R/R_*)^a}, \tag{9}$$

where $A$ is a constant, $a$ is the free parameter lying in the interval $0 \leq a < 1$, which is related to the power of the cosmic time during the inflationary phase and $R_*$ is a certain reference value of $R$. A class of models can be obtained according to the value of parameter $a$. For $a = 0$, we have exponential inflation ($\gamma = 0,$ at $R = 0$ and $p = -\rho$). For $a$ in the range of $0 < a < 1$, $\gamma$ slowly increases from $(2/3)a$ for inflationary phase ($R \ll R_*$) to $+4/3$, when we have radiation-dominated phase ($R \gg R_*$). In the limit of $R \to 0$ we have $\gamma(R) = 2a/3$ so that 1 is the maximum value of $a$ for an inflation ($\gamma < 2/3$) epoch. Therefore, the function $\gamma(R)$ is such that when the scale factor is less than a certain reference value $R_*$, we have the inflationary phase and as the scale factor increases, $\gamma$ also increases to reach the value $+4/3$ for the radiation-dominated phase.

Using (8), eq. (7) can be written as

$$\frac{\rho'}{\rho} + \frac{3\gamma(R)}{R} = \frac{9\zeta H}{\rho R}, \tag{10}$$

where prime denotes derivative with respect to the scale factor $R$. Also from eq. (5), we get

$$\frac{\rho'}{\rho} = \frac{2H'}{H}. \tag{11}$$

Substituting eqs (5) and (11) into (10), we finally get

$$H' + \frac{3\gamma(R)}{2}\frac{H}{R} = \frac{3}{2}\frac{\zeta(\rho)}{R}. \tag{12}$$

Equation (12), involving $H$ and $\zeta$, admits solution for $H$ only if $\zeta$ is specified. In the homogeneous models, $\zeta$ depends only on time and therefore we may consider it as a function of the energy density $\rho$ of the Universe. According to [12–15,19], we assume that the bulk viscosity coefficient depends on $\rho$ via a power-law of the form

$$\zeta(\rho) = \zeta_0 \rho^n, \tag{13}$$

where $\zeta_0(\geq 0)$ is a dimensional constant and $n(\geq 0)$ is a numerical constant. It is standard to assume the above law in the absence of better alternatives. Using (5) and (13), eq. (12) now reduces to

$$H' + \frac{3\gamma(R)}{2}\frac{H}{R} = \frac{3^{n+1}\zeta_0}{2}\frac{H^{2n}}{R}. \tag{14}$$

## 3. Solution of field equations

Substituting (9) into (14) and integrating, we have the solution for all $n \neq 1/2$ as

$$\frac{1}{H^{(2n-1)}[A(R/R_*)^2 + (R/R_*)^a]^{(2n-1)}} = C - \frac{(2n-1)3^{n+1}\zeta_0}{2}\int \frac{\mathrm{d}R}{R[A(R/R_*)^2 + (R/R_*)^a]^{(2n-1)}}, \tag{15}$$

where $C$ is the integration constant and $0 < a < 1$. In the following, we solve eq. (15) for two early phases of Universe – inflationary and radiation-dominated phases.

*For inflationary phase* ($R \ll R_*$), the term $(R/R_*)^a$ on both sides of eq. (15) dominate over the term $A(R/R_*)^2$ and therefore, one has the inflationary phase, which is given in terms of scale factor as

$$H^{(2n-1)} = \frac{1}{[C(R/R_*)^{a(2n-1)} + (3^{n+1}\zeta_0/2a)]}. \tag{16}$$

*For radiation-dominated phase* ($R \gg R_*$), the term $(R/R_*)^2$ on both sides of eq. (15) dominates over the term $(R/R_*)^a$ and therefore, one has the radiation phase which is given in terms of scale factor as

$$H^{(2n-1)} = \frac{1}{[CA^{(2n-1)}(R/R_*)^{2(2n-1)} + (3^{n+1}\zeta_0/4)]}. \tag{17}$$

We can integrate eqs (16) and (17) respectively for $H = \dot{R}/R$ to obtain an expression for $R$ in terms of cosmic time $t$, which are given by

$$\int \frac{[C(R/R_*)^{a(2n-1)} + (3^{n+1}\zeta_0/2a)]^{1/(2n-1)}}{R}\mathrm{d}R = t, \tag{18}$$

$$\int \frac{[CA^{(2n-1)}(R/R_*)^{2(2n-1)} + (3^{n+1}\zeta_0/4)]^{1/(2n-1)}}{R}\mathrm{d}R = t, \tag{19}$$

where constants of integration are taken to be zero for simplicity. Murphy [9] constructed a class of viscous cosmological models with $\zeta(\rho) = \zeta_0\rho$. The model obtained by Murphy possesses an interesting feature that the Big Bang singularity of infinite space time curvature does not occur at finite past. But the relationship assumed by Murphy is not acceptable for large values of density. Later on, Murphy's model was explored by Davies [34]. Belinskii and Khalatnikov [10,11] found out that the effects of viscosity would not be capable of removing the cosmological singularity, but would lead to a qualitatively new behaviour near singularity. Klimek [35] and Santos *et al* [14] investigated the viscosity effects in isotropic models by assuming relation (13).

For inflationary phase, eq. (18) can be rewritten as

$$\left(\frac{2a}{3^{n+1}\zeta_0}\right)^{1/(2n-1)} t = \int [1 + (2aC/3^{n+1}\zeta_0)(R/R_*)^{a(2n-1)}]^{1/(2n-1)} \frac{\mathrm{d}R}{R}. \tag{20}$$

For radiation-dominated phase, eq. (19) can be rewritten as

$$\left(\frac{4}{3^{n+1}\zeta_0}\right)^{1/(2n-1)} t = \int [1 + (4CA^{2n-1}/3^{n+1}\zeta_0)(R/R_*)^{2(2n-1)}]^{1/(2n-1)} \frac{\mathrm{d}R}{R}. \tag{21}$$

We observe that the expressions (20) and (21) are similar. Therefore, depending on the values of $n$, we now obtain a more general solution of (20) for inflationary phase in the following three cases. One can apply similar method to obtain the general solution of (21) for radiation-dominated phase. But we will give some solutions of (21), where it is necessary to explain the evolution of Universe for radiation phase.

If we take

$$\sinh\theta = [(2aC/3^{n+1}\zeta_0)(R/R_*)^{a(2n-1)}]^{1/2}, \tag{22}$$

then eq. (20) becomes

$$\alpha_0 t = \int \frac{(\cosh\theta)^{(2n+1)/(2n-1)}}{\sinh\theta} \mathrm{d}\theta, \tag{23}$$

where $\alpha_0 = \frac{a(2n-1)}{2}\left(\frac{2a}{3^{n+1}\zeta_0}\right)^{1/(2n-1)}$.

*Case* 1. When $2n > 1$

In this case, the general solution of (23) can be found when $(2n+1)/(2n-1)$ is an integer. When $(2n+1)/(2n-1) = 2m$, $m \in z^+$, the solution of (23) is

$$\alpha_0 t = \sum_{s=1}^{m} \frac{(\cosh\theta)^{(2s-1)}}{(2s-1)} + \ln\tanh(\theta/2). \tag{24}$$

Using (22), the complete solution of scale factor in terms of cosmic time $t$ is given by

$$\begin{aligned}\alpha_0 t = &\sum_{s=1}^{m} \frac{1}{(2s-1)}\left[1 + \frac{2aC}{3^{(n+1)}\zeta_0}\left(\frac{R}{R_*}\right)^{a(2n-1)}\right]^{(s-1/2)} \\ &+ \ln\left[\sqrt{1 + \frac{2aC}{3^{(n+1)}\zeta_0}\left(\frac{R}{R_*}\right)^{a(2n-1)}} - 1\right] + \frac{a(2n-1)}{2}\ln R \\ &+ \frac{1}{2}\ln\left[\frac{2aC}{3^{(n+1)}\zeta_0 R_*^{a(2n-1)}}\right].\end{aligned} \tag{25}$$

When $(2n+1)/(2n-1) = 2m+1$, $m \in z^+$, the solution of (23) is

$$\alpha_0 t = \sum_{s=1}^{m} \frac{(\cosh\theta)^{2s}}{2s} + \ln\sinh\theta. \tag{26}$$

Using (22), the complete solution of scale factor in terms of cosmic time $t$ is given by

$$\alpha_0 t = \sum_{s=1}^{m} \frac{1}{2s}\left[1 + \frac{2aC}{3^{(n+1)}\zeta_0}\left(\frac{R}{R_*}\right)^{a(2n-1)}\right]^{s} + \frac{a(2n-1)}{2}\ln R$$
$$+\frac{1}{2}\ln\left[\frac{2aC}{3^{(n+1)}\zeta_0 R_*^{a(2n-1)}}\right]. \tag{27}$$

The case $2n > 1$ implies that for $R \to 0$ the energy density and bulk viscous coefficient become constant with an exponential increase in the scale factor while for $R \to \infty$ the energy density becomes equivalent to that of a Universe dominated by a barotropic fluid. Thus, we observe that as $R \to 0$, $H$ approaches a finite magnitude showing a steady-state characteristic of the model and subsequently behaves like the perfect fluid Friedmann model as $t \to \infty$. The models have $R = 0$ with finite mass density at infinite past. The model is similar to those of Murphy. The bulk viscosity appears to be the effective mechanism to remove the initial singularity. A cosmological model like that of Murphy, which evolves from an initial de Sitter era into a Friedmann era, has been termed a deflationary model by Barrow [12,13]. We will discuss in detail the physical behaviour of the solutions (25) and (27) by taking particular values of $n$ after Cases 2 and 3.

*Case* 2. When $2n < 1$

In this case, the solution of (23) with $(2n+1)/(1-2n) = 2l$, $l \in z^+$ is

$$\alpha_0 t = \sum_{s=1}^{l} \frac{(\operatorname{sech}\theta)^{(2l-2s+1)}}{(2l-2s+1)} + \ln\tanh(\theta/2). \tag{28}$$

Using (22), the complete solution for scale factor is given by

$$\alpha_0 t = \sum_{s=1}^{l} \frac{1}{(2l-2s+1)}\left[1 + \frac{2aC}{3^{(n+1)}\zeta_0}\left(\frac{R}{R_*}\right)^{a(2n-1)}\right]^{(s-l-1/2)}$$
$$+\ln\left[\sqrt{1 + \frac{2aC}{3^{(n+1)}\zeta_0}\left(\frac{R}{R_*}\right)^{a(2n-1)}} - 1\right] + \frac{a(2n-1)}{2}\ln R$$
$$+\frac{1}{2}\ln\left[\frac{2aC}{3^{(n+1)}\zeta_0 R_*^{a(2n-1)}}\right]. \tag{29}$$

When $(2n+1)/(1-2n) = 2l+1$, $l \in z^+$, the solution of (23) is

$$\alpha_0 t = \sum_{s=1}^{l} \frac{(\operatorname{sech}\theta)^{2l-2s+2}}{(2l-2s+2)} + \ln\tanh(\theta/2). \tag{30}$$

Using (22), the complete solution of scale factor in terms of cosmic time $t$ is given by

$$\begin{aligned}\alpha_0 t = &\sum_{s=1}^{l} \frac{1}{(2l-2s+1)} \left[1 + \frac{2aC}{3^{(n+1)}\zeta_0}\left(\frac{R}{R_*}\right)^{a(2n-1)}\right]^{l-s-1} \\ &+ \frac{a(2n-1)}{2}\ln R + \frac{1}{2}\ln\left[\frac{2aC}{3^{(n+1)}\zeta_0 R_*^{a(2n-1)}}\right] \\ &- \frac{1}{2}\ln\left[1 + \frac{2a}{3^{(n+1)}\zeta_0}\left(\frac{R}{R_*}\right)^{a(2n-1)}\right].\end{aligned} \tag{31}$$

In this case, the Universe evolves into a viscosity-dominated steady-state era. If the coefficient of bulk viscosity decays sufficiently slowly, the late epochs of the Universe will be viscosity-dominated, and the Universe will enter a final inflationary era with steady-state characteristic. This case is distinct from Murphy's case or other values of $n$ with $2n > 1$. This can also be explained from (18). If the coefficient of bulk viscosity decays slowly, i.e. if $3^{n+1}\zeta_0/2a \gg C(R/R_*)^{a(2n-1)}$, there is exponential expansion $R \propto \exp(H_0 t)$. Thus at any finite proper time in the past, the curvature is finite. The viscosity has removed the initial singularity. On the other hand, if $C(R/R_*)^{a(2n-1)} \gg 3^{n+1}\zeta_0/2a$, we get power-law expansion $R \propto t^{1/a}$ and the effects of viscosity are negligible.

The energy density and bulk viscous coefficient in terms of scale factor for inflationary phase are respectively given by

$$\rho = 3[C(R/R_*)^{a(2n-1)} + (3^{n+1}\zeta_0/2a)]^{2/(1-2n)}, \tag{32}$$

$$\zeta = 3\zeta_0[C(R/R_*)^{a(2n-1)} + (3^{n+1}\zeta_0/2a)]^{2/(1-2n)}. \tag{33}$$

An important observational quantity is the deceleration parameter $q = -(\dot{H} + H^2)/H^2$. In the case of inflationary phase from eq. (16), the deceleration parameter can be expressed as a function of scale factor in the form

$$q = \frac{(a-1)C(R/R_*)^{a(2n-1)} - (3^{n+1}\zeta_0/2a)}{[C(R/R_*)^{a(2n-1)} + (3^{n+1}\zeta_0/2a)]}. \tag{34}$$

It may be noted that the sign of deceleration parameter indicates whether the model inflates or not. The negative value of $q$ ($q < 0$) describes the acceleration expansion while positive one ($q > 0$) describes the decelerating expansion of the Universe. In the absence of viscous term, it becomes $q = (a-1)$. Since $0 \le a < 1$, we observe that $q < 0$. Therefore, the model inflates. We also observe that $C \ge 0$ also give the negative value of $q$ showing the inflation in the model. Thus, the expansion of the Universe is accelerated through out in the evolution.

Again, if $C \geq 0$ or $a = 1$, eq. (34) reduces to

$$q = -\frac{(3^{n+1}\zeta_0/2)}{[C(R/R_*)^{(2n-1)} + (3^{n+1}\zeta_0/2)]} \tag{35}$$

This indicates that the model is accelerating with $q \to 0$ at the later stage of evolution.

Similarly, in the case of radiation-dominated phase from eq. (17), $q$ can be expressed as a function of scale factor in the form

$$q = \frac{A^{(2n-1)}C(R/R_*)^{2(2n-1)} - (3^{n+1}\zeta_0/4)}{[A^{(2n-1)}C(R/R_*)^{2(2n-1)} + (3^{n+1}\zeta_0/4)]}. \tag{36}$$

In the absence of viscous term, we have $q = 1$. This shows that the model decelerates during radiation era. If $C > 0$, we observe that $q$ is positive for $(R/R_*)^{2(2n-1)} > (3^{n+1}\zeta_0/4A^{(2n-1)}C)$, $q$ is negative for $(R/R_*)^{2(2n-1)} < (3^{n+1}\zeta_0/4A^{(2n-1)}C)$ and $q = 0$ for $(R/R_*)^{2(2n-1)} = (3^{n+1}\zeta_0/4A^{(2n-1)}C)$. Again, if $C = 0$, eq. (36) reduces to $q = -1$, which shows the acceleration of the Universe.

*Case* 3. When $2n = 1$

The solutions obtained in Cases 1 and 2 are not valid for $2n = 1$. In the case when $2n = 1$, eq. (14) reduces to

$$H' + \left[\frac{3\gamma(R)}{2} - \alpha\right]\frac{H}{R} = 0, \tag{37}$$

where $\alpha = (3\sqrt{3}\zeta_0/2)$. Substituting (9) into (37) and integrating, we have the solution for Hubble parameter

$$H = \frac{DR^{\alpha}}{[A(R/R_*)^2 + (R/R_*)^a]}. \tag{38}$$

where $D$ is the constant of integration. We can integrate eq. (38) for $0 < a \leq 1$ to obtain an expression for $t$ in terms of the scale factor, which is given by

$$Dt = \frac{1}{(a-\alpha)}\frac{R^{(a-\alpha)}}{R_*^a} + \frac{A}{(2-\alpha)}\frac{R^{(2-\alpha)}}{R_*^2}. \tag{39}$$

The constant of integration is taken to be zero for simplicity. For $R \ll R_*$, the first term in the right-hand side of eq. (39) dominates and one has a power-law inflation

$$R = [R_*^a D(a-\alpha)t]^{1/(a-\alpha)}. \tag{40}$$

On the other hand, for radiation-dominated phase ($R \gg R_*$), we have

$$R = [(1/A)R_*^2 D(2-\alpha)t]^{1/(2-\alpha)}. \tag{41}$$

For $0 < a < 1$, the model is singular with energy density varying for $R \ll R_*$ as

$$\rho = \frac{3D^2}{R_*^{-2a}}R^{-2(a-\alpha)}. \tag{42}$$

For the expansion of the Universe, we must have $a > \alpha$ for $R \ll R_*$ and $\alpha < 2$ for $R \gg R_*$. From (38), a unified expression for deceleration parameter can be given in terms of scale factor as

$$q = \frac{(1-\alpha)A(R/R_*)^2 + (a-1-\alpha)(R/R_*)^a}{[A(R/R_*)^2 + (R/R_*)^a]}. \tag{43}$$

Therefore, $q$ varies from $q = (a - \alpha - 1)$ for $R \ll R_*$ to $q = (1-\alpha)$ for radiation phase. The deceleration parameter is positive for $(a - \alpha - 1) > 0$, negative for $(a - \alpha - 1) < 0$ and $q = 0$ for $a - \alpha = 1$ during inflationary phase.

Singular solutions are obtained for $2n = 1$ with power-law expansion. The evolution begins from singularity with a Friedmann leading behaviour. The Universe explodes from singularity and increases in size in the course of time with a decrease of energy density. The energy density varies inversely as the square of the age of the Universe whereas the bulk viscosity decreases linearly as time passes. Thus, the rate of bulk viscosity is more important in the early stages of the evolution of Universe.

We now study the solution in limit $a \to 0$. In this case, eq. (38) becomes

$$H = \frac{DR^\alpha}{[A(R/R_*)^2 + 1]}. \tag{44}$$

Integration of (44) gives

$$Dt = \frac{R^{-\alpha}}{-\alpha} + \frac{A}{(2-\alpha)}\frac{R^{(2-\alpha)}}{R_*^2}, \tag{45}$$

where $\alpha \neq 2$. In the limit of very small $R$, the first term in the right-hand side of eq. (45) dominates and one has the solution of the form

$$R^\alpha = -\frac{1}{\alpha D}t^{-1}. \tag{46}$$

Again, the radiation phase is described by the same solution as obtained in eq. (41). From eq. (46), we observe that $\alpha > 0$ and $D > 0$ lead to contraction. As $t \to -\infty$, we find that $R \to 0$. The model starts from infinite past with zero proper volume. Thus, for $a = 0$ the Universe is infinitely old and we have inverse power-law. From (44), a unified expression for deceleration parameter can be given in terms of scale factor as

$$q = \frac{(1-\alpha)A(R/R_*)^2 - (1+\alpha)}{A(R/R_*)^2 + 1}. \tag{47}$$

Therefore, $q$ varies from $q = -(\alpha + 1)$ for inflationary phase to $q = (1-\alpha)$ for $R \gg R_*$ as expected.

We now calculate a number of particular simple cases according to $2n < 1$ or $2n > 1$ to discuss the behaviour of the model in details for inflationary phase. One can find out the solution for radiation-dominated phase in a similar way.

## 4. Particular cases

*Case* i: When $n = 0$, i.e. $\zeta = \zeta_0$

Equation (20) becomes

$$\left(\frac{2a}{3\zeta_0}\right)^{-1} t = \int \left[1 + \left(\frac{2aC}{3\zeta_0}\frac{R^{-a}}{R_*^{-a}}\right)\right]^{-1} \frac{\mathrm{d}R}{R}. \tag{48}$$

When $C \neq 0$, the solution of (48) is given by

$$R^a = R_*^a \left[\frac{C_1 \exp(3\zeta_0 t/2) - 2aC}{3\zeta_0}\right]. \tag{49}$$

The solutions of other physical parameters in terms of cosmic time have the following expressions after some manipulation between the constants:

$$H = (3\zeta_0/2a)[1 - 2aC\exp(-3\zeta_0 t/2)]^{-1}, \tag{50}$$

$$\rho = (3\zeta_0/2a)[1 - 2aC\exp(-3\zeta_0 t/2)]^{-2}. \tag{51}$$

We observe that the Universe starts from a non-singular state, characterized by constant and finite initial values of $R$, $H$ and $\rho$. Equation (49) shows that bulk viscosity gives rise to exponential expansion and is able to remove the initial singularity. The effect of bulk viscosity is capable of producing inflation in the early Universe. In the case where bulk viscosity vanishes, the solution reduces to $R \propto t^{1/a}$ in inflationary phase and $R \propto t^{1/2}$ for radiation phase, which exhibits singularity.

In the limit $a \to 0$, we get a hyper-inflationary phase with double exponential expansion with scale factor $R \propto \exp[\{\exp(3\zeta_0 t/2) - 1\}/3\zeta_0]$. We find that the Universe has finite dimension for $a = 0$ as $t \to -\infty$. The solution (49) can also be directly obtained from the general solution (29) or (31) after some manipulation.

*Case* ii: When $n = 1$, i.e. $\zeta \propto \rho$

Equation (20) becomes

$$\int [C(R/R_*)^a + (9\zeta_0/2a)] \frac{\mathrm{d}R}{R} = t + t_0. \tag{52}$$

Integrating (52), we get

$$\frac{C}{a}\left(\frac{R}{R_*}\right)^a + \frac{9\zeta_0}{2a}\ln R = t + t_0, \tag{53}$$

where $t_0$ is the integration constant and can be adjusted to zero. Study of asymptotic behaviour of (53) reveals that as $t \to -\infty$, $R \to 0$ and as $t \to +\infty$, $R \to \infty$. In general, it is not possible to express $R$ explicitly as a function of time. For sufficiently small $R$, the second term in eq. (53) dominates over the first, containing the viscosity term. Thus, we get exponential expansion of the form $R = \exp(H_* t)$, where $H_* = 2a/9\zeta_0$, which represents the singularity-free model and all solutions approach the de Sitter state with the expansion rate $H = H_*$. Such a non-singular behaviour is exhibited only in the presence of bulk viscosity. Such type of solution has been obtained by Murphy [9], where he attributed the viscosity effect to graviton production in the graviton–gravitons scattering. Since $H_* \propto \zeta_0^{-1}$, the smaller is the value of $\zeta_0$ the bigger is the inflation rate. During inflation both $\rho$ and $\zeta$ decrease exponentially with cosmic time and tend to zero as $t \to \infty$. The deceleration parameter has the value $q = -1$. The above solution (53) can also be directly obtained from the general solution (25) or (27) after some manipulation.

On the other hand, if the first term dominates we get the power-law expansion $R \propto t^{1/a}$ and we observe that the effect of viscous coefficient becomes negligible. The model shows singularity and the expansion continuously slows down but never reverses. Thus, we see that this model evolves from a de Sitter state as $t \to -\infty$ to zero curvature Friedmann state as $t \to +\infty$. At $t = 0$, the scale factor and energy density are finite, whereas the scale factor tends to zero and the energy density becomes infinite as $t \to -\infty$. The viscosity removes the initial singularity at finite past, i.e. moves it to the infinite past. The energy density and bulk viscosity decrease as time passes. Similarly, for radiation-dominated phase, where the bulk viscosity dominates we get $R \propto \exp(Ht)$, where $H = 4/9\zeta_0$, $\rho \propto \exp(-4Ht)$ and $\zeta \propto \exp(-4Ht)$. In the absence of viscous term, we have $R \propto t^{1/2}$, $\rho \propto t^{-2}$ and $\zeta \propto t^{-2}$. The deceleration parameter has the value $q = 1$, which shows the deceleration of the Universe.

*Case* iii: When $n = 3/2$, i.e. $\zeta \propto \rho^{3/2}$

Equation (20) becomes

$$\int [C(R/R_*)^{2a} + (3^{5/2}\zeta_0/2a)]^{1/2} \frac{\mathrm{d}R}{R} = t. \tag{54}$$

Integrating (54), we get

$$\begin{aligned} \left(\frac{2a}{3^{5/2}\zeta_0}\right)^{1/2} t = &\frac{1}{a} \ln \left[1 - \sqrt{1 + \frac{2aC}{3^{5/2}\zeta_0}\left(\frac{R}{R_*}\right)^{2a}}\right] \\ &-\frac{1}{a} \ln \left[1 + \sqrt{1 + \frac{2aC}{3^{5/2}\zeta_0}\left(\frac{R}{R_*}\right)^{2a}}\right] \\ &+\frac{1}{a}\sqrt{1 + \frac{2aC}{3^{5/2}\zeta_0}\left(\frac{R}{R_*}\right)^{2a}}. \end{aligned} \tag{55}$$

*Case* iv: When $n = 2$, i.e. $\zeta \propto \rho^2$

Equation (20) becomes

$$\int \left[C(R/R_*)^{3a} + (27\zeta_0/2a)\right]^{1/3} \frac{\mathrm{d}R}{R} = t. \tag{56}$$

Integrating (56), we get

$$\left(\frac{2a}{27\zeta_0}\right)^{1/3} t = \frac{1}{a}\left(1+\frac{2aCR^{3a}}{27\zeta_0 R_*^{3a}}\right)$$
$$+\frac{1}{3a}\ln\left[\left(1+\frac{2aC}{27\zeta_0}\left(\frac{R}{R_*}\right)^{3a}\right)^{1/3}-1\right]$$
$$-\frac{1}{6a}\ln\left[1+\left(1+\frac{2aC}{27\zeta_0}\left(\frac{R}{R_*}\right)^{3a}\right)^{2/3}\right.$$
$$\left.+\left(1+\frac{2aC}{27\zeta_0}\left(\frac{R}{R_*}\right)^{3a}\right)^{1/3}\right]$$
$$-\frac{1}{\sqrt{3}}\tan^{-1}\left[\frac{2}{\sqrt{3}}\left(1+\frac{2aC}{27\zeta_0}\left(\frac{R}{R_*}\right)^{3a}\right)^{2/3}+\frac{1}{\sqrt{3}}\right]. \quad (57)$$

*Case* v: When $n = 3/4$, i.e. $\zeta \propto \rho^{3/4}$

Equation (20) becomes

$$\int\left[1+\frac{2aC}{3^{7/4}\zeta_0}\left(\frac{R}{R_*}\right)^{a/2}\right]^2\frac{\mathrm{d}R}{R} = \left(\frac{2a}{3^{7/4}\zeta_0}\right)^2 t. \quad (58)$$

The solution of (58) is

$$\frac{a^3}{(3^{7/4}\zeta_0)^2}t = \frac{1}{4}\left(1+\frac{2aC}{3^{7/4}\zeta_0}\frac{R^{a/2}}{R_*^{a/2}}\right)^2 + \frac{1}{2}\left(1+\frac{2aC}{3^{7/4}\zeta_0}\frac{R^{a/2}}{R_*^{a/2}}\right)$$
$$+\frac{9}{4}\ln R + \frac{1}{2}\ln\left(\frac{2aC}{3^{7/4}\zeta_0 R_*^{a/2}}\right). \quad (59)$$

The last three cases show the same physical meaning as discussed in Case ii of this section.

## 5. Conclusion

We have presented a spatially homogeneous and isotropic flat Friedmann–Robertson–Walker model with bulk viscosity. We have discussed the model using ‘gamma-law’ equation of state, in which the adiabatic parameter gamma depends on the scale factor. A unified description of early evolution of Universe is presented in which an inflationary phase is followed by radiation-dominated phase. The value of free parameter $a$ used in functional form of gamma, related to inflationary phase, lies in the interval $0 \le a < 1$. As $a > 1$, the Universe shows the transition from inflationary phase to radiation-dominated phase. A particular case in the limit of $a \to 0$ has also been examined. In this case it is observed that the Universe is infinitely old and there is no real singularity. This work is more general than the author's previous works [30–33].

The concept of viscosity term has been used in a generalized form. We have analysed the consequence of the inclusion of such a dissipative term in both inflationary and radiation-dominated phases. The solutions of the model are qualitatively similar in both phases. We have observed that the bulk viscosity changes the behaviour from that of the perfect fluid model. We have seen that the introduction of viscosity term into the equation of Friedmann cosmology does not exclude automatically the appearance of singularity. Within the discussed class of models, singular and non-singular solutions appear.

It is evident from the three cases discussed in §3 that for $2n > 1$ and $2n < 1$, the strong energy condition $\rho + 3\bar{p} \geq 0$ is not satisfied for $C \leq 0$. It means that except for $C$ positive the energy conditions are violated throughout the evolution. But for $C > 0$ and $2n < 1$ the energy condition, which is equivalent to $\ddot{R} < 0$ is satisfied at the initial stage of expansion and violated at later stages, whereas for $C > 0$ and $2n > 1$ the energy condition is violated at the beginning and satisfied at later stages of evolution. When $2n = 1$, the solution is (39), where there is no de Sitter exact solution and the general solution is a power-law form, all energy conditions are satisfied. In this case there is a Big Bang-type singularity, where the evolution begins from singularity with a Friedmann leading behaviour. In case ii of §4, the weak energy condition $\rho \geq 0$ and dominant energy condition $\rho + \bar{p} \geq 0$ are always obeyed, but the strong energy condition $\rho + 3\bar{p} \geq 0$ is violated at early times, which allows the avoidance of singularity. Therefore, there is no problem with the Hawking–Penrose energy conditions [36] for $k = 0$ and $2n = 1$, because it is satisfied throughout the evolution. But the energy conditions are satisfied for part of this period either at initial stages or at later stages depending on whether $2n < 1$ or $2n > 1$. We also observe that similar type of energy condition $\mathrm{d}\bar{p}/d\rho < 1$ is violated at later stages or intial stages of evolution depending on whether $2n > 1$ or $2n < 1$, but there is no problem with $2n = 1$.

In the case of constant coefficient of bulk viscosity the solutions to the field equations can be expressed in an exact exponential form. The Universe starts from a non-singular state, characterized by constant and finite initial values of $R$, $H$ and $\rho$. In the case where the coefficient of bulk viscosity is proportional to energy density, we get a result which is similar to the solution obtained by Murphy. We observe that the bulk viscosity is capable of removing the cosmological singularity and the model approach to de Sitter Universe. The solution is quite different when $\zeta \propto \rho^{1/2}$. In this case, the solutions correspond to power-law inflation in inflationary phase and power-law expansion in radiation-dominated phase. The energy density and bulk viscosity decrease and tend to zero for large time. The Universe starts from a singular state.

We have presented the complete solutions of flat FRW model by taking the cases $2n > 1$, $0 \leq 2n < 1$ and $2n = 1$ with varying 'gamma' of 'gamma-law' equation of state, if one takes the viscosity coefficient to have the general form $\zeta(\rho) = \zeta_0 \rho^n$. The solutions with $0 \leq n < 1/2$ display the inflationary behaviour (de Sitter solution), where the Big-Bang singularity of infinite density occurs at finite past, but the solutions with $n > 1/2$ exhibit the deflationary behaviour. The special intermediate case $n = 1/2$ leads to power-law expansion and there is no de Sitter solution. Our results show that the occurrence of viscosity-driven exponential inflationary behaviour depends mainly on the values of $n$. We have observed that Murphy's model is the only special case of these general sets of solutions. The effect of bulk viscosity is more prominent at the beginning of the Universe, where the expansion scalar and energy density are quite large. In this way, we have studied flat FRW model in a unified manner and are able to find out solutions for two phases of early evolution of Universe, which explain the early expansion of the Universe driven by bulk viscous term.

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